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# THE DISTRIBUTION OF TIME SPENT BY A STANDARD EXCURSION ABOVE A GIVEN LEVEL, WITH APPLICATIONS TO RING POLYMERS NEAR A DISCONTINUITY IN POTENTIAL

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*Abstract*

*The law for the time $\tau_a$ spent by a standard Brownian excursion above a given level $a > 0$ is found using Itô excursion theory. This is achieved by conditioning the excursion to have exactly one mark of an independent Poisson process. Various excursion rates for excursions conditioned to have exactly $n \in \mathbb{Z}^+$ marks are also given in terms of generating functions. This work has applications to the theory of ring polymers and end-attached polymers near a discontinuity in potential.*

## 1 Introduction

Let $X = (X_t, 0 \le t \le 1)$ denote the standard Brownian excursion (or scaled Brownian excursion). In fact [3] $X$ is a $BES^3(0,0)$, i.e. a standard Bessel 3 bridge process. The quantity of interest is

$$\phi(a, \alpha) \equiv \mathbb{E}[\exp(-\tfrac{1}{2}\alpha^2\tau_a)], \quad \alpha \ge 0,$$

where $\tau_a$ is the time spent by $X$ above level $a > 0$. However, it is difficult to determine $\phi$ directly. It is convenient to stretch the process $X$, and consider instead a Brownian excursion $Y$ of duration $T$, i.e.

$$Y_t = \sqrt{T} X_{t/T}, \quad 0 \le t \le T.$$

In terms of $Y$, we find

$$\phi(a/\sqrt{T}, \alpha\sqrt{T}) = \mathbb{E}[\exp(-\tfrac{1}{2}\alpha^2 T_a)],$$

where $T_a$ is the time spent by $Y$ above level $a > 0$. The quantity that we shall determine is

$$\psi(a, \alpha, \nu) = \mathbb{E}[\phi(a/\sqrt{T}, \alpha\sqrt{T})],$$

where $T$ has a $gamma(\frac{1}{2}, \frac{1}{2}\nu^2)$ distribution, $\nu > 0$. Note that mixing with a gamma distribution is, at least in principle, invertible using inverse Laplace transforms.

**Theorem 1** *The time $T_a$ spent above level $a > 0$ by a Brownian excursion $Y$ with a single mark of an independent Poisson marking process of rate $\frac{1}{2}\nu^2$ satisfies*

$$\mathbb{E}[\exp(-\tfrac{1}{2}\alpha^2 T_a)] = \frac{-\alpha^2\nu(1+a\gamma) + \nu\gamma^2\cosh(2\nu a) + (\nu^2 + \frac{1}{2}\alpha^2)\gamma\sinh(2\nu a)}{\gamma(\nu\cosh(\nu a) + \gamma\sinh(\nu a))^2},$$

*where $\gamma = (\alpha^2 + \nu^2)^{\frac{1}{2}}$. The duration $T$ of the excursion $Y$, conditioned on having exactly one mark, has a $gamma(\frac{1}{2}, \frac{1}{2}\nu^2)$ distribution.*

## 2 Excursion Rates

All excursion rates, $\mathbf{n}_+\{\cdot\}$ and $\mathbf{n}_-\{\cdot\}$ for upward and downward excursions from the origin, are for standard Brownian motion, and are with respect to local time in semi-martingale normalization. The following list of well-known excursion rates are for a Brownian motion marked by the points of an independent Poisson process of rate $\frac{1}{2}\nu^2$ [3]:

$$\mathbf{n}_+\{\cdot\} = \mathbf{n}_-\{(\cdot)^-\},$$

$$\mathbf{n}_+\{\text{hits } a > 0\} = \tfrac{1}{2}a^{-1},$$

$$\mathbf{n}_+\{\text{marked}\} = \tfrac{1}{2}\nu,$$

$$\mathbf{n}_+\{\text{hits } a > 0 \text{ or marked}\} = \tfrac{1}{2}\nu\coth\nu a,$$

$$\mathbf{n}_+\{\text{hits } a > 0 \text{ before any mark}\} = \tfrac{1}{2}\nu\,\mathrm{cosech}\,\nu a,$$

$$\mathbf{n}_+\{H_0 \geq t > 0\} = (2\pi t)^{-\frac{1}{2}},$$

where $H_0$ is the duration of the excursion.

