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ON A SDE DRIVEN BY A FRACTIONAL BROWNIAN MOTION AND WITH MONOTONE DRIFT

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Abstract
Let $\{B_t^H, t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. We prove the existence of a weak solution for a stochastic differential equation of the form $X_t = x + B_t^H + \int_0^t (b_1(s,X_s) + b_2(s,X_s))\,ds$, where $b_1(s,x)$ is a Hölder continuous function of order strictly larger than $1-\frac{1}{2H}$ in x and than $H-\frac{1}{2}$ in time and b_2 is a real bounded nondecreasing and left (or right) continuous function.

1 Introduction

Let $B^H = \{B_t^H, t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. That is, B^H is a centered Gaussian process with covariance

$$R_H(t,s) = E(B_t^H B_s^H) = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}.$$

If $H = \frac{1}{2}$ the process B^H is a standard Brownian motion. Consider the following stochastic differential equation

$$X_t = x + B_t^H + \int_0^t (b_1(s,X_s) + b_2(s,X_s))\,ds, \tag{1.1}$$

where $b_1, b_2 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are Borel functions. The purpose of this paper is to prove, by approximation arguments, the existence of a weak solution to this equation if $H > \frac{1}{2}$, under the following weak regularity assumptions on the coefficients:

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$(\mathbf{H}_1)$ b_1 is Hölder continuous of order $1 > \alpha > 1 - \frac{1}{2H}$ in x and of order $\gamma > H - \frac{1}{2}$ in time:

$$|b_1(t,x) - b_1(s,y)| \leq C\left(|x-y|^{\alpha} + |t-s|^{\gamma}\right). \tag{1.2}$$

$(\mathbf{H}_2)$ $\sup_{s\in[0,T]} \sup_{x\in\mathbb{R}} | b_2(s,x) | \leq M < \infty.$

$(\mathbf{H}_3)$ $\forall s \in [0,T]\,,\ b_2(s,.)$ is a nondecreasing and left (or right) continuous function.

The same approximation arguments can be used to consider the case where b_2 satisfies the following assumptions:

$(\mathbf{H'}_2)$ $\sup_{s\in[0,T]} \sup_{x\in\mathbb{R}} | b_2(s,x) | \leq M(1+|x|)$

$(\mathbf{H'}_3)$ for all $s \in [0,T]\,,\ b_2(s,.)$ is a nonincreasing and continuous function

If $b_2 \equiv 0$ and $H = \frac{1}{2}$ (the process B^H is a standard Brownian motion), the existence of a strong solution is well-known by the results of Zvonkin [18], Veretennikov [16] and Bahlali [2]. See also the work by Nakao [11] and its generalization by Ouknine [14]. In the case of Equation (1.1) driven by the fractional Brownian motion with $b_2 \equiv 0$, the weak existence and uniqueness are established in [13] using a suitable version of Girsanov theorem; the existence of a strong solution could be deduced from an extension of Yamada-Watanabe's theorem or by a direct arguments.
In the general case $H > 1/2$, to establish existence and uniqueness result, a Hölder type space-time condition is imposed on the drift. Recently, Mishura and Nualart [9] gave an existence and uniqueness result for one discontinuous function namely the *sgn* function. Their approach relies on the Novikov criterion and it is valid for $\frac{1+\sqrt{5}}{4} > H > 1/2$.
Our aim is to establish existence and uniqueness result for general monotone function including *sgn* function and $H > 1/2$.
The paper is organized as follows. In Section 2 we give some preliminaries on fractional calculus and fractional Brownian motion. In Section 3 we formulate a Girsanov theorem and show the existence of a weak solution to Equation (1.1). As a consequence we deduce the uniqueness in law and the pathwise uniqueness. Finally Section 4 discusses the existence of a strong solution.

