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WEAK CONVERGENCE TO OCONE MARTINGALES: A REMARK

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Abstract
We show, by a simple counterexample, that the main result in a recent paper by H. Van Zanten [Electronic Communications in Probability **7** (2002), 215-222] is false. We eventually point out the origin of the error.

Throughout the following we use concepts and notation from standard semimartingale theory. The reader is referred e.g. to [3] for any unexplained notion. Every càdlàg stochastic process is defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and it is interpreted as a random element with values in $D([0,\infty))$, the Skorohod space of càdlàg functions on $[0,\infty)$. The symbol "$\Rightarrow$" indicates weak convergence (see [2]). Given a filtration $\mathcal{F}_t$ and a real-valued càdlàg $\mathcal{F}_t$-local martingale started from zero, say $M = \{M_t : t \geq 0\}$, we will denote by $[M] = \{[M]_t : t \geq 0\}$ the optional quadratic variation of M. We recall that, when M is continuous, $[M] = \langle M \rangle$, where $\langle M \rangle$ is the conditional quadratic variation of M as defined in [3, Chapter III]. Moreover, by the Dambis-Dubins-Schwarz (DDS) Theorem (see [4, Ch. V]), every continuous $\mathcal{F}_t$-local martingale M, such that $M_0 = 0$ and $\langle M \rangle_\infty = \lim_{t \to +\infty} \langle M \rangle_t = +\infty$ a.s.-$\mathbb{P}$, can be represented as

$$M_t = W^{(M)}_{\langle M \rangle_t}, \quad t \geq 0, \tag{1}$$

where $W^{(M)}_t$ is a standard Brownian motion with respect to the filtration

$$\mathcal{G}_t = \mathcal{F}_{\sigma(t)}, \quad t \geq 0, \quad \text{where} \;\; \sigma(t) = \inf\{s : \langle M \rangle_s > t\}.$$

According e.g. to [7], we say that a continuous $\mathcal{F}_t$-martingale M_t, such that $M_0 = 0$ and $\langle M \rangle_\infty = +\infty$, is a (continuous) *Ocone martingale* if the Brownian motion $W^{(M)}$ appearing in its DDS representation (1) is independent of $\langle M \rangle$.

The following statement, concerning rescaled càdlàg martingales, appears as Theorem 4.1 in [6].

Claim 1 *Let M be a martingale with bounded jumps, and let a_n, b_n be sequences of positive numbers both increasing to infinity. For each n, define*

$$M^n_t = \frac{M_{b_n t}}{\sqrt{a_n}}. \tag{2}$$

Then, the following statements hold
(i) If $M^n \Rightarrow N$ in $D([0,\infty))$, then necessarily N is a continuous Ocone martingale.
(ii) Let N be a continuous Ocone martingale. Then, $M^n \Rightarrow N$ in $D([0,\infty))$ if, and only if, $[M^n] \Rightarrow [N]$ in $D([0,\infty))$.

Both parts (i) and (ii) of Claim 1 are false, as shown by the following counterexample. Take a standard Brownian motion started from zero $W = \{W_t : t \geq 0\}$, and define

$$\begin{aligned} M_t &= W_t^2 - t \\ M_t^n &= \frac{1}{n} M_{nt} = \left(n^{-\frac{1}{2}} W_{nt}\right)^2 - t. \end{aligned}$$

Then, M is a continuous square-integrable martingale that is *not* Ocone (since it is non-Gaussian and *pure*, see [5, Proposition 2.5] and [7, p. 423]). Moreover, $M_t^n = (a_n)^{-1/2} M_{b_n t}$, for $a_n = n^2$ and $b_n = n$, and $M^n \stackrel{\text{law}}{=} M$ for each n, due to the scaling properties of Brownian motion. It follows that $M^n \Rightarrow M$, thus contradicting point (i) of Claim 1.
As for point (ii), consider the continuous Ocone martingale (see [7, p. 427])

$$N_t = 2 \int_0^t W_s d\widetilde{W}_s$$

where $\widetilde{W}$ is a standard Brownian motion independent of W. It is evident that

$$\begin{aligned} [N]_t &= 4 \int_0^t W_s^2 ds \\ [M^n]_t &= \frac{4}{n^2} \int_0^{nt} W_s^2 ds = 4 \int_0^t \left(n^{-1/2} W_{nu}\right)^2 du \end{aligned}$$

and therefore that $[M^n] \stackrel{\text{law}}{=} [N]$ for each n, although M^n converges weakly to the martingale M, which is not Ocone. This contradicts point (ii) of Claim 1.

The error comes from a misuse of the Skorohod almost sure representation theorem (see e.g. [1, p. 281]) in [6, Section 4]. Starting from p. 219, line 10 of [6], the author considers a sequence

$$\left\{ \left(W, \tau^{n'}\right) : n' \geq 1 \right\},$$

where W is a standard Brownian motion and $\tau^{n'}$ is an appropriate time-change, such that

$$\left(W, \tau^{n'}\right) \Rightarrow (B, [N]),$$

where B is a standard Brownian motion, and $[N]$ is a positive, continuous and increasing process. Then, the Skorohod theorem allows one to conclude that, on an auxiliary space, there exist random elements $\left(\overline{W}^{n'}, \overline{\tau}^{n'}\right)$ and $\left(\overline{B}, \overline{[N]}\right)$ such that

$$\left(W, \tau^{n'}\right) \stackrel{\text{law}}{=} \left(\overline{W}^{n'}, \overline{\tau}^{n'}\right) \quad \text{and} \quad (B, [N]) \stackrel{\text{law}}{=} \left(\overline{B}, \overline{[N]}\right),$$

where the Brownian motion $\overline{W}^{n'}$ *depends (in general) on* n', and $\left(\overline{W}^{n'}, \overline{\tau}^{n'}\right) \stackrel{\text{a.s.}}{\to} (B, [N])$. On the other hand, the (fallacious) conclusion of Theorem 4.1 in [6] is obtained by supposing that, on the auxiliary space, there exists a Brownian motion $\overline{W}$ such that $\overline{W}^{n'} = \overline{W}$ for each n', which is clearly not the case, due to the counterexamples constructed above.

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