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ON THE CHARACTERIZATION OF ISOTROPIC GAUSSIAN FIELDS ON HOMOGENEOUS SPACES OF COMPACT GROUPS

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Abstract
Let T be a random field weakly invariant under the action of a compact group G. We give conditions ensuring that independence of the random Fourier coefficients is equivalent to Gaussianity. As a consequence, in general it is not possible to simulate a non-Gaussian invariant random field through its Fourier expansion using independent coefficients

1 Introduction

Recently an increasing interest has been attracted by the topic of rotationally *real* invariant random fields on the sphere $\mathbb{S}^2$, due to applications to the statistical analysis of Cosmological and Astrophysical data (see [MP04], [Mar06] and [AK05]).
Some results concerning their structure and spectral decomposition have been obtained in [BM07], where a peculiar feature has been pointed out, namely that if the development into spherical harmonics

$$T=\sum_{\ell=0}^{\infty}\sum_{-m}^{m}a_{\ell,m}Y_{\ell,m}$$

of a rotationally invariant random field T is such that $a_{00}=0$ and the coefficients $a_{\ell,m}$,

$\ell = 1, 2, \ldots, 0 \le m \le \ell$ are independent, then the field is necessarily Gaussian (the other coefficients are constrained by the condition $a_{\ell,-m} = (-1)^m \overline{a}_{\ell,m}$). In particular that non Gaussian rotationally invariant random fields on the sphere *cannot* be simulated using independent coefficients.
Indeed a natural and computationally efficient procedure in order to simulate a random field on the sphere is by sampling the coefficients $a_{\ell m}$. This is the route pursued for instance in [Ka96] where it is proposed to generate a non Gaussian random field by choosing the $a_{\ell m}$'s to be chi-square complex valued and independent. The authors failed to notice that the resulting random field is not invariant, as a consequence of [BM07].
This fact (independence of the coefficients+isotropy⇒Gaussianity) is not true for isotropic random fields on other structures, as the torus or $\mathbb{Z}$ (which are situations where the action is Abelian). In this note we show that this is a typical phenomenon for homogeneous spaces of compact non-Abelian groups. This should be intended as a contribution to a much more complicated issue, i.e. the characterization of the isotropy of a random field in terms of its random Fourier expansion.
In §2 and §3 we review some background material on harmonic analysis and spectral representations for random fields. §4 contains the main results, whereas we moved to §5 an auxiliary proposition.

2 The Peter-Weyl decomposition

Let $\mathscr{X}$ be a compact topological space and G a compact group acting on $\mathscr{X}$ transitively with an action that we note $x \to g^{-1}x$, $g \in G$. We denote by m_G the Haar measure of G (see [VK91] e.g.), from which one can derive the existence on $\mathscr{X}$ of the measure $m = \int_{\mathscr{X}} \delta_{g^{-1}x}\, dm_G(g)$ that is invariant by the action of G. We assume that both m and m_G are normalized and have total mass equal to 1. We shall write $L^2(\mathscr{X})$ or simply L^2 instead of $L^2(\mathscr{X}, m)$. Unless otherwise stated the spaces L^2 are spaces of *complex valued* square integrable functions. We denote by L_g the action of G on L^2, that is $L_g f(x) = f(g^{-1}x)$.
The classical Peter-Weyl theorem (see [VK91] again) states that, for a compact topological group G, the following decomposition holds:

$$L^2(G) = \bigoplus_{\sigma \in \hat{G}} \bigoplus_{1 \le i \le \dim(\sigma)} H_{\sigma,i} \tag{2.1}$$

where $\hat{G}$ denotes the set of equivalence classes of irreducible representations of G. In this decomposition G acts irreducibly on the subspaces $H_{\sigma,i}$. The isotypical subspaces

$$H_\sigma = \bigoplus_{1 \le i \le \dim(\sigma)} H_{\sigma,i}$$

are uniquely determined, whereas their decomposition into the $H_{\sigma,i}$ is not.
From (2.1) one can deduce a similar decomposition for $L^2(\mathscr{X})$ by remarking that, if we fix $x_0 \in \mathscr{X}$, then the relation $\widetilde{f}(g) = f(g^{-1}x_0)$ uniquely identifies functions in $L^2(\mathscr{X})$ as functions in $L^2(G)$ that are invariant under the action of the isotropy group G_{x_0}. One verifies immediately that the regular representation of G acts on the subspace of $L^2(G)$ of the functions that are invariant under the action of G_{x_0}. Therefore the intersection of $H_{\sigma,i}$ and $L^2(\mathscr{X})$ is either $\{0\}$ or $H_{\sigma,i}$ itself, thus providing the Peter-Weyl decomposition for $L^2(\mathscr{X})$.