**Proposition 1 (Generating functions for excursion rates)** *Define for $n \in \mathbb{Z}^+$*

$$A_n(a,\nu) = \mathbf{n}_+\{\textit{exactly } n \textit{ marks, and does not hit } a\}$$

*and*

$$B_n(a,\nu) = \mathbf{n}_+\{\textit{exactly } n \textit{ marks before hitting } a, \textit{ and does hit } a\},$$

*then*

$$A(a,\theta,\nu) \equiv \sum_{n=1}^{\infty}\theta^n A_n(a,\nu) = \tfrac{1}{2}\nu\coth\nu a - \tfrac{1}{2}\nu(1-\theta)^{\frac{1}{2}}\coth\nu(1-\theta)^{\frac{1}{2}}a$$

*and*

$$B(a,\theta,\nu) \equiv \sum_{n=1}^{\infty}\theta^n B_n(a,\nu) = \tfrac{1}{2}\nu(1-\theta)^{\frac{1}{2}}\,\mathrm{cosech}\,\nu(1-\theta)^{\frac{1}{2}}a - \tfrac{1}{2}\nu\,\mathrm{cosech}\,\nu a.$$

*Proof.* Consider a standard Brownian motion marked by the points of an independent Poisson process of rate $\frac{1}{2}\nu^2$, and suppose that each mark is independently 'good' or 'bad' with probabilities $\theta$ and $1-\theta$. Thus $\theta^n A_n(a,\nu)$ is the rate of upward excursions with exactly $n$ good marks and no bad marks, which gives

$$A(a,\theta,\nu) \equiv \sum_{n=1}^{\infty} \theta^n A_n(a,\nu)$$

as the rate of upward marked excursions that do not hit $a>0$ for which all the marks are good. From the above list of excursion rates we find

$$\mathbf{n}_+\{\text{marked, but does not hit } a>0\} = \tfrac{1}{2}\nu \coth \nu a - \tfrac{1}{2}a^{-1}.$$

Now we need to remove any excursions that contain any bad marks and do not hit $a \geq 0$. Since bad marks occur at (real-time) rate $\frac{1}{2}\nu^2(1-\theta)$, this gives

$$A(a,\theta,\nu) = \left(\tfrac{1}{2}\nu \coth \nu a - \tfrac{1}{2}a^{-1}\right) - \left(\tfrac{1}{2}\nu(1-\theta)^{\frac{1}{2}} \coth \nu(1-\theta)^{\frac{1}{2}} a - \tfrac{1}{2}a^{-1}\right).$$

We can use a similar argument for $B(a,\theta,\nu)$. From the above excursion rates we find

$$\mathbf{n}_+\{\text{marked before hitting } a \text{, and does hit } a\} = \tfrac{1}{2}a^{-1} - \tfrac{1}{2}\nu \operatorname{cosech} \nu a.$$

Again conditioning on good marks gives the required generating function, and completes the proof. ∎

Using the generating functions from proposition 1, we can extract the required rates for single marks, given below.

**Corollary 1 (Rates for single marks)** *For $a>0$,*

$$\mathbf{n}_+\{\textit{single mark}\} = \tfrac{1}{4}\nu,$$

$$A_1(a,\nu) = \mathbf{n}_+\{\textit{single mark and does not hit } a\} = \tfrac{1}{4}\nu \coth \nu a - \tfrac{1}{4}\nu^2 a \operatorname{cosech}^2 \nu a,$$

$$B_1(a,\nu) = \mathbf{n}_+\{\textit{single mark before hitting } a\} = \tfrac{1}{4}\nu^2 a \coth \nu a \operatorname{cosech} \nu a - \tfrac{1}{4}\nu \operatorname{cosech} \nu a.$$

**Lemma 1** *The duration $T$ of an excursion with a single mark has a gamma$(\frac{1}{2},\frac{1}{2}\nu^2)$ distribution, i.e. it has density*

$$\mathbb{P}[T \in dt] = \frac{\nu}{(2\pi t)^{\frac{1}{2}}} \exp(-\tfrac{1}{2}\nu^2 t)dt.$$

*(Alternatively, $T$ is distributed as the square of a $N(0,\nu^{-2})$ random variable.)*

*Proof.* From above, the rate of upward excursions with duration greater than $T$ is

$$\mathbf{n}_+\{T \geq t > 0\} = (2\pi t)^{-\frac{1}{2}},$$

the probability that an excursion of duration $T$ has exactly one mark is $\frac{1}{2}\nu^2 T \exp(-\frac{1}{2}\nu^2 T)$, and the rate of upward Brownian excursions with a single mark is $\frac{1}{4}\nu$. Combining these gives the required result. ∎

# 3 Proof of Theorem 1

From lemma 1, we know that a Brownian excursion with a single mark has a $gamma(\frac{1}{2}, \frac{1}{2}\nu^2)$ distribution.
Suppose that in addition to the marking at rate $\frac{1}{2}\nu^2$ ($\nu$ marks) there is an independent Poisson marking process of rate $\frac{1}{2}\alpha^2$ ($\alpha$ marks), then

$$\mathbb{E}[\exp(-\tfrac{1}{2}\alpha^2 T_a)] = \frac{\mathbf{n}_+\{\text{single } \nu \text{ mark, and no } \alpha \text{ marks above } a\}}{\mathbf{n}_+\{\text{single } \nu \text{ mark}\}}.$$