2 Preliminaries

2.1 Fractional calculus

An exhaustive survey on classical fractional calculus can be found in [15]. We recall some basic definitions and results.
For $f \in L^1([a,b])$ and $\alpha > 0$ the *left fractional Riemann-Liouville integral* of f *of order* α on (a,b) is given at almost all x by

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy,$$

where Γ denotes the Euler function.
This integral extends the usual n-order iterated integrals of f for $\alpha = n \in \mathbb{N}$. We have the first composition formula

$$I^{\alpha}_{a+}(I^{\beta}_{a+}f) = I^{\alpha+\beta}_{a+}f.$$

The fractional derivative can be introduced as inverse operation. We assume $0 < \alpha < 1$ and $p > 1$. We denote by $I^{\alpha}_{a+}(L^p)$ the image of $L^p([a,b])$ by the operator I^{α}_{a+} . If $f \in I^{\alpha}_{a+}(L^p)$, the function ϕ such that $f = I^{\alpha}_{a+}\phi$ is unique in L^p and it agrees with the *left-sided Riemann-Liouville derivative of f of order* α defined by

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(y)}{(x-y)^{\alpha}}dy.$$

The derivative of f has the following Weil representation:

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}dy\right)\mathbf{1}_{(a,b)}(x), \tag{2.1}$$

where the convergence of the integrals at the singularity $x = y$ holds in L^p-sense.
When $\alpha p > 1$ any function in $I^{\alpha}_{a+}(L^p)$ is $\left(\alpha - \frac{1}{p}\right)$- Hölder continuous. On the other hand, any Hölder continuous function of order $\beta > \alpha$ has fractional derivative of order α. That is, $C^{\beta}([a,b]) \subset I^{\alpha}_{a+}(L^p)$ for all $p > 1$.
Recall that by construction for $f \in I^{\alpha}_{a+}(L^p)$,

$$I^{\alpha}_{a+}(D^{\alpha}_{a+}f) = f$$

and for general $f \in L^1([a,b])$ we have

$$D^{\alpha}_{a+}(I^{\alpha}_{a+}f) = f.$$

If $f \in I^{\alpha+\beta}_{a+}(L^1)$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq 1$ we have the second composition formula

$$D^{\alpha}_{a+}(D^{\beta}_{a+}f) = D^{\alpha+\beta}_{a+}f.$$

2.2 Fractional Brownian motion

Let $B^H = \{B^H_t, t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $0 < H < 1$ defined on the probability space $(\Omega, \mathcal{F}, P)$. For each $t \in [0,T]$ we denote by $\mathcal{F}^{B^H}_t$ the σ-field generated by the random variables B^H_s, $s \in [0,t]$ and the sets of probability zero.
We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow B^H_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B^H)$ associated with B^H. We will denote this isometry by $\varphi \longrightarrow B^H(\varphi)$.
The covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r)K_H(s,r)dr,$$

where K_H is a square integrable kernel given by (see [3]):

$$K_H(t,s) = \Gamma(H+\frac{1}{2})^{-1}(t-s)^{H-\frac{1}{2}}F(H-\frac{1}{2},\frac{1}{2}-H,H+\frac{1}{2},1-\frac{t}{s}),$$

$F(a,b,c,z)$ being the Gauss hypergeometric function. Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$(K_H^*\varphi)(s) = K_H(T,s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r,s)dr.$$

For any pair of step functions φ and ψ in $\mathcal{E}$ we have (see [1])

$$\langle K_H^*\varphi, K_H^*\psi\rangle_{L^2([0,T])} = \langle \varphi, \psi\rangle_{\mathcal{H}}.$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert spaces $\mathcal{H}$ and $L^2([0,T])$. Hence, the process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t = B^H((K_H^*)^{-1}(\mathbf{1}_{[0,t]})) \tag{2.2}$$

is a Wiener process, and the process B^H has an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s)dW_s, \tag{2.3}$$

because $\left(K_H^*\mathbf{1}_{[0,t]}\right)(s) = K_H(t,s)\,\mathbf{1}_{[0,t]}(s)$.
On the other hand, the operator K_H on $L^2([0,T])$ associated with the kernel K_H is an isomorphism from $L^2([0,T])$ onto $I_{0+}^{H+1/2}(L^2([0,T]))$ and it can be expressed in terms of fractional integrals as follows (see [3]):

$$(K_H h)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} h, \text{ if } H \le 1/2, \tag{2.4}$$

$$(K_H h)(s) = I_{0+}^{1} s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h, \text{ if } H \ge 1/2, \tag{2.5}$$

where $h \in L^2([0,T])$.
We will make use of the following definition of $\mathcal{F}_t$-fractional Brownian motion.

