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DISTRIBUTION OF THE BROWNIAN MOTION ON ITS WAY TO HITTING ZERO

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Abstract

For the one-dimensional Brownian motion $B = (B_t)_{t\ge 0}$, started at $x > 0$, and the first hitting time $\tau = \inf\{t \ge 0 : B_t = 0\}$, we find the probability density of $B_{u\tau}$ for a $u \in (0,1)$, i.e. of the Brownian motion on its way to hitting zero.

1 Introduction

The following problem has been recently addressed in [5], [6]. The authors considered a continuous time subcritical branching process $Z = (Z_t)_{t\ge 0}$, starting from the initial population of size $Z_0 = x$. As is well known, Z_t gets extinct at the random time $T = \inf\{t \ge 0 : Z_t = 0\}$, and $T < \infty$ with probability one. What can be said about $Z_{T/2}$, i.e. the population size on the half-way to its extinction? While the complete characterization of the law of Z_{uT} with $u = 1/2$, or more generally $u \in (0,1)$, does not seem to be trackable, it turns out that under quite general conditions

$$x^{u-1} Z_{uT} \xrightarrow[x\to\infty]{d} Cc^{-u}e^{-u\eta}, \tag{1.1}$$

where the convergence is in distribution, C and c are constants, explicitly computable in terms of parameters of Z and η is a random variable with Gumbel distribution.

In this note we study the analogous problem for one-dimensional Brownian motion $B = (B_t)_{t\ge 0}$, started from $x > 0$. Hereafter we assume that B is defined on the canonical probability space $(\Omega, \mathscr{F}, \mathsf{P}_x)$ and let τ denote the first time it hits zero, i.e. $\tau = \inf\{t \ge 0 : B_t = 0\}$.

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Theorem 1.1. *For $x > 0$ and $u \in (0,1)$, the distribution $\mathsf{P}_x\big(B_{u\tau} \le y\big)$ is absolutely continuous with the density*

$$p(u,x;y) = \frac{4\sqrt{u(1-u)}xy^2}{\pi\{(y-x)^2(1-u)+y^2u\}\{(y+x)^2(1-u)+y^2u\}}. \tag{1.2}$$

Remark 1.2. *Notice that $p(u,x;y)$ decays as $\propto 1/y^2$ and hence its mean is infinite. Such behavior, of course, stems from the possibility of large excursions of B from the origin, before hitting zero.*

Remark 1.3. *The formula* (1.2) *implies that $x^{-1}B_{u\tau}$ has the same law under P_x as $B_{u\tau}$ under P_1, or using different notations,*

$$x^{-1}B^x_{u\tau(x)} \stackrel{d}{=} B^1_{u\tau(1)}, \tag{1.3}$$

where B^x stands for the Brownian motion, starting at $x > 0$, and $\tau(x) = \inf\{t \ge 0 : B^x_t = 0\}$ (i.e. $B^1_{u\tau(1)}$ has the density $p(u,1,y)$). This scale invariance does not seem to be obvious at the outset and should be compared to (1.1)*, where the scaling depends on u and holds only in the limit.*

In the following section we shall give an elementary proof of Theorem 1.1. In Section 3 our result is discussed in the context of Doob's h-transform conditioning.

2 Proof

Let $\delta > 0$ and define[2] $\tau_\delta := \delta\lfloor \tau/\delta \rfloor$. Recall that τ has the probability density (see e.g. [2]):

$$f(x;t) = \frac{\partial}{\partial t}\mathsf{P}_x(T \le t) = \frac{x}{\sqrt{2\pi t^3}}e^{-x^2/2t}, \quad t \ge 0, \quad x > 0. \tag{2.1}$$

Let $\hat{M}_{s,t} := \inf_{s \le r < t} B_r$ and $\phi(\cdot)$ be a continuous bounded function, then[3]