Since the action of G commutes with the complex conjugation on $L^2(\mathscr{X},m)$, it is clear that for any irreducible subspace H, we have that $\overline{H}$, its conjugate subspace is also irreducible.
If $H=\overline{H}$, we can find orthonormal bases (ϕ_k) for H which are stable under conjugation; for instance we can choose the ϕ_k to be real.
If $H\neq\overline{H}$, then there are two cases according as the action of G on $\overline{H}$ is or is not equivalent to the action on H. If the two actions are inequivalent, then automatically $H\perp\overline{H}$. If the actions are equivalent, it is possible that H and $\overline{H}$ are not orthogonal to each other. In this case $H\cap\overline{H}=0$ as both are irreducible. The space $S:=H+\overline{H}$ is stable under G and conjugation and we can find $K\subset S$ stable under G and irreducible such that $\overline{K}\perp K$ and $S=K\oplus\overline{K}$ is an orthogonal direct sum. The proof of this is postponed to the Appendix so as not to interrupt the main flow of the argument. If we drop the dependence on σ, we obtain the following orthogonal decomposition of $L^2(\mathscr{X},m)$ into irreducible finite dimensional subspaces

$$L^2(\mathscr{X},m)=\bigoplus_{i\in\mathscr{I}^o}H_i\oplus\bigoplus_{i\in\mathscr{I}^+}(H_i\oplus\overline{H_i}) \tag{2.2}$$

where the direct sums are orthogonal and

$$i\in\mathscr{I}^o\Leftrightarrow H_i=\overline{H}_i,\qquad i\in\mathscr{I}^+\Leftrightarrow H_i\perp\overline{H}_i.$$

We can therefore choose an orthonormal basis (ϕ_{ik}) for $L^2(\mathscr{X},m)$ such that
• for $i\in\mathscr{I}^o$, $(\phi_{ik})_{1\le k\le d_i}$ is an orthonormal basis of H_i stable under conjugation;
• for $i\in\mathscr{I}^+$, $(\phi_{ik})_{1\le k\le d_i}$ is an orthonormal basis for H_i (d_i =the dimension of H_i) and $(\overline{\phi_{ik}})_{1\le k\le d_i}$ is an orthonormal basis for $\overline{H}_i$.
Such a orthonormal basis $(\phi_{ik})_{ik}$ of $L^2(\mathscr{X},m)$ has therefore the property that, if ϕ_{ik} is an element of the basis, then $\overline{\phi}_{ik}$ is also an element of the basis (possibly coinciding with ϕ_{ik}).
We say that such a basis is *compatible with complex conjugation*.

Example 2.1. $\mathscr{X}=\mathbb{S}^1$, the one dimensional torus. Here $\widehat{G}=\mathbb{Z}$ and H_k, $k\in\mathbb{Z}$ is generated by the function $\gamma_k(\theta)=\mathrm{e}^{ik\theta}$. $\overline{H}_k=H_{-k}$ and $\overline{H}_k\perp H_k$ for $k\neq 0$. All of the H_k's are one-dimensional.

Recall that the irreducible representations of a compact topological group G are all one-dimensional if and only if G is Abelian.

Example 2.2. $G=SO(3)$, $\mathscr{X}=G$ itself. The group $SO(3)$ has one and only one irreducible representation σ_ℓ of dimension $2\ell+1$ for every odd number $2\ell+1$, $\ell=0,1,\dots$. There is a popular choice of a basis for the subspaces $H_{\sigma_\ell,i}$, given by the matrix elements of the columns of the Wigner matrices (see again [VK91]). These $2\ell+1$ subspaces are usually indexed $H_{\sigma_\ell,-\ell},H_{\sigma_\ell,-\ell+1},\dots,H_{\sigma_\ell,\ell}$. We have that $\overline{H}_{\sigma_\ell,0}=H_{\sigma_\ell,0}$, whereas $\overline{H}_{\sigma_\ell,-m}=H_{\sigma_\ell,m}$, $m=1,\dots,\ell$. Therefore for every irreducible representation σ_ℓ, there is a single subspace that is self-conjugate, whereas the others are pairwise conjugated (and, in particular, they are orthogonal to their conjugate).

Example 2.3. $G=SO(3)$, $\mathscr{X}=\mathbb{S}^2$, the sphere. Among the subspaces $H_{\sigma_\ell,m}$ introduced in the Example 2.2 only $H_{\sigma_\ell,0}$ is invariant under the action of the isotropy subgroup of the north pole. Therefore $L^2(\mathbb{S}^2)$ can be identified with the direct sum of these irreducible subspaces, for $\ell=0,1,\dots$. A popular choice of a basis of $H_{\sigma_\ell,0}$ are the spherical harmonics, $(Y_{\ell,m})_{-\ell\le m\le\ell}$, $\ell\in\mathbb{N}$ (see [VK91]). $H_{\sigma_\ell,0}=\operatorname{span}((Y_{\ell,m})_{-\ell\le m\le\ell})$ are subspaces of $L^2(\mathscr{X},m)$ on which G acts irreducibly. We have $\overline{Y}_{\ell,m}=(-1)^mY_{\ell,-m}$ and $Y_{\ell,0}$ is real. In this case, therefore, all irreducible subspaces are self-conjugated.