Let $H_a$ be the first hitting time of $a > 0$, with $H_a = \infty$ if the excursion does not hit $a$, and let $K_a$ be the last exit time from $[a, \infty)$.
We can now split the numerator as follows:

$$\mathbf{n}_+\{\text{single } \nu \text{ mark, and no } \alpha \text{ marks above } a\} = \mathbf{n}_+\{\text{single } \nu \text{ mark, and does not hit } a\}$$

$$+\mathbf{n}_+\{\text{single } \nu \text{ mark, no } \alpha \text{ marks above } a\text{, and does hit } a\}.$$

Using the excursion rates of corollary 1, we find

$$\mathbf{n}_+\{\text{single } \nu \text{ mark, and does not hit } a\} = A_1(a, \nu).$$

On the event that the excursion hits $a$, the part of the excursion before $H_a$ is independent of the remainder; thus

$$\mathbf{n}_+\{\text{single } \nu \text{ mark, no } \alpha \text{ marks above } a\text{, and does hit } a\}$$

$$= \mathbf{n}_+\{\text{single } \nu \text{ mark before } H_a < \infty\}\mathbb{P}^a[\text{no } \nu \text{ marks, and no } \alpha \text{ marks above } a]$$

$$+\mathbf{n}_+\{\text{no } \nu \text{ marks before } H_a < \infty\}\mathbb{P}^a[\text{single } \nu \text{ mark and no } \alpha \text{ marks above } a]$$

$$= B_1(a, \nu)\, \mathbb{P}^a[\text{no } \nu \text{ marks, and no } \alpha \text{ marks above } a]$$

$$+\tfrac{1}{2}\nu \operatorname{cosech} \nu a\, \mathbb{P}^a[\text{single } \nu \text{ mark and no } \alpha \text{ marks above } a],$$

using the excursion rates given in the last section. To determine the remaining probabilities we consider excursions from $a$. Thus

$$\mathbb{P}^a[\text{no } \nu \text{ marks, and no } \alpha \text{ marks above } a] = \frac{\mathbf{n}_-^a\{\text{no } \nu \text{ marks, and hits } 0\}}{\mathbf{n}_-^a\{\nu \text{ marked or hits } 0\} + \mathbf{n}_+^a\{\text{marked}\}}$$

$$= \frac{\frac{1}{2}\nu \operatorname{cosech} \nu a}{\frac{1}{2}\nu \coth \nu a + \frac{1}{2}\gamma},$$

where we have used the up-down and translational symmetries of Brownian excursion rates. Similarly, since the excursions from $a$ are independent, we find

$$\mathbb{P}^a[\text{single } \nu \text{ mark and no } \alpha \text{ marks above } a]$$

$$= \mathbb{P}^a[\text{single } \nu \text{ mark before } K_a\text{, no } \nu \text{ marks after } K_a \text{ and no } \alpha \text{ marks above } a]$$

$$+\mathbb{P}^a[\text{no } \nu \text{ marks before } K_a\text{, single } \nu \text{ mark after } K_a\text{, and no } \alpha \text{ marks above } a]$$

$$= \frac{A_1(a, \nu) + \frac{1}{4}\nu^2\gamma^{-1}}{\frac{1}{2}\nu \coth \nu a + \frac{1}{2}\gamma} \frac{\frac{1}{2}\nu \operatorname{cosech} \nu a}{\frac{1}{2}\nu \coth \nu a + \frac{1}{2}\gamma} + \frac{B_1(a, \nu)}{\frac{1}{2}\nu \coth \nu a + \frac{1}{2}\gamma},$$

where we have used

$$\mathbf{n}_+^a\{\text{single mark, and the mark is of type } \nu\} = \mathbf{n}_+\{\text{single } \alpha \text{ or } \nu \text{ mark}\}\mathbb{P}[\nu \text{ mark}|\text{ mark}]$$

$$= \tfrac{1}{4}(\alpha^2+\nu^2)^{\frac{1}{2}} \frac{\frac{1}{2}\nu^2}{\frac{1}{2}\alpha^2+\frac{1}{2}\nu^2} = \tfrac{1}{4}\nu^2\gamma^{-1}.$$