2.1 Definition. Let $\{\mathcal{F}_t, t \in [0,T]\}$ be a right-continuous increasing family of σ-fields on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains the sets of probability zero. A fractional Brownian motion $B^H = \{B_t^H, t \in [0,T]\}$ is called an $\mathcal{F}_t$-fractional Brownian motion if the process W defined in (2.2) is an $\mathcal{F}_t$-Wiener process.

3 Existence of strong solution for SDE with monotone drift.

In this section we are interested by the special case $b_1 \equiv 0$. We will prove by approximation arguments that there is a strong solution of equation (1.1). We will discuss two cases:
1) $b_2(s,.)$ satisfies $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$.
2) $b_2(s,.)$ satisfies $(\mathbf{H'}_2)$ and $(\mathbf{H'}_3)$.

1– The first case:

To treat the first situation, let us suppose that $b_2(s,.)$ is nondecreasing and left continuous function. We will use the following approximation lemma:

3.1 Lemma. *Let $b : [0,T]\times\mathbb{R} \to \mathbb{R}$, a bounded measurable function such that for any $s \in [0,T]$, $b(s,.)$ is a nondecreasing and left continuous function. Then there exists a family of measurable functions*

$$\{b_n(s,x);\ \ n \geq 1,\ \ s \in [0,T], x \in \mathbb{R}\}$$

such that

$$\begin{cases} \bullet\ \textit{For any sequence } x_n \textit{ increasing to } x \in \mathbb{R},\ \textit{ we have} \\ \quad \lim\limits_{n\to\infty} b_n(s,x_n) = b(s,x), \quad ds \quad a.e. \\ \bullet\ \ x \mapsto b_n(s,x) \ \textit{ is nondecreasing, for all } n \geq 1,\ s \in [0,T] \\ \bullet\ n \mapsto b_n(s,x) \ \textit{ is nondecreasing, for all } x \in \mathbb{R},\ s \in [0,T] \\ \bullet\ |b_n(s,x) - b_n(s,y)| \leq 2n\, M\ |x-y| \ \textit{ for all } n \geq 1,\ s \in [0,T] \\ \bullet\ \sup\limits_{n\geq 1}\ \sup\limits_{s\in[0,T]}\ \sup\limits_{x\in\mathbb{R}} |b_n(s,x)| \leq M. \end{cases}$$

Proof. First assume that $b(s,.)$ is left continuous and let us choose for any $n \geq 1$

$$b_n(s,x) = n \int_{x-\frac{1}{n}}^{x} b(s,y)\, dy\,.$$

Since $b(s,.)$ is nondecreasing then $b_n(s,.)$ is also a nondecreasing function for any fixed $n \geq 1$. Let x, $y \in \mathbb{R}$, we clearly have for any $n \geq 1$,

$$\mid b_n(s,x) - b_n(s,y) \mid \leq 2\, n\, M \ \mid x - y \mid\ . \tag{3.1}$$

Obviously, we get that b_n is uniformly bounded by the constant M. Let $n < m$, $s \in [0,T]$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} b_m(s,x) - b_n(s,x) &= (m-n) \int_{x-\frac{1}{m}}^{x} b(s,y)\, dy - n \int_{x-\frac{1}{n}}^{x-\frac{1}{m}} b(s,y)\, dy \\ &\geq (m-n) \int_{x-\frac{1}{m}}^{x} b(s,y)\, dy - \frac{m-n}{m}\, b(s,x-\frac{1}{m})\,, \\ &= (m-n) \int_{x-\frac{1}{m}}^{x} \left(b(s,y) - b(s,x-\frac{1}{m}) \right)\, dy \geq 0\,. \end{aligned}$$

Now let $x_0 \in \mathbb{R}$ and take an increasing sequence of real numbers x_n converging to x_0. We want to show that for any $s \in [0,T]$, $\lim\limits_{n\to\infty} b_n(s,x_n) = b(s,x_0)$. It is enough to prove that there exists a subsequence $b_{\varphi(n)}(s,x_{\varphi(n)})$ which converges to $b(s,x_0)$. To do this, remark first that since $b(s,.)$ is left continuous we have $\lim\limits_{n\to\infty} b_n(s,x_0) = b(s,x_0)$. Now let us consider any strictly increasing sequence x'_n converging to x_0 such that $x_0 - x'_n = o(\frac{1}{n})$. We clearly get by (3.1)

$$\forall s \in [0,T]\,,\ \ \lim_{n\to\infty} b_n(s,x'_n) = b(s,x_0)\,. \tag{3.2}$$