By choosing $\phi_{\ell,m} = Y_{\ell,m}$ for $m \geq 0$ and $\phi_{\ell,m} = (-1)^m Y_{\ell,m}$ for $m < 0$, we find a basis of H_ℓ such that if ϕ is an element of the basis, then the same is true for $\overline{\phi}$. Here $\dim(H_\ell) = 2\ell + 1$, $\overline{H}_\ell = H_\ell$, so that in the decomposition (2.2) there are no subspaces of the form H_i for $i \in \mathscr{I}^+$.

Example 2.4. $G = SU(2)$, $\mathscr{X} = G$ itself. There is exactly one (up to equivalences) irreducible representation of dimension j, $j = 0, 1, \dots$ Again, if one chooses the basis given by the columns of the Wigner matrices, for j even the subspaces $H_{\sigma_j,i}$ are pairwise conjugated, whereas for j odd the situation is the same as $SO(3)$ (one subspace self-conjugated and the other ones pairwise conjugated). Actually, as it is well known, $SO(3)$ is a quotient group of $SU(2)$.

The arguments of this section also apply to the particular case of finite groups and their homogeneous spaces. Of course in this case in the developments above only a finite number of irreducible subspaces appears.

3 The Karhunen-Loève expansion

We consider on $\mathscr{X}$ a real *centered* square integrable random field $(T(x))_{x\in\mathscr{X}}$. We assume that there exists a probability space $(\Omega, \mathscr{F}, \mathrm{P})$ on which the r.v.'s $T(x)$ are defined and that $(x,\omega) \to T(x,\omega)$ is $\mathscr{B}(\mathscr{X}) \otimes \mathscr{F}$ measurable, $\mathscr{B}(\mathscr{X})$ denoting the Borel σ-field of $\mathscr{X}$. We assume that

$$\mathrm{E}\Big[\int_{\mathscr{X}} T(x)^2\, dm(x)\Big] = M < +\infty \tag{3.3}$$

which in particular entails that $x \to T_x(\omega)$ belongs to $L^2(m)$ a.s. Let us recall the main elementary facts concerning the Karhunen-Loève expansion for such fields. We can associate to T the bilinear form on $L^2(m)$

$$T(f,g) = \mathrm{E}\Big[\int_{\mathscr{X}} T(x) f(x)\, dm(x) \int_{\mathscr{X}} T(y) g(y)\, dm(y)\Big] \tag{3.4}$$

By (3.3) and the Schwartz inequality one gets easily that

$$|T(f,g)| \leq M\|f\|_2 \|g\|_2 \ ,$$

M being as in (3.3). Therefore, by the Riesz representation theorem there exists a function $R \in L^2(\mathscr{X} \times \mathscr{X}, m \otimes m)$ such that

$$T(f,g) = \int_{\mathscr{X}\times\mathscr{X}} f(x) g(y) R(x,y)\, dm(x) dm(y) \ .$$

We can therefore define a continuous linear operator $R : L^2(m) \to L^2(m)$

$$Rf(x) = \int_{\mathscr{X}} R(x,y) f(y)\, dm(y) \ .$$

It is immediate that the linear operator R is trace class and therefore compact (see [Par05] for details). Since it is self-adjoint there exists an orthonormal basis of $L^2(\mathscr{X}, m)$ that is formed by eigenvectors of R.
Let us define, for $\phi \in L^2(\mathscr{X}, m)$,

$$a(\phi) = \int_{\mathscr{X}} T(x)\phi(x)\, dm(x) \ ,$$

Let λ be an eigenvalue of R, that is $Rw = \lambda w$, for some non-zero function $w \in L^2(\mathscr{X})$ and denote by E_λ the corresponding eigenspace. Then the following is well-known.

Proposition 3.5. *Let $\phi \in E_\lambda$.*

a) If $\psi, \phi \in L^2(\mathscr{X}, m)$ are orthogonal, $a(\psi)$ and $a(\phi)$ are orthogonal in $L^2(\Omega, \mathrm{P})$. Moreover $\mathrm{E}[|a(\psi)|^2] = \lambda \|\psi\|_2^2$.

b) If ϕ is orthogonal to $\overline{\phi}$, then the r.v.'s $\Re a(\phi)$ and $\Im a(\phi)$ are orthogonal and have the same variance.

c) If the field T is Gaussian, $a(\phi)$ is a Gaussian r.v. If moreover ϕ is orthogonal to $\overline{\phi}$, then $a(\phi)$ is a complex centered Gaussian r.v. (that is $\Re a_i$ and $\Im a_i$ are centered, Gaussian, independent and have the same variance).

