Elect. Comm. in Probab. **11** (2006), 24–34

ELECTRONIC COMMUNICATIONS in PROBABILITY

INVARIANT MEASURES FOR STOCHASTIC CAUCHY PROBLEMS WITH ASYMPTOTICALLY UNSTABLE DRIFT SEMIGROUP

ONNO VAN GAANS[1]
Mathematical Institute
Leiden University
P.O. Box 9512
2300 RA Leiden
The Netherlands
email: `vangaans@math.leidenuniv.nl`

JAN VAN NEERVEN[2]
Delft Institute of Applied Mathematics
Technical University of Delft
P.O. Box 5031
2600 GA Delft
The Netherlands
email: `J.M.A.M.vanNeerven@tudelft.nl`

Submitted 25 August 2005, *accepted in final form* 23 March 2006

AMS 2000 Subject classification: 35R15, 47D06, 60H05
Keywords: Invariant measures, stochastic evolution equations in Hilbert spaces

Abstract
We investigate existence and permanence properties of invariant measures for abstract stochastic Cauchy problems of the form

$$dU(t) = (AU(t) + f)\,dt + B\,dW_H(t), \quad t \geqslant 0,$$

governed by the generator A of an asymptotically unstable C_0-semigroup on a Banach space E. Here $f \in E$ is fixed, W_H is a cylindrical Brownian motion over a separable real Hilbert space H, and $B : H \to E$ is a bounded operator. We show that if $c_0 \not\subseteq E$, such invariant measures fail to exist generically but may exist for a dense set of operators B. It turns out that many results on invariant measures which hold under the assumption of uniform exponential stability of S break down without this assumption.

[1]RESEARCH SUPPORTED BY THE RESEARCH TRAINING NETWORK HPRN-CT-2002-00281
[2]RESEARCH SUPPORTED BY THE NWO VIDI SUBSIDIE 639.032.201 AND THE RESEARCH TRAINING NETWORK HPRN-CT-2002-00281

1. Introduction

Let A be the infinitesimal generator of a C_0-semigroup $S = \{S(t)\}_{t\geqslant 0}$ on a real Banach space E and let $W_H = \{W_H(t)\}_{t\geqslant 0}$ be a cylindrical Brownian motion over a separable real Hilbert space H. In this note we study invariant measures for the stochastic abstract Cauchy problem of the form

$$dU(t) = (AU(t) + f)\,dt + B\,dW_H(t), \quad t \geqslant 0, \tag{1.1}$$

where $f \in E$ is a fixed vector and $B \in \mathscr{L}(H, E)$ is a bounded operator. We are interested in the situation where the semigroup S fails to be uniformly exponentially stable and intend to answer such questions as for 'how many' operators B an invariant measure exists and what can be said about its properties.

If the problem (1.1) with initial condition $U(0) = x$ has a weak solution $U = U^x$, then this solution is unique up to modification and it is given explicitly by the stochastic convolution integral

$$U^x(t) = S(t)x + \int_0^t S(t-s)f\,ds + \int_0^t S(t-s)B\,dW_H(s), \quad t \geqslant 0. \tag{1.2}$$

We refer to [11] for more details and unexplained terminology. A Radon probability measure μ on E is called an *invariant measure* for the problem (1.1) if for all $t \geqslant 0$ and all bounded real-valued Borel functions $\varphi \in C_b(E)$ we have

$$\int_E P(t)\varphi\,d\mu = \int_E \varphi\,d\mu,$$

where $P(t)\varphi \in C_b(E)$ is given as

$$P(t)\varphi(x) = \mathbb{E}\varphi(U^x(t)), \quad x \in E.$$

As is well known [3, 4, 11] a unique invariant measure μ exists if a weak solution U^x exists for some (all) $x \in E$ and the semigroup generated by A is uniformly exponentially stable. It is obtained as the weak limit $\mu = \lim_{t\to\infty} \mu(t)$, where $\mu(t)$ is the distribution of $U(t) := U^0(t)$ as given by (1.2) with initial value $x = 0$.

If the operators $S(t)$ are compact for all $t > 0$, the existence of a nondegenerate invariant measure for the problem (1.1) with $f = 0$ implies that the semigroup S is uniformly exponentially stable [8, Theorem 2.6]. Recall that an invariant measure μ is said to be *nondegenerate* if $\mu(O) > 0$ for every nonempty open set $O \subseteq E$. The following example, adapted from [4, Chapter 7], shows that in general the uniform exponential stability is by no means a necessary condition for the existence of a nondegenerate invariant measure, even if E is a Hilbert space. Although more refined examples will be presented below, this one is included because of its particular simplicity.

Example 1. Let $H = E = L^2(\mathbb{R}_+)$ and let S be the rescaled left translation semigroup defined by

$$S(t)f(x) = e^t f(x+t), \quad x \in \mathbb{R}_+,\ t \geqslant 0.$$

Define, for $n = 1, 2, \ldots$, the functions $f_n \in E$ by $f_n(x) := p_n(x)e^{-x^2}$, where the polynomials p_n are chosen in such a way that $(f_n)_{n\geqslant 1}$ is an orthonormal basis for E. The fact that such polynomials exist can be deduced, e.g., from [7, Theorem 9.1]. Choose constants $\lambda_n > 0$ such that $\sum_{n\geqslant 1} \lambda_n^2 < \infty$. The operator $B := \sum_{n\geqslant 1} \lambda_n B_n$, where $B_n f := [f, f_n]_E f_n$, is well defined, Hilbert-Schmidt, and has dense range.

