Elect. Comm. in Probab. **14** (2009), 55–65

ELECTRONIC COMMUNICATIONS in PROBABILITY

A CONNECTION BETWEEN THE STOCHASTIC HEAT EQUATION AND FRACTIONAL BROWNIAN MOTION, AND A SIMPLE PROOF OF A RESULT OF TALAGRAND

CARL MUELLER[1]
Dept. of Mathematics
University of Rochester
Rochester, NY 14627
email: `cmlr@math.rochester.edu`

ZHIXIN WU
Dept. of Mathematics
DePauw University
Greencastle, IN 46135
email: `zhixinwu@depauw.edu`

Submitted September 10, 2008, *accepted in final form* January 8, 2009

AMS 2000 Subject classification: Primary, 60H15; Secondary, 35R60, 35K05.
Keywords: heat equation, white noise, stochastic partial differential equations.

Abstract

We give a new representation of fractional Brownian motion with Hurst parameter $H \leq \frac{1}{2}$ using stochastic partial differential equations. This representation allows us to use the Markov property and time reversal, tools which are not usually available for fractional Brownian motion. We then give simple proofs that fractional Brownian motion does not hit points in the critical dimension, and that it does not have double points in the critical dimension. These facts were already known, but our proofs are quite simple and use some ideas of Lévy.

1 Introduction

Our main result is a new representation for fractional Brownian motion using stochastic partial differential equations, described in this section. As an application, in Section 2, Theorems 1 and 2, we state some known results about when fractional Brownian motion hits points and has double points. Our representation allows us to give simple proofs of these results.

In recent years there has been an upsurge of interest in fractional Brownian motion, see Nualart, Chapter 5 [Nua06]. The most common model for noise in physical systems is white noise $\dot{B}_t$, the derivative of Brownian motion. The central limit theorem gives some justification for using a Gaussian process such as B_t. Furthermore, $\dot{B}_s, \dot{B}_t$ are independent if $s \neq t$. In many situations, however, there are correlations between noise at different times. A natural correlated Gaussian

[1] RESEARCH SUPPORTED BY THE NSF AND NSA

model to consider is fractional Brownian motion $X_t = X_t^H : t \geq 0$ taking values in $\mathbf{R}^n$, with Hurst parameter $H \in (0,1]$. The process X_t is uniquely specified by the following axioms.

1. $X_0 = 0$ with probability 1.
2. $X_t : t \geq 0$ is a Gaussian process with stationary increments. That is, for $t, h > 0$ the probability distribution of the increment $X_{t+h} - X_t$ is independent of t.
3. For $c > 0$ we have $X_{ct} \stackrel{\mathcal{D}}{=} c^H X_t$, where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution.
4. X_1 has the standard normal distribution in $\mathbf{R}^n$.

Note that Brownian motion is a fractional Brownian motion with Hurst parameter $H = 1/2$.
Next, we describe a seemingly unrelated process, the solution of the heat equation with additive Gaussian noise. Then we show that fractional Brownian motion can be recovered from this solution. There are several representations of fractional Brownian motion, see Nualart [Nua06], Chapter 5. One advantage of our representation is that we can use the Markov property and time reversal, tools which fail for the fractional Brownian motion alone. Using these extra tools, we give a simple proof of some hitting properties of fractional Brownian motion, and a result of Talagrand about double points. Throughout the paper we will write SPDE for “stochastic partial differential equation”.
Informally speaking, we wish to consider solutions $u(t,x) : (t,x) \in \mathbf{R} \times \mathbf{R}^n$ to the following equation.

$$\begin{aligned} \partial_t u &= \Delta u + \dot{F}(t,x) \\ u(-\infty, x) &= 0 \end{aligned}$$

where $\dot{F}(t,x)$ is a generalized Gaussian field with the following covariance:

$$E\left[\dot{F}(t,x)\dot{F}(s,y)\right] = \delta(t-s)h(x-y)$$

where

$$h(x) = \begin{cases} \delta(x) & \text{if } H = \frac{1}{2} \\ |x|^{-\alpha} & \text{otherwise} \end{cases}$$

and

$$H = \frac{1-\alpha n}{2}$$

so that

$$\alpha = \frac{1-2H}{n}.$$

We will require

$$0 < H \leq \frac{1}{2}$$

so that

$$0 \leq \alpha < \frac{1}{n}.$$

Note that if $H > \frac{1}{2}$ then $\alpha < 0$ and $h(0) = 0$, and then h is not a proper covariance. Our goal is to show that $X_t \stackrel{\mathcal{D}}{=} u(t,0)$, but this will not be literally true.