Proof. a) We have

$$\mathrm{E}[a(\phi)\overline{a}(\psi)] = \mathrm{E}\Big[\int_{\mathscr{X}} T(x)\phi(x)\, dm(x) \int_{\mathscr{X}} T(y)\overline{\psi}(y)\, dm(y)\Big] =$$
$$= \int_{\mathscr{X}\times\mathscr{X}} R(x,y)\phi(x)\overline{\psi}(y)\, dm(x)\, dm(y) = \lambda \int_{\mathscr{X}} \phi(y)\overline{\psi}(y)\, dm(y) = \lambda\langle \phi, \psi\rangle\ .$$

From this relation, by choosing first ψ orthogonal to ϕ and then $\psi = \phi$, the statement follows.

b) From the computation in a), as $a(\overline{\phi}) = \overline{a(\phi)}$, one gets $\mathrm{E}[a(\phi)^2] = \lambda\langle \phi, \overline{\phi}\rangle$. Therefore, if ϕ is orthogonal to $\overline{\phi}$, $\mathrm{E}[a(\phi)^2] = 0$ which is equivalent to $\Re a(\phi)$ and $\Im a(\phi)$ being orthogonal and having the same variance.

c) It is immediate that $a(\phi)$ is Gaussian. If ϕ is orthogonal to $\overline{\phi}$, $a(\phi)$ is a complex centered Gaussian r.v., thanks to b).

■

If $(\phi_k)_k$ is an orthonormal basis that is formed by eigenvectors of R, then under the assumption (3.3) it is well-known that the following expansion holds

$$T(x) = \sum_{k=1}^{\infty} a(\phi_k)\phi_k(x) \tag{3.5}$$

which is called the Karhunen-Loève expansion. This is intended in the sense of $L^2(\mathscr{X}, m)$ a.s. in ω. Stronger assumptions (continuity in square mean of $x \to T(x)$, e.g.) ensure also that the convergence takes place in $L^2(\Omega, \mathrm{P})$ for every x (see [SW86], p.210 e.g.)

More relevant properties are true if we assume in addition that the random field is invariant by the action G. Recall that the field T is said to be (weakly) *invariant* by the action of G if, for $f_i, \dots f_m \in L^2(\mathscr{X})$ the joint laws of $(T(f_1), \dots, T(f_m))$ and $(T(L_g f_1), \dots, T((L_g f_m))$ are equal for every $g \in G$. Here we write

$$T(f) = \int_{\mathscr{X}} T(x) f(x)\, dm(x), \qquad f \in L^2(\mathscr{X})\ .$$

If, in addition, the field is assumed to be continuous in square mean, this implies that for every $x_1, \dots, x_m \in \mathscr{X}$, $(T(x_1), \dots, T(x_m))$ and $(T(g^{-1}x_1), \dots, T(g^{-1}x_m))$, have the same joint laws for every $g \in G$. If the field is invariant then it is immediate that the covariance function R enjoys the invariance property

$$R(x,y) = R(g^{-1}x, g^{-1}y) \qquad \text{a.e. for every } g \in G \tag{3.6}$$

which also reads as

$$L_g(Rf) = R(L_g f)\ . \tag{3.7}$$

Then, thanks to (3.7), it is clear that G acts on E_λ. As moreover the E_λ's are stable under conjugation and

$$L^2(\mathscr{X}) = \bigoplus_k E_{\lambda_k}\ ,$$

it is clear that one can choose the H_i's introduced in (2.2) in such a way that each E_λ is the direct sum of some of them. It turns out therefore that the basis $(\phi_{ik})_{ik}$ of $L^2(\mathscr{X})$ introduced in the previous section can always be chosen to be formed by eigenvectors of R.
Moreover, if some of the H_i's are of dimension > 1, some of the eigenvalues of R have necessarily a multiplicity that is strictly larger than 1. As pointed out in §2, this phenomenon is related to the non commutativity of G. For more details on the Karhunen-Loève expansion and group representations see [PP05].
Remark that if the random field is isotropic and satisfies (3.3), then (3.5) follows by the Peter-Weyl theorem. Actually (3.3) entails that, for almost every ω, $x \to T(x)$ belongs to $L^2(\mathscr{X}, m)$.

Remark 3.6. An important issue when dealing with isotropic random fields is simulation. In this regard, a natural starting point is the Karhunen-Loève expansion: one can actually sample random r.v.'s $\alpha(\phi_k)$, (centered and standardized) and write

$$T_n(x) = \sum_{k=1}^n \sqrt{\lambda_k}\,\alpha(\phi_k)\phi_k \tag{3.8}$$

where the sequence $(\lambda_k)_k$ is summable. The point of course is what conditions, in addition to those already pointed out, should be imposed in order that (3.8) defines an isotropic field. In order to have a real field, it will be necessary that

$$\alpha(\overline{\phi}_k) = \overline{\alpha(\phi_k)} \tag{3.9}$$

Our main result (see next section) is that if the $\alpha(\phi_k)$'s are independent r.v.'s (abiding nonetheless to condition (3.9)), then the coefficients, and therefore the field itself are Gaussian.