For $t \geqslant 0$ define the operators $Q_t \in \mathscr{L}(E)$ by

$$Q_t f := \int_0^t S(s)BB^*S^*(s)f\,ds, \quad f \in E.$$

A simple computation using the orthonormality of the f_n gives

$$\begin{aligned}\operatorname{tr}(Q_t) &= \sum_{m\geqslant 1}[Q_t f_m, f_m]_E = \sum_{m\geqslant 1}\sum_{n\geqslant 1}\lambda_n^2 \int_0^t [f_m, S(s)f_n]_E^2\,ds \\ &= \sum_{n\geqslant 1}\lambda_n^2\int_0^t \|S(s)f_n\|_E^2\,ds \leqslant \sum_{n\geqslant 1}\lambda_n^2 \int_0^\infty\int_0^\infty p_n(x+s)^2 e^{2s-2(x+s)^2}\,dx\,ds.\end{aligned}$$

If we let $\lambda_n \to 0$ fast enough the right hand side is finite and we infer that

$$\sup_{t\geqslant 0}\ \operatorname{tr}(Q_t) < \infty.$$

By [3, Theorem 11.7] this implies that the stochastic Cauchy problem

$$dU(t) = AU(t)\,dt + B\,dW_H(t), \quad t \geqslant 0,$$

admits an invariant measure μ_∞ whose covariance operator is given by the strong operator limit $Q_\infty = \lim_{t\to\infty} Q_t$. Since B (and therefore also BB^*) has dense range, it follows from [6, Lemma 5.2] that the operator Q_∞ has dense range as well. By standard results on Gaussian measures this implies that μ_∞ is nondegenerate.

2. Linear equations with additive noise

In this section we consider the problem (1.1) for $f = 0$, that is, we study the linear stochastic Cauchy problem

$$dU(t) = AU(t)\,dt + B\,dW_H(t), \quad t \geqslant 0. \tag{2.1}$$

We begin our discussion with recalling some definitions from the theory of C_0-semigroups. Our notations are standard and may be looked up in e.g. [5]. The spectral bound and growth bound of A are denoted by $s(A)$ and $\omega_0(A)$, respectively. The *abscissa of uniform boundedness* of the resolvent of A is defined as

$$s_0(A) := \inf\Big\{\omega \in \mathbb{R}:\ \{\operatorname{Re}\lambda > \omega\} \subseteq \varrho(A),\ \sup_{\operatorname{Re}\lambda>\omega}\|R(\lambda, A)\| < \infty\Big\}.$$

One has $s(A) \leqslant s_0(A) \leqslant \omega_0(A)$, and both inequalities may be strict. As a consequence of the Pringsheim-Landau theorem one has $s(A) = s_0(A)$ for positive C_0-semigroups on Banach lattices E. The celebrated Gearhart-Herbst-Prüss theorem asserts that for C_0-semigroups on Hilbert spaces E one has $s_0(A) = \omega_0(A)$.

Let $\mathscr{H}$ be a separable real Hilbert space with orthonormal basis $(h_n)_{n\geqslant 1}$. In the applications below, $\mathscr{H}$ will be either H or $L^2(\mathbb{R}_+; H)$. Let $(\gamma_n)_{n\geqslant 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathbb{P})$. A bounded operator $R \in \mathscr{L}(\mathscr{H}, E)$ is called *γ-radonifying* if the sum $\sum_{n\geqslant 1}\gamma_n Rh_n$ converges in $L^2(\Omega; E)$. The space $\gamma(\mathscr{H}, E)$ of all γ-radonifying operators in $\mathscr{L}(\mathscr{H}, E)$ is a Banach space with respect to the norm

$$\|R\|_{\gamma(\mathscr{H},E)} := \Big(\mathbb{E}\Big\|\sum_{n\geqslant 1}\gamma_n Rh_n\Big\|^2\Big)^{1/2}.$$

Moreover, $\gamma(H, E)$ is an *operator ideal* in $\mathscr{L}(H, E)$, i.e., as a Banach space it is continuously embedded into $\mathscr{L}(H, E)$, it contains all finite rank operators in $\mathscr{L}(H, E)$, and for every separable real Hilbert space $\tilde{H}$, every real Banach space $\tilde{E}$, and all operators $R \in \gamma(H,E)$, $T \in \mathscr{L}(\tilde{H},H)$, and $S \in \mathscr{L}(E,\tilde{E})$ we have $SRT \in \gamma(\tilde{H},\tilde{E})$ and

$$\|SRT\|_{\gamma(\tilde{H},\tilde{E})} \leqslant \|S\|_{\mathscr{L}(\tilde{H},H)}\|R\|_{\gamma(H,E)}\|T\|_{\mathscr{L}(E,\tilde{E})}.$$

For Hilbert spaces E one has $\gamma(\mathscr{H},E) = \mathscr{L}_2(\mathscr{H},E)$ with identical norms, where $\mathscr{L}_2(\mathscr{H},E)$ denotes the operator ideal of Hilbert-Schmidt operators from $\mathscr{H}$ to E.
The following necessary and sufficient condition for the existence of an invariant measure was noted in [11, Proposition 4.4]:

Proposition 2. *For an operator $B \in \mathscr{L}(H,E)$ the following assertions are equivalent:*

(i) *The problem* (2.1) *admits an invariant measure;*

(ii) *The operator $I_B : C_c(\mathbb{R}_+;H) \to E$ defined by*

$$I_B f := \int_0^\infty S(t)Bf(t)\,dt$$

extends to a bounded operator $I_B \in \gamma(L^2(\mathbb{R}_+;H),E)$.