The above description is not rigorous. To be precise, $\dot{F}$ is a centered Gaussian random linear functional on $\mathbf{C}_c^\infty(\mathbf{R}^{n+1})$, the set of infinitely differentiable functions with compact support on $(t,x)\in\mathbf{R}\times\mathbf{R}^n$, with covariance

$$Q(f,g) := E\left[F(f)F(g)\right] = \int_{\mathbf{R}}\int_{\mathbf{R}^n} f(t,x)g(t,x)dxdt \tag{1.1}$$

if $H=\frac{1-n}{2}$, and

$$Q(f,g) := E\left[F(f)F(g)\right] = \int_{\mathbf{R}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(t,x)g(t,y)h(x-y)dydxdt \tag{1.2}$$

if $H\neq\frac{1-n}{2}$. Note that in either case, the integral in (1.1) or (1.2) is nonnegative definite. Thus, we can extend $F(f)$ to all functions f satisfying

$$Q(f,f)<\infty.$$

We call this class of functions $\mathbf{H}$. Note that $\mathbf{H}$ implicitly depends on α, n.
Next, for $t>0$ and $x\in\mathbf{R}^n$ let

$$G(t,x) := \begin{cases} (4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right) & \text{if } t>0 \\ 0 & \text{if } t\leq 0\end{cases}$$

be the heat kernel on $\mathbf{R}^n$.
We would be tempted to define $u(t,x)$ by

$$u(t,x) = \int_{-\infty}^{t}\int_{\mathbf{R}^n} G(t-s,x-y)F(dyds)$$

but the integral will not converge. However, $u(t,x)-u(0,0)$ looks more promising. When $H=1/4$ the stationary pinned string was defined in [MT02] as

$$U(t,x) := \int_{-\infty}^{t}\int_{\mathbf{R}^n}\left[G(t-s,x-y)-G(-s,-y)\right]F(dyds). \tag{1.3}$$

This definition also works for other values of H, provided $g\in\mathbf{H}$, where $g(s,y)=g_{t,x}(s,y):=G(t-s,x-y)-G(-s,-y)$.

Lemma 1. *Let g be as in the previous paragraph. For all $t>0$ we have*

$$g(s,y)\mathbf{1}_{(s\leq t)}\in\mathbf{H}.$$

We will prove Lemma 1 in the Appendix.
From the covariance of $\dot{F}$ one can easily deduce the following scaling property. We leave the proof to the reader.

Lemma 2. *The noise $\dot{F}$ obeys the following scaling relation,*

$$\dot{F}(ct,c^{1/2}x)\stackrel{\mathcal{D}}{=}c^{-(1+\alpha n)/2}\dot{F}(t,x).$$

Turning to the SPDE, define $v(t,x)$ by $av(t,x) = U(ct, c^{1/2}x)$. The reader can verify the following calculation using (1.3).

$$\begin{aligned} a\partial_t v &= c\partial_t U \\ &= c\left(\Delta U + \dot{F}(ct, c^{1/2}x)\right) \\ &\stackrel{\mathscr{D}}{=} a\Delta v + c^{1-(1+\alpha n)/2}\dot{F}(t,x) \\ &\stackrel{\mathscr{D}}{=} a\Delta v + c^{(1/2)-(\alpha n/2)}\dot{F}(t,x) \end{aligned}$$

where the equality in distribution holds for the entire random field indexed by t, x. Thus, we can cancel out the constants c, a provided

$$a = c^{\frac{1-\alpha n}{2}}$$

and then v satisfies the same equation as u. Thus,

$$aU(t,x) \stackrel{\mathscr{D}}{=} U(ct, c^{1/2}x).$$

Setting $x = 0$ gives us the scaling relation for $U(t,0)$. Thus we find

Lemma 3. *$U(t,0)$ obeys the following scaling relation. For $c > 0$ we have*

$$U(ct,0) \stackrel{\mathscr{D}}{=} c^{\frac{1-\alpha n}{2}} U(t,0)$$

where the equality in distribution holds for the entire process indexed by t.