If $H_i \subset L^2(\mathscr{X}, m)$ is a subspace on which G acts irreducibly, then one can consider the random field

$$T_{H_i}(x) = \sum a(\phi_j)\phi_j(x)$$

where the ϕ_j are an orthonormal basis of H_i. As remarked before, all functions in H_i are eigenvectors of R associated to the same eigenvalue λ.
Putting together this fact with (3.5) and (2.2) we obtain the decomposition

$$T = \sum_{i\in\mathscr{I}^\circ} T_{H_i^\circ} + \sum_{i\in\mathscr{I}^+} (T_{H_i^+} + T_{H_i^-})\ . \tag{3.10}$$

Example 3.7. Let T be a centered random field satisfying assumption (3.3) over the torus $\mathbb{T}$, whose Karhunen-Loève expansion is

$$T(\theta) = \sum_{k=-\infty}^{\infty} a_k\, e^{ik\theta}, \qquad \theta \in \mathbb{T}\ . \tag{3.11}$$

Then, if T is invariant by the action of $\mathbb{T}$ itself, the fields $(T(\theta))_\theta$ and $(T(\theta+\theta'))_\theta$ are equidistributed, which implies that the two sequences of r.v.'s

$$(a_k)_{-\infty<k<+\infty} \qquad \text{and} \qquad (e^{ik\theta'}a_k)_{-\infty<k<+\infty} \tag{3.12}$$

have the same finite distribution for every $\theta' \in \mathbb{T}$. Actually one can restrict the attention to the coefficients $(a_k)_{0\le k<+\infty}$, as necessarily $a_{-k} = \overline{a}_k$.
Conversely it is clear that if the two sequences in (3.12) have the same distribution for every $\theta' \in \mathbb{T}$, then the field is invariant.
Condition (3.12) implies in particular that, for every $k \in \mathbb{Z}, k \neq 0$ the distribution of a_k must be invariant by rotation (i.e. by the multiplication of a complex number of modulus 1).
If we assume moreover that the coefficients a_k appearing in (3.11) are independent, then the discussion above implies that the random field T is invariant by the action of $\mathbb{T}$ *if and only if* each of the complex r.v.'s a_k has a distribution which is invariant with respect to rotations of the complex plane. Hence, as it is easy to imagine a non Gaussian distribution satisfying this constraint, in the case of the torus it is possible to have independent coefficients for a non Gaussian random field.

4 Independent coefficients and non-Abelian groups

In this section we prove our main results showing that, if the group G is non commutative and under some mild additional assumptions, independence of the coefficients of the Fourier development implies their Gaussianity and, therefore, also that the random field must be Gaussian. We stress that we *do not* assume independence of the real and imaginary parts of the random coefficients.

Proposition 4.8. *Let $\mathscr{X}$ be an homogeneous space of the compact group G. Let $H_i^+ \subset L^2(\mathscr{X}, m)$ be a subspace on which G acts irreducibly, having a dimension ≥ 2 and such that if $f \in H_i^+$ then $\overline{f} \notin H_i^+$. Let $(\phi_k)_k$ be an orthonormal basis of H_i^+ and consider the random field*

$$T_{H_i^+}(x) = \sum_k a_k \phi_k(x) \ .$$

for a family of r.v.'s $(a_k)_k \subset L^2(\Omega, \mathrm{P})$. Then, if the r.v.'s a_i are independent, the field $T_{H_i^+}$ is G-invariant if and only if the r.v.'s $(a_k)_k$ are jointly Gaussian and $\mathrm{E}(|a_k|^2) = c$ (and therefore also the field $T_{H_i^+}$ is Gaussian).

Proof. Since G acts irreducibly on H_i^+, we have

$$\phi_k(g^{-1}x) = \sum_{\ell=1}^{d_i} D_{k,\ell}(g)\phi_\ell(x) \ ,$$

d_i being the dimension of H_i^+ and $D(g)$ being the representative matrix of the action of $g \in G$. Therefore

$$T(g^{-1}x) = \sum_{\ell=1}^{d_i} \tilde{a}_\ell \phi_\ell(x)$$

where

$$\tilde{a}_\ell = \sum_{k=1}^{d_i} D_{k,\ell}(g) a_k \ .$$

If the field is G-invariant, then the vectors $(\tilde{a}_\ell)_\ell$ have the same joint distribution as $(a_k)_k$ and in particular the $(\tilde{a}_\ell)_\ell$ are independent. One can then apply the Skitovich-Darmois theorem