Concerning uniqueness, in [11] it was shown that if there exists a weak*-sequentially dense subspace F of E^* such that weak*-$\lim_{t\to\infty} S^*(t)x^* = 0$ for all $x^* \in F$, then the problem (2.1) admits at most one invariant measure. In passing we mention the following application of this result:

Proposition 3. *Let $\mathscr{B}$ be a subset of $\mathscr{L}(H,E)$ such that*

$$\bigcup_{B\in\mathscr{B}} \mathrm{ran}(B) = E.$$

If for all $B \in \mathscr{B}$ the problem (2.1) *admits an invariant measure, then for all $B \in \mathscr{L}(H,E)$ the problem* (2.1) *admits at most one invariant measure. In particular, for all $B \in \mathscr{B}$ the problem* (2.1) *then admits a unique invariant measure.*

Proof. Let $x^* \in \mathscr{D}(A^*)$ be arbitrary. By the result quoted above it suffices to prove that weak*-$\lim_{t\to\infty} S^*(t)x^* = 0$. Let $x \in E$ be arbitrary. Choose $B \in \mathscr{B}$ and $h \in H$ such that $Bh = x$. We claim that $\lim_{t\to\infty}\langle x, S^*(t)x^*\rangle = 0$. For $y^* \in E^*$ let $f_{y^*} : \mathbb{R}_+ \to H$ be defined as $f_{y^*}(t) = \langle x, S^*(t)y^*\rangle$. By the assumptions and Proposition 2 we have that for all $y^* \in E^*$, $f_{y^*} \in L^2(\mathbb{R}_+)$ and

$$\|f_{y^*}\|_{L^2(\mathbb{R}_+)} \leqslant \|S(\cdot)B\|_{\gamma(\mathbb{R}_+;H,E)}\|h\|\|x^*\|.$$

Since $x^* \in \mathscr{D}(A^*)$, the function $g(t) := |f_{x^*}(t)|^2$ is continuously differentiable on $\mathbb{R}_+$ with $g'(t) = 2f_{x^*}(t)f_{A^*x^*}(t)$. Hence by the Cauchy-Schwartz inequality, $g' \in L^1(\mathbb{R}_+)$. From

$$\lim_{t,s\to\infty}|g(t)-g(s)| \leqslant \lim_{t,s\to\infty}\int_s^t |g'(u)|\,du = 0$$

it follows that the limit $L = \lim_{t\to\infty} g(t)$ exists. If $L > 0$, there exist $\varepsilon > 0$ and $T > 0$ such that for all $t \geqslant T$ we have $g(t) \geqslant \varepsilon$, which contradicts the fact that $g \in L^1(\mathbb{R}_+)$. We conclude that

$$\lim_{t\to\infty}|\langle x, S^*(t)x^*\rangle|^2 = \lim_{t\to\infty} g(t) = L = 0.$$

Since $x \in E$ was arbitrary, this proves that weak*-$\lim_{t\to\infty} S^*(t)x^* = 0$. ☐

A subset of a topological space is called *residual* if it is the intersection of a countable family of open dense sets. By the Baire category theorem, every residual set in a complete metric space is dense.

Theorem 4. *Let E be a Banach space not containing a closed subspace isomorphic to c_0 and let $\mathscr{I}$ be an operator ideal in $\mathscr{L}(H,E)$. Let A be the generator of a C_0-semigroup on E. If $s_0(A) > 0$, then the set $\mathscr{J}$ of all $B \in \mathscr{I}$ such that the problem (2.1) does not admit an invariant measure is residual in $\mathscr{I}$. If, moreover, the finite rank operators are dense in $\mathscr{I}$, then the finite rank operators of $\mathscr{J}$ are dense in $\mathscr{I}$.*

Proof. For $k = 1, 2, \ldots$ let

$$G_k := \big\{B \in \mathscr{I} : \ \|I_B\|_{\gamma(L^2(\mathbb{R}_+;H),E)} > k\big\}$$

where we put $\|I_B\|_{\gamma(L^2(\mathbb{R}_+;H),E)} = \infty$ in case $I_B \notin \gamma(L^2(\mathbb{R}_+;H),E)$. We shall prove that each G_k is open and dense in $\mathscr{I}$. The residual set $G := \bigcap_{k\geqslant 1} G_k$ is precisely the set of all $B \in \mathscr{I}$ for which $I_B \notin \gamma(L^2(\mathbb{R}_+;H),E)$, or equivalently, for which the problem (2.1) has no invariant measure.