Remark 1. *Let $V_{t,x}(s,y) = U(t+s, x+y) - U(t,x)$. It follows immediately from (1.3) that the random fields $V_{t,x}(s,y)$ and $U(s,y)$ are equal in distribution.*

Let

$$X_t = K_\alpha U(t,0)$$

where

$$K_\alpha = \left[\frac{(2-\alpha)\Gamma(\frac{n}{2})}{2^{-\frac{3\alpha}{2}+1}\Gamma(\frac{n-\alpha}{2})}\right]^{1/2} \quad \text{if } \alpha \neq 1$$

and

$$K_\alpha = 2^{-1/2}(4\pi)^{d/4} \text{ if } \alpha = 1.$$

We claim that

Proposition 1. *Assume that $0 \le \alpha < \frac{1}{n}$, and let $X_t = K_\alpha U(t,0)$. Then X_t, as defined above, is a fractional Brownian motion with Hurst parameter*

$$H = \frac{1-\alpha n}{2}.$$

Proof. We only need to verify the four axioms for fractional Brownian motion. It follows from (1.3) that $X_0 = 0$, so axiom 1 is satisfied. Axiom 2 follows from Remark 1. Axiom 3 follows from the scaling properties of fractional Brownian motion and Lemma 3. Finally, Axiom 4 follows from (1.3) and the integral of the covariance h, which we verify in the Appendix. $\square$

Remark: Proposition 1 is related to a recent preprint of Lei and Nualart [LN08].

2 Critical dimension for hitting points, and for double points

The rest of the paper is devoted to the following questions.

1. For which values of d,H does X_t hit points?

2. For which values of d,H does X_t have double points?

Recall that we say X_t hits points if for each $z \in \mathbf{R}^n$, there is a positive probability that $X_t = z$ for some $t > 0$. We say that X_t has double points if there is a positive probability that $X_s = X_t$ for some positive times $t \neq s$. Here are our main results.

Theorem 1. *Assume $0 < H \leq \frac{1}{2}$, and that $\frac{1}{H}$ is an integer. For the critical dimension $n = \frac{1}{H}$, fractional Brownian motion does not hit points.*

Theorem 2. *Assume $0 < H \leq \frac{1}{2}$, and that $\frac{2}{H}$ is an integer. For the critical dimension $n = \frac{2}{H}$, fractional Brownian motion does not have double points.*

In fact, Talagrand answered the question of double points in [Tal98], Theorem 1.1, and the assertion about hitting points was already known. Techniques from Gaussian processes, such as Theorem 22.1 of [GH80], can usually answer such questions except in the critical case, which is much more delicate. The critical case is the set of parameters n,H which lie on the boundary of the parameter set where the property occurs. For example, $n = 2, H = \frac{1}{2}$ falls in the the critical case for fractional Brownian motion to hit points. But for $H = \frac{1}{2}$ we just have standard Brownian motion, which does not hit points in $\mathbf{R}^2$. This illustrates the usual situation, that hitting does not occur, or double points do not occur, in the critical case.

It is not hard to guess the critical parameter set for fractional Brownian motion hitting points or having double points. Heuristically, the range of a process with scaling $X_{\alpha t} \stackrel{\mathscr{D}}{=} \alpha^H X_t$ should have Hausdorff dimension $\frac{1}{H}$, if X_t takes values in a space of dimension at least $\frac{1}{H}$. For example, Brownian motion satisfies $B_{\alpha t} \stackrel{\mathscr{D}}{=} \alpha^{1/2} B_t$, and Brownian motion has range of Hausdorff dimension 2, at least if the Brownian motion takes values in $\mathbf{R}^n$ with $n \geq 2$. The critical parameter of H for a process to hit points should be when the dimension of the range equals the dimension of the space. Thus, the critical case for fractional Brownian motion taking values in $\mathbf{R}^n$ should be when the Hurst parameter is $H = 1/n$. For double points, we consider the 2-parameter process $V(s,t) = X_t - X_s$. This process hits zero at double points of X_t, except when $t = s$. The Hausdorff dimension of the range of V should be $n = \frac{2}{H}$, and so the critical Hurst parameter for double points of X_t should be $H = \frac{1}{2n}$.