$$\begin{aligned}
\mathsf{E}_x\phi\big(B_{u\tau_\delta}\big) &= \sum_{k=0}^\infty \mathsf{E}_x\phi\big(B_{u\tau_\delta}\big)I\Big(\tau \in \big[\delta k, \delta(k+1)\big)\Big) \\
&= \sum_{k=0}^\infty \mathsf{E}_x\phi\big(B_{u\delta k}\big)I\Big(\tau \in \big[\delta k, \delta(k+1)\big)\Big) \\
&= \sum_{k=0}^\infty \mathsf{E}_x\phi\big(B_{u\delta k}\big)I\Big(\hat{M}_{0,\delta k} > 0, \hat{M}_{\delta k,\delta(k+1)} \le 0\Big) \\
&= \sum_{k=0}^\infty \mathsf{E}_x\phi\big(B_{u\delta k}\big)I\Big(\hat{M}_{0,u\delta k} > 0\Big)\mathsf{P}_x\Big(\hat{M}_{u\delta k,\delta k} > 0, \hat{M}_{\delta k,\delta(k+1)} \le 0\Big|\mathscr{F}^B_{u\delta k}\Big) \\
&= \sum_{k=0}^\infty \mathsf{E}_x\phi\big(B_{u\delta k}\big)I\Big(\hat{M}_{0,u\delta k} > 0\Big)\mathsf{P}_x\Big(\hat{M}_{u\delta k,\delta k} > 0, \hat{M}_{\delta k,\delta(k+1)} \le 0\Big|B_{u\delta k}\Big) \\
&= \phi(x)\mathsf{P}_x\Big(\tau \in [0,\delta)\Big) + \sum_{k=1}^\infty \mathsf{E}_x\phi\big(B_{u\delta k}\big)I\Big(\hat{M}_{0,u\delta k} > 0\Big) \times \\
&\qquad\qquad \mathsf{P}_{B_{u\delta k}}\Big(\tau \in \big[(1-u)\delta k, (1-u)\delta k + \delta\big)\Big)
\end{aligned}$$

[2] $\lfloor x \rfloor$ stands for the integer part of $x \in \mathbb{R}$ and $\lceil x \rceil := \lfloor x \rfloor + 1$

[3] $I(\cdot)$ denotes the indicator function

$$
\begin{aligned}
&= \phi(x)\int_0^{\delta} f(x;t)\,dt + \sum_{k=1}^{\infty} \mathsf{E}_x \phi(B_{u\delta k}) I\left(\hat{M}_{0,u\delta k} > 0\right) \int_{(1-u)\delta k}^{(1-u)\delta k+\delta} f(B_{u\delta k};t)\,dt \\
&= \phi(x)\int_0^{\delta} f(x;t)\,dt + \int_0^{\infty} \phi(y)\left\{\sum_{k=1}^{\infty}\int_{\delta k}^{\delta(k+1)} q(x,u\delta k,y) f(y;t-u\delta k)\,dt\right\}dy \\
&= \phi(x)\int_0^{\delta} f(x;t)\,dt + \int_0^{\infty} \phi(y)\left\{\int_{\delta}^{\infty} q(x,u\lfloor t/\delta\rfloor\delta,y) f(y;t-u\lfloor t/\delta\rfloor\delta)\,dt\right\}dy \\
&= \phi(x)\int_0^{\delta} f(x;t)\,dt + \int_0^{\infty} \phi(y)\left\{\int_{0}^{\infty} q(x,u\lceil t/\delta\rceil\delta,y) f(y;t+\delta-u\lceil t/\delta\rceil\delta)\,dt\right\}dy, \qquad (2.2)
\end{aligned}
$$

where $q(x,t,y)$ is the probability density of $\mathsf{P}_x\left(\hat{M}_{0,t} > 0, B_t \in dy\right)$ with respect to the Lebesgue measure (see e.g. formula 1.2.8 page 126, [2]):

$$
q(x,t,y) = \left\{\frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t} - \frac{1}{\sqrt{2\pi t}} e^{-(y+x)^2/2t}\right\}, \quad x,y > 0. \qquad (2.3)
$$

By continuity of the densities (2.1) and (2.3), for any fixed $x > 0$ and $u \in (0,1)$, the function

$$
F_\delta(t,y) := q(x,u\lceil t/\delta\rceil\delta,y) f\left(y;t+\delta-u\lceil t/\delta\rceil\delta\right)
$$

converges to

$$
\lim_{\delta\to 0} F_\delta(t,y) = q(x,ut,y) f(y;t-ut), \quad \forall t \ge 0, \ \ y \ge 0.
$$