Fix $k \geqslant 1$. First we check that G_k is open in $\mathscr{I}$, or equivalently, that the complement $\complement G_k$ is closed. Suppose $\lim_{n\to\infty} B_n = B$ in $\mathscr{I}$ with each $B_n \in \complement G_k$. Then $\|I_{B_n}\|_{\gamma(L^2(\mathbb{R}_+;H,E))} \leqslant k$ for all n. Since for all $t \geqslant 0$ we have $\lim_{n\to\infty} S(t)B_n = S(t)B$ in $\mathscr{L}(H,E)$, from [10, Theorem 4.1] (here we use that $c_0 \not\subseteq E$) we infer that $I_B \in \gamma(L^2(\mathbb{R}_+;H),E)$ and $\|I_B\|_{\gamma(L^2(\mathbb{R}_+;H),E)} \leqslant k$. Hence, $B \in \complement G_k$.

Next we check that G_k is dense in $\mathscr{I}$. Suppose the contrary. Then there exist $B_0 \in \mathscr{I}$ and $r > 0$ such that $\mathscr{B}(B_0,r) \subseteq \complement G_k$, where $\mathscr{B}(B_0,r)$ is the open ball in $\mathscr{I}$ of radius r centred at B_0. Fix a real number $0 < \delta < s_0(A)$. By [11, Theorem 1.2], for all $B \in \mathscr{B}(B_0,r)$ the $\mathscr{L}(H,E)$-valued function $\lambda \mapsto R(\lambda,A)B$ admits a uniformly bounded analytic extension to the half-plane $\{\operatorname{Re}\lambda > \delta\}$, and by linearity this conclusion holds for all $B \in \mathscr{I}$. Fixing an arbitrary norm one vector $h_0 \in H$ and taking for B the rank one operators of the form $h \mapsto [h,h_0]_H x$ with $x \in E$, we see that the E-valued functions $\lambda \mapsto R(\lambda,A)x$ admits a uniformly bounded analytic extension to the half-plane $\{\operatorname{Re}\lambda > \delta\}$. From the uniform boundedness theorem we conclude that $\{\operatorname{Re}\lambda > \delta\} \subseteq \varrho(A)$ and $\sup_{\operatorname{Re}\lambda>\delta} \|R(\lambda,A)\| < \infty$. But this implies that $s_0(A) \leqslant \delta$, a contradiction.

Suppose next that the finite rank operators are dense in $\mathscr{I}$. Let $B \in \mathscr{I}$, fix $\varepsilon > 0$ arbitrary, and let $\tilde{B} \in \mathscr{I}$ be a finite rank operator satisfying $\|\tilde{B} - B\|_{\mathscr{I}} < \frac{\varepsilon}{2}$. If the problem (2.1), with B replaced by $\tilde{B}$, does not admit an invariant measure we are done. Otherwise, write $\tilde{B}h = \sum_{n=1}^N c_n[h,h_n]_H x_n$ with $h_1,\ldots,h_N$ orthonormal in H. Let H_N be the linear span in H of the vectors $h_1,\ldots,h_N$ and let W_{H_N} be the restriction of W_H to H_N. Denote by $\mathscr{I}_N$ the space of all linear operators from H_N to E endowed with the norm inherited from $\mathscr{I}$. We now apply the first part of the theorem, with H, W_H, $\mathscr{I}$ replaced by H_N, W_{H_N}, $\mathscr{I}_N$. This results in an operator $\tilde{\tilde{B}} \in \mathscr{I}_N$ with $\|\tilde{\tilde{B}} - \tilde{B}\|_{\mathscr{I}_N} < \frac{\varepsilon}{2}$ for which the problem $dU(t) = AU(t)\,dt + \tilde{\tilde{B}}\,dW_{H_N}(t)$ has no invariant measure. Extending $\tilde{\tilde{B}}$ identically zero on the orthogonal complement of H_N, we obtain an operator in $\mathscr{I}$ with the desired properties. $\square$

As an immediate consequence we see that if $s_0(A) > 0$, the presence of an invariant measure can be destroyed by an arbitrary small perturbation of B in $\mathscr{I}$.

An obvious example of an operator ideal for which the first part of the theorem applies is $\mathscr{I} = \mathscr{L}(H,E)$. In the special case $H = \mathbb{R}^N$ (in which case W_H is a standard $\mathbb{R}^N$-valued Brownian motion W) we have $\mathscr{I} = \mathscr{L}(\mathbb{R}^N,E) = E^N$ and problem (2.1) may be written in the form

$$dU(t) = AU(t)\,dt + d[W(t),x], \quad t \geqslant 0,$$

where $[W(t),x] = \sum_{n=1}^N W_n(t)x_n$.