First note that the supercritical case can be reduced to the critical case. That is, if $X_t = (X_t^{(1)}), \dots, X_t^{(n+m)}) = 0$, then it is also true that the projection $(X_t^{(1)}), \dots, X_t^{(n)}) = 0$. Furthermore, the subcritical case is easier than the critical case, and it can be analyzed using Theorem 22.1 of Geman and Horowitz [GH80]. Therefore, we concentrate on the critical case $H = 1/n$.

Below we give a simple argument inspired by [MT02] which settles the critical case. The argument goes back to Lévy, and an excellent exposition is given in Khoshnevisan [Kho03]. It is based on scaling properties of the process, the Markov property, and time reversal. Although fractional Brownian motion is not a Markov process, $U(t,x)$ does have the Markov property with respect to time. Furthermore, it is time-reversible.

3 Summary of Lévy's argument

Here is a brief summary of Lévy's argument that 2-dimensional Brownian motion does not hit points. Let $m(dx)$ denote Lebesgue measure on $\mathbf{R}^n$ and let B_t denote Brownian motion on $\mathbf{R}^n$. For this section, let $n=2$. Furthermore, let $B[a,b] := \{B_t : a \le t \le b\}$. It suffices to show that

$$E\Big[m(B[0,2])\Big] = 0 \tag{3.1}$$

since then we would have

$$\begin{aligned} 0 &= E\left[\int_{\mathbf{R}^2} \mathbf{1}(z \in B[0,2])dz\right] \\ &= \int_{\mathbf{R}^2} P(z \in B[0,2])dz \end{aligned}$$

and so $P(z \in B[0,2]) = 0$ for almost every z.
Next, for $0 \le t \le 1$, let

$$\begin{aligned} Y_t &= B_{1+t} - B_1 \\ Z_t &= B_{1-t} - B_1. \end{aligned}$$

Recall that $Y_t, Z_t : 0 \le t \le 1$ are independent standard 2-dimensional Brownian motions. This is a standard property of Brownian motion, which can be verfied by examining the covariances of $Y_t, Z_t : 0 \le t \le 1$. Then $Y[0,1], Z[0,1]$ are independent random sets. Furthermore, by Brownian scaling and translation

$$\begin{aligned} E\Big[m(B[0,2])\Big] &= 2E\Big[m(B[0,1])\Big] = 2E\Big[m(B[1,2])\Big] \\ &= E\Big[m(B[0,1])\Big] + E\Big[m(B[1,2])\Big] \\ &= E\Big[m(Y[0,1])\Big] + E\Big[m(Z[0,1])\Big]. \end{aligned}$$

On the other hand, set theory gives us

$$\begin{aligned} &E\Big[m(B[0,2])\Big] = E\left[m(Y[0,1] \cup Z[0,1])\right] \\ &= E\Big[m(Y[0,1])\Big] + E\Big[m(Z[0,1])\Big] - E\Big[m(Y[0,1] \cap Z[0,1])\Big] \end{aligned}$$

and therefore

$$E\Big[m(Y[0,1] \cap Z[0,1])\Big] = 0.$$

By Fubini's theorem,

$$\begin{aligned} 0 &= E\Big[m(Y[0,1] \cap Z[0,1])\Big] \\ &= E\left[\int_{\mathbf{R}^2} \mathbf{1}_{Y[0,1]}(z)\mathbf{1}_{Z[0,1]}(z)dz\right] \\ &= \int_{\mathbf{R}^2} E\left[\mathbf{1}_{Y[0,1]}(z)\mathbf{1}_{Z[0,1]}(z)\right] dz. \end{aligned} \tag{3.2}$$