We may choose a sequence $\varphi(n) \geq n$ such that $x'_n \leq x_{\varphi(n)}$. Since $(b_n(s,x))_{n\geq 1}$ is increasing and for any fixed $n \geq 1$ the function $b_n(s,.)$ is nondecreasing, we have

$$b_n(s,x'_n) \leq b_{\varphi(n)}(s,x_{\varphi(n)}) \leq b(s,x_{\varphi(n)})\,. \tag{3.3}$$

We deduce by (3.2) and the left continuity of $b(s,.)$,

$$\lim_{n\to\infty} b_{\varphi(n)}(s, x_{\varphi(n)}) = b(s, x_0)\,.$$

Which ends the proof. □

Let $(B^H)_{t\geq 1}$ be a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. We consider the following SDE

$$X_t = x + B_t^H + \int_0^t b_2(s, X_s)\, ds\,, \quad 0 \leq t \leq T\,. \tag{3.4}$$

3.2 Theorem. *Suppose that b_2 satisfies the assumptions* $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$*. Then there exists a strong solution to the equation (3.4).*

Proof. Assume that $b_2 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable and bounded function which is nondecreasing and left continuous with respect to the space variable x. For $n \geq 1$, let b_n be as in lemma 3.1 and consider the following SDE

$$X_t^n = x + B_t^H + \int_0^t b_n(s, X_s^n)\, ds\,, \quad 0 \leq t \leq T\,. \tag{3.5}$$

By standard Picard's iteration argument, one may show that for any $n \geq 1$, the equation (3.5) has a strong solution which we denote by X^n.
Let $n > m$, we denote by $\Delta_t = X_t^n - X_t^m$. Using the monotony argument on b_n, we have

$$\begin{aligned}
\Delta_t &\geq \int_0^t b_m(s, X_s^n) - b_m(s, X_s^m)\, ds\,,\\
&\geq \int_0^t \left(b_m(s, X_s^n) - b_m(s, X_s^m)\right)\, I_{\{\Delta_s \leq 0\}}\, ds\,,\\
&\geq 2\,mM \int_0^t \Delta_s\, I_{\{\Delta_s \leq 0\}}\, ds \geq -2\,mM \int_0^t \Delta_s^-\, ds\,.
\end{aligned} \tag{3.6}$$

We then get

$$\Delta_t^- \leq 2\,mM \int_0^t \Delta_s^-\, ds\,. \tag{3.7}$$

By Gronwall's lemma, we have for almost all w and for any $t \in [0,T]$, the sequence $(X_t^n(w))$ is a nondecreasing function of n which is bounded since b_n is. Therefore it has a limit when $n \to \infty$ and we set

$$\lim_{n\to\infty} X_t^n(\omega) = X_t(\omega)\,,$$

which entails in particular that X is $\mathcal{F}_t^{B^H}-$ adapted. Applying the convergence result in Lemma 3.1 and the boundedness of b_n we get by Lebesgue's dominated convergence theorem,

$$X_t = x + B_t^H + \int_0^t b_2(s, X_s)\, ds\,.$$

□

3.1 Remark. To show that Equation (3.4) has a weak solution, a continuity condition is imposed on the drift in [13]. Here, the function b_2 may have a countable set of discontinuity points. The solution constructed in Theorem 3.2 is the minimal one.

3.2 Remark. Let $b_2 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded measurable function, which is nondecreasing and right continuous. In this case we consider a decreasing sequence of Lipschitz continuous functions which approximate the drift. One may take

$$b_n(s,x) = n \int_x^{x+\frac{1}{n}} b_2(s,y)\, dy\,.$$

For any fixed $(s,x) \in [0,T] \times \mathbb{R}$, the sequence $(b_n(s,x))_{n\geq 1}$ is nonincreasing and for any fixed $n \geq 1$ and $s \in [0,T]$ the function $b_n(s,.)$ is nondecreasing. The same arguments as in Lemma 3.1 can be used to prove that for any sequence $(x_n)_{n\geq 1}$ decreasing to x, we have

$$\lim_{n\to\infty} b_n(s,x_n) = b_2(s,x)\,.$$

This allows us to construct the maximal solution to the equation (3.4).