Both parts of the theorem apply to the operator ideal $\mathscr{I} = \gamma(H,E)$. The interest of this particular example is explained by the fact that roughly speaking there is a correspondence between operators $B \in \gamma(H,E)$ on the one hand and E-valued Brownian motions on the other. To be more precise let W_H be a cylindrical Brownian motion on a probability space $(\Omega,\mathbb{P})$. If $(h_n)_{n\geqslant 1}$ is an orthonormal basis for H, then for each $B \in \gamma(H,E)$ and $t \geqslant 0$ the sum $W_B(t) := \sum_{n\geqslant 1} W_H(t)h_n\, Bh_n$ converges in $L^2(\Omega;E)$ and the resulting process W_B is an E-valued Brownian motion on $(\Omega,\mathbb{P})$ which is independent of the choice of $(h_n)_{n\geqslant 1}$. Conversely, every E-valued Brownian motion W arises in such a way by taking for H the so-called reproducing kernel Hilbert space associated with W and for B the (γ-radonifying) inclusion mapping from H into E. Although in general the problem (1.1) may fail to have a solution even if $B \in \gamma(H,E)$ (an example is presented in [9]), a solution always exists if in addition to $B \in \gamma(H,E)$ we assume that either E has type 2 (in particular, if E is a Hilbert space or if $E = L^p$ for $2 \leqslant p < \infty$) or the semigroup generated by A is analytic. See [9, 10, 11] for more details.

Our next aim is to exhibit an example of a C_0-semigroup generator A on a Hilbert space E with the following properties:

(a) The spectral bound and growth bound of A satisfy $s(A) = \omega_0(A) > 0$;
(b) The set of all $B \in \gamma(H,E) = \mathscr{L}_2(H,E)$ for which (2.1) has an invariant measure is dense.

Its construction is based on [11, Example 4] which we recall first.

Example 5. For $2 < p < \infty$ consider the space $F = L^2(1,\infty) \cap L^p(1,\infty)$ endowed with the norm $\|f\| := \max\{\|f\|_2, \|f\|_p\}$. On F we define the C_0-semigroup S^F by

$$(S^F(t)f)(x) = f(xe^t), \quad x > 1,\ t \geqslant 0.$$

It was shown by Arendt [1] that its generator A^F satisfies $s_0(A^F) = -\frac{1}{2}$ and $\omega_0(A^F) = -\frac{1}{p}$. Put $S^F_\beta(t) := e^{\beta t}S^F(t)$, where $\frac{1}{p} < \beta < \frac{1}{2}$ is an arbitrary but fixed number. As is shown in [11], for every $B \in \gamma(H,F)$ the stochastic Cauchy problem (2.1) associated with the operator $A^F_\beta := A^F + \beta$ admits a unique invariant measure. Note that $\omega_0(A^F_\beta) = -\frac{1}{p} + \beta$, which is strictly positive by the choice of β.

Example 6. We construct a Hilbert space semigroup with the properties (a) and (b) announced above. The idea is to embed the space F of Example 5 into a suitable weighted L^2-space in such a way that the relevant properties of the semigroup S^F_β are preserved.

We have contractive and dense embeddings

$$F = L^2(1,\infty) \cap L^p(1,\infty) \hookrightarrow L^2(1,\infty) \hookrightarrow L^2(1,\infty;\tfrac{dx}{x}) =: E.$$

The semigroup S^F_β on F defined in Example 5 extends to a C_0-semigroup S on E. To see this, note that for $f \in F$ and $t \geqslant 0$ we have

$$\int_1^\infty |S^F_\beta(t)f(x)|^2\,\frac{dx}{x} = e^{2\beta t}\int_{e^t}^\infty |f(\xi)|^2\,\frac{d\xi}{\xi} \leqslant e^{2\beta t}\int_1^\infty |f(\xi)|^2\,\frac{d\xi}{\xi}.$$

Thus $S^F_\beta(t)$ extends to a bounded operator $S(t)$ on E of norm $\|S(t)\| \leqslant e^{\beta t}$. In combination with the strong continuity of S on the dense subspace F of E it follows that S is a C_0-semigroup on E. For the function $f_c := \mathbf{1}_{(e^t,ce^t)}$ with $c > 1$ we have

$$\|f_c\|^2_E = \int_{e^t}^{ce^t} \frac{dx}{x} = \ln c$$

and

$$\|S(t)f_c\|_E^2 = e^{2\beta t}\int_{e^t}^{ce^t} \frac{dx}{x} = e^{2\beta t}\ln c.$$

Hence $\|S(t)\| \geqslant e^{\beta t}$, and we conclude that $\|S(t)\| = e^{\beta t}$. Stated differently, the generator A of S satisfies $\omega_0(A) = \beta$. Since S is positive and E is a Hilbert space, we have $s(A) = s_0(A) = \omega_0(A)$ and property (a) holds.

To prove that property (b) holds we make the simple observation that the dense embedding $j : F \hookrightarrow E$ induces a dense embedding

$$j : \gamma(H,F) \hookrightarrow \gamma(H,E) = \mathscr{L}_2(H,E).$$

The density of this embedding follows from the fact that the finite rank operators with values in F are dense in both spaces. Now if $B \in \gamma(H,F)$ is given, let μ_β^F denote an invariant measure of the linear stochastic Cauchy problem in F associated with A_β^F and B. Then the image measure $\mu := j(\mu_\beta^F)$ is an invariant measure for the linear stochastic Cauchy problem in E associated with A and jB.

Notice that in the previous example the invariant measure μ is nondegenerate whenever B has dense range.