By the independence of $Y[0,1]$ and $Z[0,1]$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 0 &= \int_{\mathbf{R}^2} E\left[\mathbf{1}_{Y[0,1]}(z)\mathbf{1}_{Z[0,1]}(z)\right] dz \qquad (3.3)\\ &= \int_{\mathbf{R}^2} E\left[\mathbf{1}_{Y[0,1]}(z)\right] E\left[\mathbf{1}_{Z[0,1]}(z)\right] dz \\ &= \int_{\mathbf{R}^2} \left(E\left[\mathbf{1}_{Y[0,1]}(z)\right]\right)^2 dz \\ &\geq \left(\int_{\mathbf{R}^2} E\left[\mathbf{1}_{Y[0,1]}(z)\right] dz\right)^2 \\ &= \Big(Em(Y[0,1])\Big)^2. \end{aligned}$$

Therefore $E[m(Y[0,1])] = 0$ and (3.1) follows from the definition of Y.

4 Proof of Theorems 1 and 2

Now we use Lévy's argument to prove our main theorems.

4.1 Hitting points, Theorem 1

The argument exactly follows that in Section 3, except that $\mathbf{R}^2$ is replaced by $\mathbf{R}^n$. Also, by axiom (3),

$$X_{ct} \stackrel{\mathscr{D}}{=} c^H X_t$$

for $c > 0$. However, since X_t takes values in $\mathbf{R}^n$ and $H = 1/n$, we still have

$$E\Big[m(X[0,2])\Big] = 2E\Big[m(X[0,1])\Big] = 2E\Big[m(X[1,2])\Big].$$

Recall that $m(\cdot)$ denotes Lebesgue measure in $\mathbf{R}^n$. As before, let

$$\begin{aligned} Y_t &= X_{1+t} - X_1 \\ Z_t &= X_{1-t} - X_1. \end{aligned}$$

It is no longer true that $Y[0,1], Z[0,1]$ are independent. Now we use the fact that X_t is equal in distribution to $u(t,0)$, where $u(t,x)$ is the stationary pinned string. Changing the probability space if necessary, let us write $X_t = K_\alpha U(t,0)$, and let $\mathscr{H}_t$ denote the σ-field generated by $U(t,x)$: $x \in \mathbf{R}^n$. Then we have the following lemma.

Lemma 4. *Let us use the above notation. Then $Y[0,1], Z[0,1]$ are conditionally independent and identically distributed given $\mathscr{H}_1$.*

Proof of Lemma 4. The lemma is proved in [MT02], Corollary 1, for $H = 1/4$, and the proof for other values of H uses similar ideas.
To show that $Y[0,1], Z[0,1]$ are identically distributed, we merely use the definition of Y, Z and make a change of variable. Below, equality in distribution means that the processes indexed by t

are equal in distribution.

$$\begin{aligned} Y_t &= X_{1+t} - X_1 \\ &= K_\alpha U(1+t,0) - K_\alpha U(1,0) \\ &= K_\alpha \int_{-\infty}^{1+t} \int_{\mathbf{R}^n} \left[G(1+t-s,-y) - G(1-s,-y)\right] F(dyds) \\ &\stackrel{\mathscr{D}}{=} K_\alpha \int_{-\infty}^{t} \int_{\mathbf{R}^n} \left[G(t-s,-y) - G(-s,-y)\right] F(dyds). \end{aligned}$$

But by the definition of Z_t,

$$\begin{aligned} Z_t &= X_{1-t} - X_1 \\ &= K_\alpha \Big(U(1-t,0) - U(1,0)\Big) \\ &= -K_\alpha \int_{-\infty}^{1} \int_{\mathbf{R}^n} \left[G(1-s,-y) - G(1-t-s,-y)\right] F(dyds) \\ &\stackrel{\mathscr{D}}{=} K_\alpha \int_{-\infty}^{t} \int_{\mathbf{R}^n} \left[G(t-s,-y) - G(-s,-y)\right] F(dyds) \end{aligned}$$

and therefore Y_t, Z_t are identically distributed processes.
Next we discuss the conditional independence of $Y[0,1], Z[0,1]$ given $\mathscr{H}_1$. First we claim that the stationary pinned string $U(t,x)$ enjoys the Markov property with respect to t. This is a general fact about stochastic evolution equations, and we refer the reader to [Wal86], Chapter 3. It follows that $Z[0,1]$ is conditionally independent of $Y[0,1]$ given $\mathscr{H}_1$.
This proves Lemma 4. □