below (see [KLR73] e.g.) as soon as it is proved that $g \in G$ can be chosen so that $D_{k,\ell}(g) \neq 0$ for every k, ℓ. This will follow from the considerations below, where it is proved that the set $Z_{k,\ell}$ of the zeros of $D_{k,\ell}$ has measure zero.
Indeed, let G_1 be the image of G in the representation space so that G_1 is also a connected compact group, and is moreover a Lie group since it is a closed subgroup of the unitary group $\mathrm{U}(d_i)$. If the representation is non trivial, then $G_1 \neq \{1\}$ and in fact has positive dimension, and the $D_{k,\ell}$ are really functions on G_1. For any fixed k, ℓ the irreducibility of the action of G_1 implies that $D_{k,\ell}(g)$ is not identically zero on G. Indeed, if this were not the case, we must have $\langle \phi_\ell(g\cdot), \phi_k\rangle = 0$ for all $g \in G_1$, so that the span of the $g\phi_\ell$ is orthogonal to ϕ_k; this span, being G_1-invariant and nonzero, must be the whole space by the irreducibility, and so we have a contradiction.
Since $D_{k\ell}$ is a non zero analytic function on G_1, it follows from standard results that $Z_{k\ell}$ has measure zero. Hence $\bigcup_{k\ell} Z_{k\ell}$ has measure zero also, and so its complement in G_1 is non empty. ■

We use the following version of the Skitovich-Darmois theorem, which was actually proved by S. G. Ghurye and I. Olkin [GO62] (see also [KLR73]).

Theorem 4.9. *Let $X_1, \dots, X_r$ be mutually independent random vectors in $\mathbb{R}^n$. If the linear statistics*

$$L_1 = \sum_{j=1}^r A_j X_j, \qquad L_2 = \sum_{j=1}^r B_j X_j \ ,$$

are independent for some real nonsingular $n \times n$ matrices A_j, B_j, $j = 1, \dots, r$, then each of the vectors $X_1, \dots, X_r$ is normally distributed.

We now investigate the case of the random field T_H, when H is a subspace such that $\overline{H} = H$. In this case we can consider a basis of the form $\phi_{-k}, \dots, \phi_k$, $k \leq \ell$, with $\phi_{-k} = \overline{\phi}_k$. The basis may contain a real function ϕ_0, if $\dim H$ is odd. Let us assume that the random coefficients a_k, $k \geq 0$ are independent. Recall that $a_{-k} = \overline{a_k}$.
The argument can be implemented along the same lines as in Proposition 4.8. More precisely, if $m_1 \geq 0$, $m_2 \geq 0$, the two complex r.v.'s

$$\begin{aligned} \widetilde{a}_{m_1} &= \sum_{m=-\ell}^{\ell} D_{m,m_1}(g) a_m \\ \widetilde{a}_{m_2} &= \sum_{m=-\ell}^{\ell} D_{m,m_2}(g) a_m \end{aligned} \tag{4.13}$$

have the same joint distribution as a_{m_1} and a_{m_2}. Therefore, if $m_1 \neq m_2$, they are independent. Moreover $a_{-m} = \overline{a_m}$, so that the previous relation can be written

$$\begin{aligned} \widetilde{a}_{m_1} &= D_{0m_1}(g) a_0 + \sum_{m=1}^{\ell} \Big(D_{m,m_1}(g) a_m + D_{-m,m_1}(g) \overline{a_m} \Big) \\ \widetilde{a}_{m_2} &= D_{0,m_2}(g) a_0 + \sum_{m=1}^{\ell} \Big(D_{m,m_2}(g) a_m + D_{-m,m_2}(g) \overline{a_m} \Big) \end{aligned}$$

In order to apply the Skitovich-Darmois theorem, we must ensure that $g \in G$ can be chosen so that the real linear $\mathbb{R}^2 \to \mathbb{R}^2$ applications

$$z \to D_{m,m_i}(g) z + D_{-m,m_i}(g) \overline{z}, \qquad m = 1, \dots, \ell, i = 1, 2 \tag{4.14}$$

are all non singular. It is immediate that this condition is equivalent to imposing that $|D_{m,m_i}(g)| \neq |D_{-m,m_i}(g)|$.
We show below that (4.14) is satisfied for some well-known examples of groups and homogeneous spaces. We do not know whether (4.14) is always satisfied for every compact group. We are therefore stating our result conditional upon (4.14) being fulfilled.

Assumption 4.10. *There exist $g \in G$ and $0 \leq m_1 < m_2 \leq \ell$ such that*

$$|D_{m,m_i}(g)| \neq |D_{-m,m_i}(g)|$$

for every $0 < m \leq \ell$.

We have therefore proved the following.