It was shown in [8] that the existence of a nondegenerate invariant measure for the problem (2.1) implies that the adjoint operator A^* has no point spectrum in the closed right half-plane $\{\mathrm{Re}\,\lambda \geqslant 0\}$. If in addition we assume that the semigroup generated by A is uniformly bounded, then one has

$$\sigma_{\mathrm{p}}(A) \cap i\mathbb{R} \subseteq \sigma_{\mathrm{p}}(A^*) \cap i\mathbb{R}$$

and the existence of a nondegenerate invariant measure for the problem (2.1) implies that A has no point spectrum in $\{\mathrm{Re}\,\lambda \geqslant 0\}$. As was shown in [8, Theorem 4.4] this implies that there is at most one nondegenerate invariant measure for (2.1). The following example shows that for semigroups with linear growth and $0 \in \sigma_{\mathrm{p}}(A)$, a continuum of nondegenerate invariant measures may exist.

Example 7. Let $2 < p < \infty$ and fix $\frac{1}{p} < \beta < \frac{1}{2}$. Put

$$w(x) := \frac{x^{2\beta-1}}{1+\log^2 x}, \quad x > 1,$$

and let $E_w := L^2(1,\infty; w(x)\,dx)$. The space F of Example 5 is continuously and densely embedded in E_w, and the semigroup S_β^F extends to a C_0-semigroup S_w on E_w. We check that

$$\|S_w(t)\| = \big(\tfrac{1}{2}t^2 + 1 + \tfrac{1}{2}t\sqrt{t^2+4}\,\big)^{1/2}, \quad t \geqslant 0,$$

so S_w grows linearly. Indeed, for $f \in E_w$ and $t \geqslant 0$,

$$\begin{aligned}\int_1^\infty |S_w(t)f(x)|^2 w(x)\,dx &= \int_{e^t}^\infty f(\xi)^2 \frac{\xi^{2\beta-1}}{1+\log^2(\xi e^{-t})}\,d\xi \\ &= \int_{e^t}^\infty f(\xi)^2 w(\xi)\frac{1+\log^2\xi}{1+(\log\xi - t)^2}\,d\xi.\end{aligned}$$

It is easy to compute that the function $\xi \mapsto \dfrac{1+\log^2\xi}{1+(\log\xi-t)^2}$ attains its maximal value on (e^t,∞) at the point

$$\xi_t = \exp(\tfrac{t}{2} + \tfrac{1}{2}\sqrt{t^2+4})$$

and that the maximum equals

$$\alpha_t = \tfrac{1}{2}t^2 + 1 + \tfrac{1}{2}t\sqrt{t^2+4}.$$

Hence $\|S_w(t)f\|^2 \leqslant \alpha_t \|f\|^2$. For $t \geqslant 0$ and $\varepsilon > 0$, let $f_{t,\varepsilon} := \mathbf{1}_{(\xi_t, \xi_t+\varepsilon)}$. A straightforward computation shows that

$$\frac{\|S_w(t)f_{t,\varepsilon}\|^2}{\|f_{t,\varepsilon}\|^2} \to \frac{1+\log^2 \xi_t}{1+(\log \xi_t - t)^2} = \alpha_t \text{ as } \varepsilon \downarrow 0.$$

Thus $\|S_w(t)\| = \alpha_t^{1/2}$ as claimed.
Let

$$b(x) := x^{-\beta}, \quad x > 1. \tag{2.2}$$

An elementary computation shows that $b \in E_w$ and that $S_w(t)b = b$. It follows that $b \in \mathscr{D}(A_w)$ and $A_w b = 0$. Since b is nonzero, this shows that $0 \in \sigma_{\mathrm{p}}(A_w)$.
As in Example 6, in E_w the problem (2.1) admits an invariant measure for every operator $B \in \gamma(H,F)$, where we identify $\gamma(H,F)$ with a dense subspace of $\gamma(H,E_w) = \mathscr{L}_2(H,E_w)$, and a nondegenerate invariant measure exists whenever B has dense range. If μ is such a measure, then for all $c \in \mathbb{R}$ the translated measure $\mu_c(C) := \mu(C+cb)$ is a nondegenerate invariant measure for (2.1); here b is the function defined in (2.2). Thus, a continuum of such measures exists.

3. The inhomogeneous problem with additive noise

Next we consider the inhomogeneous problem (1.1),

$$dU(t) = (AU(t) + f)dt + B\, dW_H(t), \quad t \geqslant 0,$$

where $f \in E$ is a fixed vector. Following the arguments of [3, Propositions 11.2 and 11.5] one sees that a Radon probability measure μ on E is invariant if and only if there exists a stationary solution V of (1.1) (on a possibly larger probability space) such that μ is the distribution of $V(t)$ for all $t \geqslant 0$. If μ has a first moment, i.e., if there exists an element $m(\mu) \in E$ such that for all $x^* \in E^*$ we have $x^* \in L^1(E,\mu)$ and

$$\langle m(\mu), x^* \rangle = \int_E \langle x, x^* \rangle \, d\mu(x),$$

then by applying x^* on both sides of the identity

$$V(t) = S(t)V(0) + \int_0^t S(t-s)f\, ds + \int_0^t S(t-s)B\, dW_H(s),$$

and taking expectations, the Hahn-Banach theorem shows that $m(\mu)$ satisfies the identity

$$m(\mu) = S(t)m(\mu) + \int_0^t S(t-s)f\, ds = S(t)m(\mu) + \int_0^t S(s)f\, ds, \quad t \geqslant 0. \tag{3.1}$$

The following proposition relates invariant measures with first moments of the problem (1.1) to the invariant measure of the homogeneous problem (2.1) with $f = 0$. Its proof follows readily from the identity on [3, p. 122], which extends without change to the present Banach space setting.