2– The second case:

In this case we use the following lemma:

3.3 Lemma. *Let $b(.,.) : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with linear growth, that is there exists a constant $M < \infty$ such that $\forall (s,x) \in [0,T] \times \mathbb{R}$, $\mid b(s,x) \mid \leq M\,(1+ \mid x \mid)$. Then the sequence of functions*

$$b_n(s,x) = \sup_{y\in Q} \left(b(s,y)\ - n\,|x-y|\right)\ ,$$

is well defined for $n \geq M$ and it satisfies

$$\begin{cases} \bullet\ \textit{For any sequence}\, x_n \textit{ converging to}\, x \in \mathbb{R},\ \textit{ we have} \\ \quad \lim_{n\to\infty} b_n\left(s,x_n\right) = b\left(s,x\right)\ , \\ \bullet\ n \mapsto b_n\left(s,x\right)\ \textit{ is nonincreasing, for all } x \in \mathbb{R},\ \ s \in [0,T] \\ \bullet\ \ |b_n\left(s,x\right) - b_n\left(s,y\right)| \leq n\ |x-y|\ \textit{ for all } n \geq M,\ \ s \in [0,T]\,,\ \ x,y \in \mathbb{R} \\ \bullet\ \ |b_n\left(s,x\right)| \leq M(1+ \mid x \mid)\,,\ \textit{for all}\,(s,x) \in [0,T] \times \mathbb{R},\ \ n \geq M\,. \end{cases}$$

For the proof of this lemma we refer for example to [8].

3.4 Theorem. *Assume that b_2 satisfies conditions $\mathbf{H_2}'$ and $\mathbf{H_3}'$. Then there exists a unique strong solution to the equation (3.4).*

Proof. For any $n \geq 1$, let b_n be as in Lemma 3.3. Since b_n is Lipschitz and linear growth, the result in [13] assures the existence of a strong solution X^n to the equation

$$X_t^n = x + B_t^H + \int_0^t b_n(s, X_s^n)\, ds\,.$$

Since $(b_n)_{n\geq 1}$ is nonincreasing, comparison theorem entails that $(X^n)_{n\geq 1}$ is a.s nonincreasing. By the linear growth condition on b_n and Gronwall's lemma we may deduce that X^n converges a.s to X, which is clearly a strong solution to the SDE (3.4). Moreover, if X^1 and X^2 are two solutions of (3.4), using the fact that $b_2(s,.)$ is nonincreasing, we get by applying Tanaka's formula to the continuous semi-martingale $X^1 - X^2$,

$$(X_t^1 - X_t^2)^+ = \int_0^t sign(X_s^1 - X_s^2)\,\left(b_2(s,X_s^1) - b_2(s,X_s^2)\right)\,ds \leq 0\,.$$

Then we have the pathwise uniqueness of the solution. □

4 Existence of a weak solution

4.1 Girsanov transform

As in the previous section, let B^H be a fractional Brownian motion with Hurst parameter $0 < H < 1$ and denote by $\left\{\mathcal{F}_t^{B^H}, t \in [0,T]\right\}$ its natural filtration.
Given an adapted process with integrable trajectories $u = \{u_t, t \in [0,T]\}$ and consider the transformation

$$\widetilde{B}_t^H = B_t^H + \int_0^t u_s ds. \tag{4.1}$$

We can write

$$\begin{aligned}\widetilde{B}_t^H &= B_t^H + \int_0^t u_s ds = \int_0^t K_H(t,s)dW_s + \int_0^t u_s ds \\ &= \int_0^t K_H(t,s)d\widetilde{W}_s,\end{aligned}$$

where

$$\widetilde{W}_t = W_t + \int_0^t \left(K_H^{-1}\left(\int_0^\cdot u_s ds\right)(r)\right) dr. \tag{4.2}$$