Proposition 4.11. *Let $\mathscr{X}$ be an homogeneous space of the compact group G. Let $H_i \subset L^2(\mathscr{X}, m)$ be a subspace on which G acts irreducibly, having a dimension $d > 2$ and such that $\overline{H}_i = H_i$. Let $(\phi_k)_k$ be an orthonormal basis of H_i such that $\phi_{-k} = \overline{\phi}_k$ and consider the random field*

$$T_{H_i}(x) = \sum_k a_k \phi_k(x)$$

where the r.v.'s $a_k, k \geq 0$ are centered, square integrable, independent and $a_{-k} = \overline{a}_k$. Then T_{H_i} is G-invariant if and only if the r.v.'s $(a_k)_{k\geq 0}$ are jointly Gaussian and $\mathrm{E}(|a_k|^2) = c$ (and therefore also the field T_{H_i} is Gaussian).

Putting together Propositions 4.8 and 4.11 we obtain our main result.

Theorem 4.12. *Let $\mathscr{X}$ be an homogeneous space of the compact group G. Consider the decomposition (2.2) and let $\big((\phi_{ik})_{i\in\mathscr{I}^\circ}, (\phi_{ik}, \overline{\phi}_{ik})_{i\in\mathscr{I}^+}\big)$ be a basis of $L^2(G)$ adapted to that decomposition. Let*

$$T = \sum_{i\in\mathscr{I}^\circ}\sum_k a_{ik}\phi_{ik} + \sum_{i\in\mathscr{I}^+}\sum_k \big(a_{ik}\phi_{ik} + \overline{a}_{ik}\overline{\phi}_{ik}\big)$$

be a random field on $\mathscr{X}$, where the series above are intended to be converging in square mean. Assume that T is isotropic with respect to the action of G and that the coefficients $(a_{ik})_{i\in\mathscr{I}^\circ,k\geq 0}, (a_{ik})_{i\in I^+}$ are independent. If moreover

a) the only one-dimensional irreducible representation appearing in (2.2) are the constants;

b) there are no 2-dimensional subspaces $H \subset L^2(\mathscr{X})$, invariant under the action of G and such that $\overline{H} = H$;

c) The random coefficient corresponding to the trivial representation vanishes.

d) For every $H \subset L^2(\mathscr{X})$, irreducible under the action of G and such that $\overline{H} = H$, Assumption 4.10 holds.

Then the coefficients $(a_{ik})_{i\in\mathscr{I}^\circ,k\geq 0}, (a_{ik})_{i\in\mathscr{I}^+}$ are Gaussian and the field itself is Gaussian.

Theorem 4.12 states that if one wants to simulate a random field via the sampling of independent coefficients, then in the decomposition (3.10) all the fields T_H are necessarily Gaussian with the only possible exception of those corresponding to subspaces H

- having dimension 1;
- having dimension 2 and such that $\overline{H} = H$;

- such that $\overline{H} = H$, but not satisfying Assumption 4.10.

Let us stress with the following statements the meaning of assumption a)–d). The following result gives a condition ensuring that assumption b) of Theorem 4.12 is satisfied.

Proposition 4.13. *Let U be an irreducible unitary 2-dimensional representation of G and let H_1 and H_2 be the two corresponding subspaces of $L^2(G)$ in the Peter-Weyl decomposition. Then if U has values in $SU(2)$, then $\overline{H}_1 = H_2 \neq H_1$.*

Proof. If we note

$$U(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$$

then one can assume that H_1 is generated by the functions a and c, whereas H_2 is generated by b and d. It suffices now to remark that, the matrix $U(g)$ belonging to $SU(2)$, we have $\overline{a(g)} = d(g)$ and $\overline{b(g)} = -c(g)$.

■

Recall that the commutator G_0, of a topological group G is the closed group that is generated by the elements of the form $xyx^{-1}y^{-1}$

Corollary 4.14. *Let G be a compact group such that its commutator G_0 coincides with G himself. Then assumptions a) and b) of Theorem 4.12 are satisfied. In particular these assumptions are satisfied if G is a semisimple Lie group.*

Proof. Recall that if $G_0 = G$, G cannot have a quotient that is an Abelian group. If there was a unitary representation U with a determinant not identically equal to 1, then $g \to \det(U(G))$ would be an homomorphism onto the torus $\mathbb{T}$ and therefore G would possess $\mathbb{T}$ as a quotient. The same argument proves that G cannot have a one dimensional unitary representation other than the trivial one. One can therefore apply Proposition 4.13 and b) is satisfied.

■

Remark 4.15. It is easy to prove that Assumption 4.10 is satisfied when $\mathscr{X} = \mathbb{S}^2$ and $G = SO(3)$, if we consider the basis given by the spherical harmonics. As mentioned in [BM07], this can be established using explicit expressions of the representation coefficients as provided e.g. in [VMK88].
In the same line of arguments it is also easy to check the same in the cases $\mathscr{X} = G = SO(3)$ and $\mathscr{X} = G = SU(2)$, with respect to the basis given by the columns of the Wigner matrices. Actually in the Peter-Weyl decomposition of $SO(3)$, the irreducible spaces H appearing in addition to those in the decomposition of $\mathbb{S}^2$ are of the type $\overline{H} \perp H$ and a similar situation appears when switching from $SO(3)$ to $SU(2)$.