Proposition 8. *The inhomogeneous equation* (1.1) *admits an invariant measure with first moment if and only if* $f \in \operatorname{ran}(A)$ *and the homogeneous equation* (2.1) *admits an invariant measure (and then also a Gaussian one, which has first moment). Moreover, if* $f \in \operatorname{ran}(A)$*, then* V *is a stationary solution of* (1.1) *if and only if* $V = U + a$ *for some* $a \in D(A)$ *with* $-Aa = f$ *and some stationary solution* U *of* (2.1)*.*

We proceed with a Hilbert space example which shows that even if $s(A) > 0$ it may happen that the inhomogeneous problem (1.1) has an invariant measure for all choices of $f \in E$.

Example 9. We show that Example 6 displays the stated properties. For the proof we fix $B \in \gamma(H, F)$, where F is the space of Example 5. Let U be a stationary solution of the problem (2.1) in E, which exists according to the facts proved in Example 6 and the observations made above. Let $f \in E$ be arbitrary and define for $c \in \mathbb{R}$,

$$a_c(x) := cx^{-\beta} - x^{-\beta}\int_1^x \xi^{\beta-1} f(\xi)\, d\xi, \quad x > 1. \tag{3.2}$$

Observe that

$$\begin{aligned}|a_0(x)| &\leqslant x^{-\beta}\left(\int_1^x \xi^{\beta-1} d\xi\right)^{1/2}\left(\int_1^x \xi^{\beta-1} f(\xi)^2\, d\xi\right)^{1/2}\\ &\leqslant \beta^{-1/2} x^{-\beta/2}\left(\int_1^x \xi^{\beta-1} f(\xi)^2\, d\xi\right)^{1/2}, \quad x > 1,\end{aligned}$$

so that, by integration by parts, for all $T > 1$ we obtain

$$\begin{aligned}\int_1^T a_0(x)^2 x^{-1}\, dx &\leqslant \beta^{-1} T^{-\beta}(-\beta)^{-1}\int_1^T \xi^{\beta-1} f(\xi)^2\, d\xi + \beta^{-2}\int_1^T x^{-\beta} x^{\beta-1} f(x)^2\, dx\\ &\leqslant \beta^{-2}\int_1^\infty x^{-1} f(x)^2\, dx.\end{aligned}$$

Since $b(x) := x^{-\beta}$ belongs to E it follows that $a_c = cb + a_0 \in E$ for all $c \in \mathbb{R}$. By an elementary computation we check that

$$S(t)a_c + \int_0^t S(s) f\, ds = a_c.$$

We infer that $a_c \in \mathscr{D}(A)$ and $-Aa_c = f$. This shows that $f \in \operatorname{ran}(A)$. Thus by Proposition 8, the inhomogeneous problem admits a stationary solution and hence an invariant measure.

More can be said in the above example. If μ is an invariant measure for the inhomogeneous problem with first moment $m(\mu)$, then necessarily $m(\mu)$ is given by the right hand-side of (3.2) for some $c \in \mathbb{R}$. Indeed, since $m(\mu)$ satisfies (3.1) it suffices to show that the only elements $b \in E$ satisfying $b - S(t)b = 0$ for all $t \geqslant 0$ are given by $b(x) = cx^{-\beta}$ for some $c \in \mathbb{R}$. Since by assumption for all $t \geqslant 0$ we have $e^{\beta t} b(xe^t) = b(x)$ for almost all $x > 1$, it follows that for all $\tau \geqslant 1$ we have $\tau^\beta b(x\tau) = b(x)$ for almost all $x > 1$. Since $(x, \tau) \mapsto \tau^\beta b(x\tau)$ is measurable, Fubini's theorem yields that for almost all $x > 1$ the equality $\tau^\beta b(x\tau) = b(x)$ holds for almost all $\tau \geqslant 1$. Consider a fixed $x = x_0$ with this property. Then, with $\theta := x_0\tau$, we obtain $b(\theta) = c\theta^{-\beta}$ for almost all $\theta \geqslant x_0$, where $c = b(x_0)x_0^\beta$. By letting $x_0 \downarrow 1$ we infer that c is independent of x_0 and that $b(x) = cx^{-\beta}$ for almost all $x > 1$.
Summarizing what we proved so far, we see that for every $f \in E$ the set of means of all invariant measures having a first moment is a one-parameter family parametrized by the real parameter c. Note that even for the homogeneous problem (2.1), invariant measures may exist whose first moment does not exist. Indeed, any weak limit of convex combinations of invariant measures is invariant as well. Returning to Example 6, if μ is an invariant measure of the homogeneous

problem with mean zero and if we put $\mu_n(C) := \mu(C + 2^n b)$ and $\nu := \sum_{n\geqslant 1} 2^{-n}\mu_n$, where $b \in E$ is defined by (2.2), then ν is an invariant measure and

$$\int_E \big|[x,b]_E\big|\,d\nu(x) \geqslant \sum_{n\geqslant 1} 2^{-n}\Big|\int_E [x,b]_E\,d\mu_n(x)\Big| = \sum_{n\geqslant 1} 2^{-n}\big|[2^n b,b]_E\big| = \infty.$$