Notice that $K_H^{-1}\left(\int_0^\cdot u_s ds\right)$ belongs a.s to $L^2([0,T])$ if and only if $\int_0^\cdot u_s ds \in I_{0+}^{H+1/2}(L^2([0,T]))$. As a consequence we deduce the following version of the Girsanov theorem for the fractional Brownian motion, which has been obtained in [3, Theorem 4.9]:

4.1 Theorem. *Consider the shifted process (4.1) defined by a process $u = \{u_t, t \in [0,T]\}$ with integrable trajectories. Assume that:*

i) $\int_0^\cdot u_s ds \in I_{0+}^{H+1/2}(L^2([0,T]))$, *almost surely.*

ii) $E(\xi_T) = 1$, *where*

$$\xi_T = \exp\left(-\int_0^T \left(K_H^{-1}\int_0^\cdot u_s ds\right)(s)dW_s - \frac{1}{2}\int_0^T \left(K_H^{-1}\int_0^\cdot u_s ds\right)^2(s)ds\right).$$

Then the shifted process $\widetilde{B}^H$ is an $\mathcal{F}_t^{B^H}$ - fractional Brownian motion with Hurst parameter H under the new probability $\widetilde{P}$ defined by $\frac{d\widetilde{P}}{dP} = \xi_T$.

Proof. By the standard Girsanov theorem applied to the adapted and square integrable process $K_H^{-1}\left(\int_0^{\cdot} u_s ds\right)$ we obtain that the process $\widetilde{W}$ defined in (4.2) is an $\mathcal{F}_t^{B^H}$- Brownian motion under the probability $\widetilde{P}$. Hence, the result follows. □

From (2.5) the inverse operator K_H^{-1} is given by

$$K_H^{-1} h = s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h', \text{ if } H > 1/2 \tag{4.3}$$

for all $h \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$. Then if $H > \frac{1}{2}$ we need $u \in I_{0+}^{H-1/2}(L^2([0,T]))$, and a sufficient condition for i) is the fact that the trajectories of u are Hölder continuous of order $H - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$.

4.2 Existence of a weak solution

Consider the stochastic differential equation:

$$X_t = x + B_t^H + \int_0^t \left(b_1(s, X_s) + b_2(s, X_s)\right) ds, \ 0 \le t \le T, \tag{4.4}$$

where b_1 and b_2 are Borel functions on $[0,T] \times \mathbb{R}$ satisfying the conditions $\mathbf{H_1}$ for b_1 and $\mathbf{H_2}$ and $\mathbf{H_3}$ (resp. $\mathbf{H}_2'$ and $\mathbf{H}_3'$) for b_2. By a *weak solution* to equation (4.4) we mean a couple of adapted continuous processes $\left(B^H, X\right)$ on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0,T]\})$, such that:

i) B^H is an $\mathcal{F}_t$ -fractional Brownian motion in the sense of Definition 2.1.

ii) X and B^H satisfy (4.4).

4.2 Theorem. *Suppose that b_1 and b_2 are Borel functions on $[0,T]\times\mathbb{R}$ satisfying the conditions $\mathbf{H_1}$ for b_1, $\mathbf{H_2}$ and $\mathbf{H_3}$ (resp. $\mathbf{H_2}'$ and $\mathbf{H_3}'$) for b_2. Then Equation (4.4) has a weak solution.*

Proof. Let X^2 be the strong solution of (3.4) and set $\widetilde{B}_t^H = B_t^H - \int_0^t b_1(s, X_s^2) ds$. We claim that the process $u_s = -b_1(s, X_s^2)$ satisfies conditions i) and ii) of Theorem 4.1. If this claim is true, under the probability measure $\widetilde{P}$, $\widetilde{B}^H$ is an $\mathcal{F}_t^{B^H}$-fractional Brownian motion, and $\left(\widetilde{B}^H, X^2\right)$ is a weak solution of (4.4) on the filtered probability space $\left(\Omega, \mathcal{F}, \widetilde{P}, \left\{\mathcal{F}_t^{B^H}, t \in [0,T]\right\}\right)$.
Set

$$v_s = -K_H^{-1}\left(\int_0^{\cdot} b_1(r, X_r^2) dr\right)(s).$$

We will show that the process v satisfies conditions i) and ii) of Theorem 4.1. Along the proof c_H will denote a generic constant depending only on H. Let $H > \frac{1}{2}$, by (4.3), the process v is clearly adapted and we have