Remark 4.16. Assumption 4.10 is a sufficient condition for Proposition 4.11 to be true. Actually, in the case $\mathscr{X} = \mathbb{S}^2$ and $G = SO(3)$, it is easy to see that Assumption 4.10 cannot hold for the irreducible representation of dimension 3, for which however an ad hoc argument can be developed. Besides this phenomenon, that is typical of the representations of dimension 3, we do not know of any example in which Assumption 4.10 is not satisfied and we are led to conjecture that it holds in general. Remark that the validity of Assumption 4.10 depends not only on the equivalence class of representations that is considered, but also on the particular basis of the irreducible subspace H under consideration, i.e. we do not know whether Assumption 4.10 remains valid under a change of basis.

5 Appendix

Proposition 5.17. *Let V be a finite dimensional Hilbert space on which G acts unitarily, and let V be equipped with a conjugation $\sigma(v \to \bar{v})$ commuting with the action of G. Let H be an irreducible G-invariant subspace and let $V = H + \overline{H}$.*

a) If the actions of G on H and $\overline{H}$ are inequivalent, then $\overline{H} \perp H$ and $V = H \oplus \overline{H}$.

b) If the actions of G on H and $\overline{H}$ are equivalent, then either $H = \overline{H}$ or we can find an irreducible G-invariant subspace K of V such that $\overline{K} \perp K$ and $V = K \oplus \overline{K}$.

Proof. Let P be the orthogonal projection $V \to \overline{H}$ and A its restriction to H. Then, for every $h \in H$, $h' \in \overline{H}$ and $g \in G$, we have

$$\langle g(Ah), h'\rangle = \langle Ah, gh'\rangle = \langle h, gh'\rangle = \langle gh, h'\rangle = \langle A(gh), h'\rangle$$

From this we get that G acts on $A(H)$. The action of G on $\overline{H}$ being irreducible, we have either $A(H) = \{0\}$ or $A(H) = \overline{H}$. In the first case H is already orthogonal to $\overline{H}$. Otherwise A intertwines the actions on H and on $\overline{H}$, so that these are equivalent and $V = H \oplus H^{\perp}$.
V being the sum of two copies of the representation on H, there is a *unitary* isomorphism $V \simeq H \otimes \mathbb{C}^2$ where $\mathbb{C}^2$ is given the standard scalar product. So we assume that $V = H \otimes \mathbb{C}^2$. G acts only on the first component, so that G acts irreducibly on every subspace of the form $H \otimes Z$, Z being a one dimensional subspace of $\mathbb{C}^2$.
Let us identify the action of σ on $H \otimes \mathbb{C}^2$. Let σ_0 be the conjugation on V defined by $\sigma_0(u \otimes v) = u \otimes \bar{v}$ where $v \to \bar{v}$ is the standard conjugation $(z_1, z_2) \to (\overline{z_1}, \overline{z_2})$. Then $\sigma\sigma_0$ is a *linear operator* commuting with G and so is of the form $1 \otimes L$ where $L(\mathbb{C}^2 \to \mathbb{C}^2)$ is a linear operator. Hence

$$\sigma(u \otimes v) = \sigma\sigma_0(u \otimes \overline{v}) = u \otimes L\bar{v}.$$

If Z is any one dimensional subspace of $\mathbb{C}^2$, $H \otimes Z$ is G-invariant and irreducible, and we want to show that for some Z, $H \otimes Z \perp H \otimes Z^{\sigma}$, i.e., $Z \perp Z^{\sigma}$. Here $Z^{\sigma} = \sigma(Z)$.
For any such Z, let v be a nonzero vector in it; then the condition $Z \perp Z^{\sigma}$ becomes $(v, L\bar{v}) = 0$ where $(,)$ is the scalar product in $\mathbb{C}^2$. Since $(,)$ is Hermitian, $B(v, w) := (v, L\bar{w})$ is *bilinear* and we want v to satisfy $B(v, v) = 0$. This is actually standard: indeed, replacing B by $B + B^T$ (which just doubles the quadratic form) we may assume that B is symmetric.
If B is degenerate, there is a nonzero v such that $B(v, w) = 0$ for all w, hence $B(v, v) = 0$. If B is nondegenerate, there is a basis v_1, v_2 for $\mathbb{C}^2$ such that $B(v_i, v_j) = \delta_{ij}$. Then, if $w = v_1 + iv_2$ where $i = \sqrt{-1}$, $B(w, w) = 0$.

■

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