From here we duplicate the argument in Section 3, replacing expectation by conditional expectation given $\mathscr{H}_1$. Briefly, it suffices to show that

$$E\Big[m(Y[0,1])\Big|\mathscr{H}_1\Big] = 0. \tag{4.1}$$

But, following the same argument as before, we conclude that with probability one,

$$E\Big[m(Y[0,1]\cap Z[0,1])\Big|\mathscr{H}_1\Big] = 0.$$

We leave it to the reader to verify that (3.2) and (3.3) still hold, provided expectation is replaced by conditional expectation given $\mathscr{H}_1$. This verifies (4.1), and finishes the proof of Theorem 1.

4.2 Double points, Theorem 2

To show that X_t does not have double points in the critical case, we use the same argument, but applied to the two-parameter process

$$V(s,t) := X(t) - X(s).$$

We need to show that $V(s,t)$ has no zeros except if $s = t$. To simplify the argument, we will show that $V(s,t)$ has no zeros for $(s,t) \in \mathscr{R}$, where

$$\mathscr{R} := [0,2] \times [4,6].$$

The same argument would apply to any other rectangle whose intersection with the diagonal has measure 0. Let us subdivide $\mathscr{R}$ into 4 subrectangles $\mathscr{R}_i : i = 1, \ldots, 4$ each of which is a translation of $[0,1]^2$. Again we argue as in Section 3. By scaling, we see that for each $i = 1, \ldots, 4$

$$E\left[m(V(\mathscr{R}))\right] = 4E\left[m(V(\mathscr{R}_i))\right].$$

Next, let $\mathscr{H}_1$ be the σ-field generated by $\{u(1,x) : x \in \mathbf{R}^n\}$, and suppose we have labeled the $\mathscr{R}_i$ such that $\mathscr{R}_1 = [0,1] \times [4,5]$ and $\mathscr{R}_2 = [1,2] \times [4,5]$. Thus, as before, for each pair $i \neq j \in \{1, \ldots, 4\}$ we have

$$E\left[m(V(\mathscr{R}_i) \cap V(\mathscr{R}_j))\middle|\mathscr{H}_1\right] = 0.$$

Now in [MT02], Corollary 1, it was shown that for $H = 1/4$, $V(\mathscr{R}_1), V(\mathscr{R}_2)$ are conditionally i.i.d. given $\mathscr{H}_1$. For other values of H, the argument is very similar to the proof of Lemma 4, and we leave the details to the reader.

Therefore, as in Section 3, we conclude that with probability one,

$$E\left[m(V(\mathscr{R}_i))\middle|\mathscr{H}_1\right] = 0.$$

Also as in Section 3, this finishes the proof of Theorem 2.

A Appendix

Proof of Lemma 1. In the case of $H = \frac{1}{2}$, where $h(x) = \delta(x)$ is not difficult to show that

$$\int_0^t \int_{\mathbf{R}^n} G(r, x-y)^2 dy dr = 2(4\pi)^{-d/2} t^{1/2}$$

and we leave it to the reader.

Moving on to the case of $H \neq \frac{1-n}{2}$, we calculate the following integral.

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} G(r, x-z) G(r, x-z') |z-z'|^{-\alpha} dz dz'$$

We make use the Fourier transform, with the following definition.

$$\hat{f}(\xi) = \int_{R^n} f(x) e^{ix\cdot\xi} dx$$

for $x, \xi \in \mathbf{R}^n$.

If $f(x-z) = G(r, x-z)$, $g(x-z') = G(r, x-z')$, $h(z-z') = |z-z'|^{-\alpha}$, $H(z) = g*h(x-z) = J(x-z)$, then

$$\begin{aligned} \hat{H}(z) &= \int_{\mathbf{R}^n} J(x-z) e^{iz\cdot\xi} = e^{iz\cdot\xi} \hat{J}(-\xi) \\ &= e^{ix\cdot\xi} \hat{g}(-\xi) \hat{h}(-\xi). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} G(r,x-z)G(r,x-z')|z-z'|^{-\alpha}dzdz' \\
&= \int_{\mathbf{R}^n} f(x-z)(\int_{\mathbf{R}^n} g(x-z')h(z-z')dz')dz \\
&= \int_{\mathbf{R}^n} f(x-z)H(z)dz = (f*H)(x) \\
&= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \widehat{f*H}(\xi)e^{-ix\cdot\xi}d\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \hat{f}(\xi)\hat{g}(-\xi)\hat{h}(-\xi)d(\xi).
\end{aligned} \tag{A.1}
$$