In Lemma 2.1 below we exhibit a function $G(t,y)$, independent of δ, such that

$$
\mathscr{F}_\delta(t,y) \le G(t,y), \quad \forall (t,y) \in \mathbb{R}_+^2 \quad \text{and} \quad \int_{\mathbb{R}_+^2} G(t,y)\,dt\,dy < \infty, \qquad (2.4)
$$

and hence, the dominated convergence and (2.2) imply

$$
\lim_{\delta\to 0} \mathsf{E}_x \phi\left(B_{u\tau_\delta}\right) = \lim_{\delta\to 0}\int_{\mathbb{R}_+^2} \phi(y) F_\delta(t,y)\,dy\,dt = \int_{\mathbb{R}_+^2} \phi(y) q(x,ut,y) f(y;t-ut)\,dt\,dy. \qquad (2.5)
$$

On the other hand, $\lim_{\delta\to 0}\tau_\delta = \tau$, P_x-a.s. and thus by continuity of B_t, $\lim_{\delta\to 0} B_{u\tau_\delta} = B_{u\tau}$, P_x-a.s. for any $u \in (0,1)$. Thus, by arbitrariness of ϕ, (2.5) implies that the distribution of $B_{u\tau}$ has the density:

$$
p(u,x;y) := \int_0^{\infty} q(x,ut,y) f(y;t-ut)\,dt.
$$

A calculation now yields:

$$
\begin{aligned}
p(u,x;y) &= \int_0^{\infty} \frac{y}{\sqrt{2\pi}\big(t(1-u)\big)^{3/2}} e^{-y^2/2t(1-u)} \left\{\frac{1}{\sqrt{2\pi u t}} e^{-(y-x)^2/2ut} - \frac{1}{\sqrt{2\pi u t}} e^{-(y+x)^2/2ut}\right\} dt \\
&= \frac{y}{2\pi(1-u)^{3/2}u^{1/2}} \int_0^{\infty} \frac{1}{t^2}\left\{e^{-(y-x)^2/2ut - y^2/2t(1-u)} - e^{-(y+x)^2/2ut - y^2/2t(1-u)}\right\} dt,
\end{aligned}
$$

and by a change of variables

$$\begin{aligned}
p(u,x;y) &= \frac{y}{2\pi(1-u)^{3/2}u^{1/2}}\left\{\Big(\frac{(y-x)^2}{2u}+\frac{y^2}{2(1-u)}\Big)^{-1}-\Big(\frac{(y+x)^2}{2u}+\frac{y^2}{2(1-u)}\Big)^{-1}\right\}\\
&= \frac{2y\sqrt{u}}{2\pi\sqrt{1-u}}\left\{\frac{1}{(y-x)^2(1-u)+y^2u}-\frac{1}{(y+x)^2(1-u)+y^2u}\right\}\\
&= \frac{\sqrt{u(1-u)}4xy^2}{\pi\Big\{(y-x)^2(1-u)+y^2u\Big\}\Big\{(y+x)^2(1-u)+y^2u\Big\}}.
\end{aligned}$$

The statement of the Theorem 1.1 now follows from:

Lemma 2.1. (2.4) *holds with* $G(t,x)$ *defined in* (2.6) *below.*

Proof. Set $t^\delta := \lceil t/\delta\rceil\delta$, so that $t\le t^\delta\le t+\delta$, and

$$\begin{aligned}
F_\delta(t,y) &\le \frac{1}{\sqrt{ut^\delta}}\left\{e^{-(y-x)^2/2ut^\delta}-e^{-(y+x)^2/2ut^\delta}\right\}\frac{y}{(t+\delta-ut^\delta)^{3/2}}e^{-\frac12 y^2/(t+\delta-ut^\delta)}\\
&\le I(t\le\delta)\frac{1}{\sqrt{u\delta}}e^{-(y-x)^2/(2u\delta)}\frac{y}{\big((1-u)\delta\big)^{3/2}}e^{-\frac12 y^2/(\delta+(1-u)\delta)}+\\
&\quad I(t>\delta)\frac{1}{\sqrt{ut}}\left\{e^{-(y-x)^2/2ut^\delta}-e^{-(y+x)^2/2ut^\delta}\right\}\frac{y}{\big((t+\delta)(1-u)\big)^{3/2}}e^{-\frac12 y^2/(t(1-u)+\delta)}\\
&=: I(t\le\delta)A+I(t>\delta)B.
\end{aligned}$$