Let U denote the solution of (1.1) with initial condition $U(0) = 0$; thus

$$U(t) = \int_0^t S(s)f\,ds + \int_0^t S(t-s)\,dW(s). \tag{3.3}$$

For each $t \geqslant 0$ we denote by $\mu(t)$ the distribution of $U(t)$. The following result is a consequence of a standard result on weak convergence of Gaussian measures [2, Theorem 3.8.9]:

Proposition 10. *The weak limit* $\mu := \lim_{t\to\infty} \mu(t)$ *exists if and only if the limit* $\lim_{t\to\infty}\int_0^t S(s)f\,ds$ *exists in* E *and the homogeneous problem* (2.1) *admits an invariant measure. In this situation,* μ *is an invariant measure for the problem* (1.1) *and we have*

$$m(\mu) = \int_0^\infty S(s)f\,ds.$$

If $\omega_0(A) < 0$, then this proposition shows that the measures $\mu(t)$ converge weakly to an invariant measure μ of (1.1). We will show next that, even in the presence of invariant measures, this convergence may fail if the semigroup has linear growth.

Example 11. We continue with Example 7 and show that for certain functions f an invariant measure for (1.1) exists, although the integrals $\int_0^t S_w(s)f\,ds$ fail to converge in E_w as $t \to \infty$. An appeal to Proposition 10 then shows that the measures $\mu(t)$ fail to converge weakly.
Consider the function

$$f(x) := \begin{cases} \dfrac{x^{-\beta}}{(\log\log x)\log x} & \text{for } x > e^e \\[2ex] 0 & \text{for } 1 < x \leqslant e^e. \end{cases}$$

Then $|f(x)| \leqslant x^{-\beta}$, so $f \in E_w$. The function

$$x^{-\beta}\int_1^x \xi^{\beta-1} f(\xi)\,d\xi = \begin{cases} x^{-\beta}\log\log\log x & \text{for } x > e^e \\ 0 & \text{for } 1 < x \leqslant e^e \end{cases}$$

is a member of E_w and in the same way as in Example 9 we infer that $f \in \operatorname{ran}(A_w)$. Due to Proposition 8 and the existence of an invariant measure with first moment for the homogeneous problem, there exists an invariant measure for (1.1) with A replaced by A_w.
For all $t \geqslant e$ and $x > 1$,

$$\begin{aligned}\int_0^t S_w(s)f(x)\,ds &= \int_{(e-\log x)^+}^t \frac{x^{-\beta}}{(\log(s+\log x))(s+\log x)}\,ds \\ &= x^{-\beta}\log\Big(\frac{\log(t+\log x)}{\log((e-\log x)^+ + \log x)}\Big).\end{aligned}$$

From this we infer that the integrals $\int_0^t S_w(s)f\,ds$ diverge in E_w as $t \to \infty$.

Acknowledgment – Proposition 3 was obtained in a discussion with Mark Veraar.

References

[1] W. Arendt, Spectrum and growth of positive semigroups, in: "Evolution Equations" (Baton Rouge, LA, 1992), Lecture Notes in Pure and Appl. Math., Vol. 168, Dekker, New York, 1995, pp. 21-28.

[2] V.I. Bogachev, "Gaussian Measures", Math. Surveys and Monographs, Vol. 62, Amer. Math. Soc., 1998.

[3] G. Da Prato and J. Zabczyk, "Stochastic Equations in Infinite Dimensions", Encyclopedia of Mathematics and its Applications, Vol. 44, Cambridge University Press, Cambridge, 1992.

[4] G. Da Prato and J. Zabczyk, "Ergodicity for Infinite-Dimensional Systems", London Mathematical Society Lecture Note Series, Vol. 229, Cambridge University Press, Cambridge, 1996.

[5] K.-J. Engel and R. Nagel, "One-Parameter Semigroups for Linear Evolution Equations", Graduate Texts in Mathematics, Vol. 194, Springer-Verlag, New York, 2000.

[6] B. Goldys, F. Gozzi, and J.M.A.M. van Neerven, On closability of directional gradients, *Potential Anal.* **18** (2003), 289-310.

[7] P. Lax, "Functional Analysis", Pure and Applied Mathematics, John Wiley & Sons, New York, 2002.

[8] J.M.A.M. van Neerven, Uniqueness of invariant measures for the stochastic Cauchy problem in Banach spaces, in: "Recent Advances in Operator Theory and Related Topics: The Béla Szökefalvi-Nagy Memorial Volume", Operator Theory: Advances and Applications, Vol. 127, Birkhäuser, 2001, pp. 491-517.

[9] J.M.A.M. van Neerven and L. Weis, Stochastic integration of functions with values in a Banach space, *Studia Math.* **166** (2005), 131-170.

[10] J.M.A.M. van Neerven and L. Weis, Weak limits and integrals of Gaussian covariances in Banach spaces, *Probab. Math. Statist.* **25** (2005), 55-74.

[11] J.M.A.M. van Neerven and L. Weis, Invariant measures for the linear stochastic Cauchy problem and R-boundedness of the resolvent, to appear in *J. Evolution Eq.* (2006).