It is known that

$$\hat{f}(\xi)=\hat{g}(-\xi)=e^{-r|\xi|^2} \tag{A.2}$$

and by [Str03], page 51, we get

$$\hat{h}(\xi)=\hat{h}(-\xi)=c(\alpha)|\xi|^{-(n-\alpha)} \quad \text{where} \quad c(\alpha)=\frac{\pi^{\frac{n}{2}}2^{n-\alpha}\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}. \tag{A.3}$$

by (A.2) and (A.3), (A.1) becomes

$$
\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} G(r,x-z)G(r,x-z')|z-z'|^{-\alpha}dzdz' \\
&= \frac{\pi^{\frac{n}{2}}2^{n-\alpha}\Gamma(\frac{n-\alpha}{2})}{(2\pi)^n\Gamma(\frac{\alpha}{2})}\int_{\mathbf{R}^n} e^{-2r|\xi|^2}|\xi|^{-(n-\alpha)}d(\xi).
\end{aligned}
$$

To evaluate the integral in $\mathbf{R}^n$, we use polar coordinates. Then $|\xi|=u$, $d(\xi)=u^{n-1}du$, where $0<u<\infty$, and thus if $\sigma=2(\sqrt{\pi})^n/\Gamma(\frac{n}{2})$ is the area of the unit sphere in $\mathbf{R}^n$, we get

$$
\begin{aligned}
&\int_{\mathbf{R}^n} e^{-2r|\xi|^2}|\xi|^{-(n-\alpha)}d(\xi) \\
&= \sigma\int_0^\infty e^{-2ru^2}u^\alpha\frac{du}{u} \\
&= \sigma 2^{-1-\frac{\alpha}{2}}r^{-\frac{\alpha}{2}}\int_0^\infty e^{-y}y^{\frac{\alpha}{2}-1}dy \\
&= \frac{2^{-\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}r^{-\frac{\alpha}{2}}.
\end{aligned} \tag{A.4}
$$

Integrating (A.4) over r from 0 to t, we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} G(r,x-z)G(r,x-z')|z-z'|^{-\alpha}dzdz'dr \\
&\qquad = \frac{2^{-\frac{3\alpha}{2}+1}\Gamma\left(\frac{n-\alpha}{2}\right)}{(2-\alpha)\Gamma\left(\frac{n}{2}\right)}t^{\frac{2-\alpha}{2}}.
\end{aligned}
$$

□

References

[GH80] Donald Geman and Joseph Horowitz. Occupation densities. *Ann. Probab.*, 8(1):1–67, 1980. MR0556414

[Kho03] Davar Khoshnevisan. Intersections of Brownian motions. *Expo. Math.*, 21(2):97–114, 2003. MR1978059

[LN08] Pedro Lei and David Nualart. A decomposition of bifractional Brownian motion and some applications. arXive:0803.2227, 2008.

[MT02] C. Mueller and R. Tribe. Hitting properties of a random string. *Electronic J. Prob.*, 7:1–29, 2002. Paper no. 10. MR1902843

[Nua06] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006. MR2200233

[Str03] Robert S. Strichartz. *A guide to distribution theory and Fourier transforms*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003. MR2000535

[Tal98] Michel Talagrand. Multiple points of trajectories of multiparameter fractional Brownian motion. *Probab. Theory Related Fields*, 112(4):545–563, 1998. MR1664704

[Wal86] J.B. Walsh. An introduction to stochastic partial differential equations. In P. L. Hennequin, editor, *École d'été de probabilités de Saint-Flour, XIV-1984*, Lecture Notes in Mathematics 1180, pages 265–439, Berlin, Heidelberg, New York, 1986. Springer-Verlag. MR0876085