Since the function z^2e^{-Cz} with $C>0$ attains its maximum $4e^{-2}/C^2$ on the interval $[0,\infty)$ at $z:=2/C$,

$$A=\frac{y}{\sqrt{u(1-u)^3}}\frac{1}{\delta^2}\exp\left\{-\left(\frac{(y-x)^2}{2u}+\frac12\frac{y^2}{(2-u)}\right)\frac{1}{\delta}\right\}\le \frac{y}{\sqrt{u(1-u)^3}}\left(\frac{(y-x)^2}{2u}+\frac12\frac{y^2}{(2-u)}\right)^{-2}.$$

Similarly, for $t>\delta$,

$$\begin{aligned}
B &= \frac{y}{\sqrt{u(1-u)^3t(t+\delta)^3}}e^{-(y-x)^2/2ut^\delta}\left\{1-e^{-2xy/ut^\delta}\right\}e^{-\frac12 y^2/(t(1-u)+\delta)}\\
&\le \frac{y}{\sqrt{u(1-u)^3}}\frac{1}{t^2}e^{-(y-x)^2/2u(t+\delta)}\left\{1-e^{-2xy/ut}\right\}e^{-\frac12 y^2/(t(1-u)+\delta)}\\
&\le \frac{y}{\sqrt{u(1-u)^3}}\frac{1}{t^2}e^{-(y-x)^2/4ut}\left\{1-e^{-2xy/ut}\right\}e^{-\frac12 y^2/(t(2-u))}.
\end{aligned}$$

Hence for $\delta\in(0,1]$ we have the bound

$$\begin{aligned}
F_\delta(t,y) &\le \frac{y}{\sqrt{u(1-u)^3}}\left(\frac{(y-x)^2}{2u}+\frac12\frac{y^2}{(2-u)}\right)^{-2}I(t\le 1)+\\
&\quad \frac{y}{\sqrt{u(1-u)^3}}\frac{1}{t^2}e^{-(y-x)^2/4ut}\left\{1-e^{-2xy/ut}\right\}e^{-\frac12 y^2/(t(2-u))} =: G(t,y). \qquad (2.6)
\end{aligned}$$

Since for $u \in (0,1)$ and $x > 0$, the quadratic function is lower bounded:

$$\frac{(y-x)^2}{u} + \frac{y^2}{(2-u)} \geq \frac{x^2}{2},$$

the first function in the right hand side of (2.6) is integrable on $\mathbb{R}_+^2$. Further,

$$\begin{aligned}
&\int_0^\infty \frac{y}{\sqrt{u(1-u)^3}} \frac{1}{t^2} e^{-(y-x)^2/4ut} \left\{1 - e^{-2xy/ut}\right\} e^{-\frac{1}{2}y^2/(t(2-u))} dt \\
&= \frac{y}{\sqrt{u(1-u)^3}} \left\{ \left(\frac{(y-x)^2}{4u} + \frac{y^2}{2(2-u)} \right)^{-1} - \left(\frac{(y-x)^2}{4u} + \frac{y^2}{2(2-u)} + \frac{2xy}{u} \right)^{-1} \right\} \\
&= \frac{4yu(2-u)}{\sqrt{u(1-u)^3}} \left\{ \frac{1}{(y-x)^2(2-u) + 2y^2u} - \frac{1}{(y-x)^2(2-u) + 2y^2u + 8xy(2-u)} \right\} \\
&= \frac{32u(2-u)^2xy^2/\sqrt{u(1-u)^3}}{\left\{(y-x)^2(2-u) + 2y^2u\right\}\left\{(y-x)^2(2-u) + 2y^2u + 8xy(2-u)\right\}}.
\end{aligned}$$

The latter function decays as $\propto 1/y^2$ as $y \to \infty$ and is bounded away from zero, uniformly in $y \geq 0$, and thus is integrable on $\mathbb{R}_+$. Since the last term in the right hand side of (2.6) is nonnegative, by Fubini theorem it is an integrable function on $\mathbb{R}_+^2$ for all $u \in (0,1)$ and $x > 0$. $\square$