Putting the pieces together, we find

$$\mathbf{n}_+\{\text{single } \nu \text{ mark, and no } \alpha \text{ marks above } a\} = A_1(a,\nu) + B_1(a,\nu)\frac{\frac{1}{2}\nu\,\text{cosech}\,\nu a}{\frac{1}{2}\nu\coth\nu a + \frac{1}{2}\gamma}$$

$$+\tfrac{1}{2}\nu\,\text{cosech}\,\nu a \frac{A_1(a,\nu)+\frac{1}{4}\nu^2\gamma^{-1}}{\frac{1}{2}\nu\coth\nu a+\frac{1}{2}\gamma}\frac{\frac{1}{2}\nu\,\text{cosech}\,\nu a}{\frac{1}{2}\nu\coth\nu a+\frac{1}{2}\gamma} + \tfrac{1}{2}\nu\,\text{cosech}\,\nu a \frac{B_1(a,\nu)}{\frac{1}{2}\nu\coth\nu a+\frac{1}{2}\gamma},$$

After a little straightforward, but tedious, algebra, the required result follows. ■

# 4 Special Case

A known special case of theorem 1 is in the limit $\alpha \to \infty$.

**Corollary 2**

$$\mathbb{P}\left[\sup_{0\le t\le T} Y_t \le a\right] = \coth\nu a - \nu a\,\text{cosech}^2\nu a, \quad a > 0.$$

The right-hand side of the last equation is simply, in terms of excursion rates, equal to $A_1(a,\nu)/(\frac{1}{4}\nu)$.

By expanding the last equation in terms of exponentials, and 'unmixing' $T$ (which involves an inverse Laplace transform) to recover the corresponding result for $X$ we find the following.

**Theorem 2 (Kennedy (1976) [1])** *For a standard Brownian excursion $X$,*

$$\mathbb{P}\left[\sup_{0\le t\le 1} X_t \le a\right] = 1 - 2\sum_{n=1}^{\infty}(4a^2n^2-1)\exp(-2a^2n^2), \quad a \ge 0.$$

We could perform the same procedure to produce an explicit version of theorem 1 for the process $X$, although it is rather messy in this form.

# 5 Applications to Ring Polymers and to End-Attached Polymers

The relationship between Brownian excursions and Brownian bridges, and therefore their relationship to flexible ring polymers is given in the next theorem.

**Theorem 3 (Vervaat (1979) [2])** *Let $(W_t, 0\le t\le 1)$ be a standard Brownian bridge and $\tau$ the location of its absolute minimum. For $0\le t\le 1$, let $(\tau+t)_1$ denote $\tau+t$ reduced mod 1. Then*

$$(W((\tau+t)_1) - W_\tau, 0\le t\le 1) = X \quad \textit{in distribution},$$

*where $X$ is the standard Brownian excursion.*

Modeling a ring polymer (of 'duration' 1) $X$ by a Brownian bridge [4], then leads to the conclusion that

$$Z \equiv \phi(a,\alpha) \equiv \mathbb{E}[\exp(-\tfrac{1}{2}\alpha^2\tau_a)]$$

is the partition function for a flexible ring polymer in solution near a plane interface between two regions of constant potential. This could then be used to determine the structure of the depletion layer at the interface of a 'hostile' region.
If instead we model a ring polymer by the process $Y$, it is easy to show that this corresponds to a ring polymer in equilibrium with a bath that contains linear polymers with an exponential distribution of 'durations' (or 'lengths') which are marked with occasional 'special monomers'. However, the distribution of durations in three dimensions for ring polymers with two marks corresponds to the distribution of durations for a single mark in one dimension. Thus the one-dimensional process $Y$ is the projection of a three-dimensional ring polymer with two special monomers.
For a $d$ dimensional Brownian motion $W$ with a duration $T$ having a $gamma(n, \frac{1}{2}\nu^2)$ distribution, after conditioning on $W_0 = W_T$, the distribution of durations is $gamma(n - \frac{1}{2}d, \frac{1}{2}\nu^2)$. This leads to the following result.

**Proposition 2 (Ring Polymers)** *For a $d$-dimensional ring polymer (Brownian ring) $W = (W^{(1)}, W^{(2)}, \ldots, W^{(d)})$ with a duration $T$ having a $gamma(n - \frac{1}{2}d, \frac{1}{2}\nu^2)$ law*

$$I\!P^0[\sup_{0\le t\le T} W_t^{(i)} \le a] = \frac{A_{n-\frac{1}{2}d+\frac{1}{2}}(a,\nu)}{A_{n-\frac{1}{2}d+\frac{1}{2}}(\infty,\nu)}, \quad \textit{for any } i, \textit{ odd } d > 0, \textit{ and } n > \tfrac{1}{2}d.$$

*The distribution of $T$ corresponds to conditioning on exactly $n$ special monomers (marks).*

*Proof.* After projecting down to 1 dimension, the required result follows directly from the definition of $A_{n-\frac{1}{2}d+\frac{1}{2}}$. ∎
The corresponding result for $d$ even could be derived by considering the excursions of a Bessel 2 process.
Also, the Brownian excursion can be used directly to model a linear polymer that is attached at its ends to a plane surface.

# References

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