$$\begin{aligned} v_s &= -s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} b_1(s, X_s^2) \\ &:= -c_H\left(\alpha(s) + \beta(s)\right), \end{aligned}$$

where

$$\begin{aligned}\alpha(s) &= b_1(s,X_s^2)s^{\frac{1}{2}-H} \\ &+(H-\frac{1}{2})s^{H-\frac{1}{2}}b_1(s,X_s^2)\int_0^s \frac{s^{\frac{1}{2}-H}-r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}}dr \\ &+(H-\frac{1}{2})s^{H-\frac{1}{2}}\int_0^s \frac{b_1(s,X_s^2)-b_1(r,X_s^2)}{(s-r)^{\frac{1}{2}+H}}r^{\frac{1}{2}-H}dr.\end{aligned}$$

and

$$\beta(s)=(H-\frac{1}{2})s^{H-\frac{1}{2}}\int_0^s \frac{b_1(r,X_s^2)-b_1(r,X_r^2)}{(s-r)^{\frac{1}{2}+H}}r^{\frac{1}{2}-H}dr.$$

Using the estimate

$$\left|b_1(s,X_s^2)\right| \le |b(0,x)| + C\left(|s|^\gamma + \left|X_s^2\right|^\alpha\right)$$

and the equality

$$\int_0^s \frac{r^{\frac{1}{2}-H}-s^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}}dr = c_H s^{1-2H},$$

we obtain

$$\begin{aligned}|\alpha(s)| &\le c_H\left(s^{\frac{1}{2}-H}\left[|b_1(0,x)|+C\left(|s|^\gamma+\left|X_s^2\right|^\alpha\right)\right]+Cs^{\gamma+\frac{1}{2}-H}\right) \\ &\le c_H s^{\frac{1}{2}-H}\left(C\left\|X^2\right\|_\infty^\alpha + Cs^\gamma + |b_1(0,x)|\right).\end{aligned}$$

As consequence, taking into account that $\alpha<1$, we have for any $\lambda>1$

$$E\left(\exp\left(\lambda\int_0^T \alpha(s)^2 ds\right)\right)<\infty. \tag{4.5}$$

In order to estimate the term $\beta(s)$, we apply the Hölder continuity condition (1.2) and we get

$$\begin{aligned}|\beta(s)| &\le c_H s^{H-\frac{1}{2}}\int_0^s\left(\frac{\left|X_s^2-X_r^2\right|^\alpha}{(s-r)^{H+\frac{1}{2}}}+\frac{|r-s|^\gamma}{(s-r)^{\frac{1}{2}+H}}\right)r^{\frac{1}{2}-H}dr \\ &\le c_H\, s^{H-\frac{1}{2}}\int_0^s\left(\frac{\left|B_s^H-B_r^H\right|^\alpha}{(s-r)^{H+\frac{1}{2}}}+(s-r)^{\alpha-H-\frac{1}{2}}+\frac{|r-s|^\gamma}{(s-r)^{\frac{1}{2}+H}}\right)r^{\frac{1}{2}-H}dr \\ &\le c_H s^{\frac{1}{2}-H+\alpha(H-\varepsilon)}G^\alpha,\end{aligned}$$

where we have fixed $\varepsilon<H-\frac{1}{\alpha}(H-\frac{1}{2})$ and we denote

$$G=\sup_{0\le s<r\le 1}\frac{\left|B_s^H-B_r^H\right|}{|s-r|^{H-\varepsilon}}.$$

By Fernique's Theorem, taking into account that $\alpha<1$, for any $\lambda>1$ we have

$$E\left(\exp\left(\lambda\int_0^T \beta(s)^2 ds\right)\right)<\infty,$$

and we deduce condition ii) of Theorem 4.1 by means of Novikov criterion. □

4.3 Uniqueness in law and pathwise uniqueness

In this subsection we will prove uniqueness in law of weak solution under the condition $\mathbf{H_1}$ for b_1, $\mathbf{H_2}'$ and $\mathbf{H_3}'$ for b_2. The main result is

4.3 Theorem. *Suppose that b_1 and b_2 are Borel functions on $[0,T]\times\mathbb{R}$. satisfying the conditions $\mathbf{H_1}$ for b_1, $\mathbf{H_2}'$ and $\mathbf{H_3}'$ for b_2. Then we have the uniqueness in distribution for the solution of Equation (4.4).*

Proof. It is clear that X^2 is pathwise unique, hence the uniqueness in law holds when $b_1 \equiv 0$. Let $\left(X, B^H\right)$ be a solution of the stochastic differential equation (4.4) defined in the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0,T]\})$. Define

$$u_s = \left(K_H^{-1} \int_0^{\cdot} b_1(r, X_r))dr\right)(s).$$

Let $\widetilde{P}$ defined by

$$\frac{d\widetilde{P}}{dP} = \exp\left(-\int_0^T u_s dW_s - \frac{1}{2}\int_0^T u_s^2 ds\right). \tag{4.6}$$

We claim that the process u_s satisfies conditions i) and ii) of Theorem 4.1. In fact, u_s is an adapted process and taking into account that X_t has the same regularity properties as the fBm we deduce that $\int_0^T u_s^2 ds < \infty$ almost surely. Finally, we can apply again Novikov theorem in order to show that $E\left(\frac{d\widetilde{P}}{dP}\right) = 1$, because by Gronwall's lemma

$$\|X\|_\infty \leq \left(|x| + \left\|B^H\right\|_\infty + C_1 T\right) e^{C_2 T},$$

and

$$|X_t - X_s| \leq |B_t^H - B_s^H| + C_3|t-s|(1 + \|X\|_\infty)$$

for some constants C_i, $i = 1,2,3$.
By the classical Girsanov theorem the process

$$\widetilde{W}_t = W_t + \int_0^t u_r dr$$

is an $\mathcal{F}_t$-Brownian motion under the probability $\widetilde{P}$. In terms of the process $\widetilde{W}_t$ we can write

$$X_t = x + \int_0^t K_H(t,s)\, d\widetilde{W}_s + \int_0^t b_2(s, X_s)\, ds\,,$$

Set

$$\widetilde{B}_s^H = \int_0^t K_H(t,s) d\widetilde{W}_s\,.$$

Then X satisfies the following SDE,

$$X_t = x + \widetilde{B}_t^H + \int_0^t b_2(s, X_s) ds$$

As a consequence, the processes X and X^2 have the same distribution under the probability P. In fact, if Ψ is a bounded measurable functional on $C([0,T])$, we have

$$\begin{aligned}
E_P(\Psi(X)) &= \int_\Omega \Psi(\xi)\frac{dP}{d\widetilde{P}}(\xi)d\widetilde{P} \\
&= E_{\widetilde{P}}\left(\Psi(X)\exp\left(\int_0^T\left(K_H^{-1}\int_0^{\cdot} b_1(r,X_r)dr\right)(s)dW_s\right.\right. \\
&\quad \left.\left.+\frac{1}{2}\int_0^T\left(K_H^{-1}\int_0^{\cdot} b_1(r,X_r)dr\right)^2(s)ds\right)\right) \\
&= E_{\widetilde{P}}\left(\Psi(X)\left(\exp\int_0^T\left(K_H^{-1}\int_0^{\cdot} b_1(r,X_r)dr\right)(s)d\widetilde{W}_s\right.\right. \\
&\quad \left.\left.-\frac{1}{2}\int_0^T\left(K_H^{-1}\int_0^{\cdot} b_1(r,X_r)dr\right)^2(s)ds\right)\right) \\
&= E_P\left(\Psi(X^2)\left(\exp\int_0^T\left(K_H^{-1}\int_0^{\cdot} b_1(r,X_r^2)dr\right)(s)dW_s\right)\right) \\
&\quad \left.\left.-\frac{1}{2}\int_0^T\left(K_H^{-1}\int_0^{\cdot} b_1(r,X_r^2)dr\right)^2(s)ds\right)\right) \\
&= E_P(\Psi(X^2)).
\end{aligned}$$

In conclusion we have proved the uniqueness in law, which is equivalent to pathwise uniqueness (see [13] Theorem 5) □

4.1 Remark. In the case $H < 1/2$, a deep study is made between stochastic differential equation with continuous coefficient and unit drift and anticipating ones (cf [4]).

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