3 A connection to Doob's h-transform

In this section we show that the random variable $B_{u\tau}$ has the same density as the so called *scaled Brownian excursion* at the corresponding time, averaged over its length. The latter process is defined by conditioning in the sense of Doob's h-transform, and it would be natural to identify this formal conditioning with the usual conditional probability. While in the analogous discrete time setting, such identification is evident, its precise justification in our case remains an open problem. For a fixed time $T > 0$, let $R = (R_t)_{t \leq T}$ be the 3-dimensional Bessel bridge $R = (R_t)_{t \leq T}$, starting at $R_0 = x$ and ending at zero. Namely, R is the radial part[4]

$$R_t = \|V_t\|, \quad t \in [0,T], \tag{3.1}$$

of the 3-dimensional Brownian bridge $V = (V_t)_{t \leq T}$ with $V_0 = v$ and $V_T = 0$:

$$V_t = v + W_t - \frac{t}{T}(W_T + v), \quad t \in [0,T],$$

where $v \in \mathbb{R}^3$ with $\|v\| = x$ and W is a standard Brownian motion in $\mathbb{R}^3$.
The law of R coincides with the law of the scaled Brownian excursion process, which is defined as "the Brownian motion, started at $x > 0$ and conditioned to hit zero for the first time at time T". Here the conditioning is understood in the sense of Doob's h-transform (see Ch. IV, §39, [7], and [1], [3] for the in depth treatment).
On the other hand, one can speak on the regular conditional measure induced on the space of Brownian excursions (started from $x > 0$), given $\tau = \inf\{t \geq 0 : B_t = 0\}$. More precisely, let E

[4] $\|\cdot\|$ denotes the Euclidian norm in $\mathbb{R}^n$

be a subset of continuous functions $C_{[0,\infty)}(\mathbb{R})$, such that for all $\omega \in E$, $\omega(0) = x$ and for each ω there is a positive number $\ell(\omega)$, called the *excursion length*, such that $\omega(t) > 0$ for $0 < t < \ell(\omega)$ and $\omega(t) \equiv 0$ for all $t \geq \ell(\omega)$. E together with the smallest σ-algebra $\mathscr{E}$, making all coordinate mappings measurable, is called the *excursion* space (see §3, [2] for the brief reference and [1] for more details). Let $\mu_x(T, \cdot)$, $T \in [0, \infty)$ be a probability kernel on the excursion space $(E, \mathscr{E})$, i.e. a family of measures such that $T \mapsto \mu_x(T, A)$ is a measurable function for all $A \in \mathscr{E}$ and $\mu_x(T, \cdot)$ is a probability measure on $\mathscr{E}$ for each $T \geq 0$. By definition, $\mu_x(T, \cdot)$ is a regular conditional probability of $B_{t\wedge\tau}$ given τ, if for any bounded and measurable functional F on $(E, \mathscr{E})$:

$$\mathsf{E}_x F(B_{\cdot\wedge\tau}) I(\tau \in A) = \int_A \int_E F(\omega)\mu_x(s, d\omega) f(x; s) ds, \quad \forall A \in \mathscr{B}(\mathbb{R}),$$

where $f(x; t)$ is the density of τ, defined in (2.1). In particular, for any bounded measurable function ϕ and some $u \in (0, 1)$,

$$\mathsf{E}_x \phi(B_{u\tau}) = \int_0^\infty \int_E \phi\left(\omega(us)\right)\mu_x(s, d\omega) f(x; s) ds. \tag{3.2}$$

We were not able to trace any general result, from which the identification of $\mu_x(T, d\omega)$ with the probability $\nu_x(T, d\omega)$, induced on $(E, \mathscr{E})$ by the aforementioned Bessel bridge R, could be deduced. While the latter, of course, is intuitively appealing, its precise justification remains elusive (some relevant results can be found in [4]). The calculations below show that

$$\int_0^\infty \int_E \phi\left(\omega(us)\right)\nu_x(s, d\omega) f(x; s) ds = \mathsf{E}_x \phi(B_{u\tau}), \tag{3.3}$$

indicating in favor of such identification.

For a fixed $T > 0$ and $u \in (0, 1)$, the distribution of R_{uT}, i.e. the restriction of $\nu_x(T, d\omega)$ to the time $t := uT$, has a density $q_{uT}(x; y)$ with respect to the Lebesgue measure dy, which can be computed as follows. We have

$$EV_t = \nu(1 - t/T), \quad \operatorname{cov}(V_t) = I\frac{t(T-t)}{T},$$

where I is 3-by-3 identity matrix. Notice that the law of the Bessel bridge R in (3.1) doesn't depend on the particular ν as long as $\|\nu\| = x$ and it will be particularly convenient to carry out the calculations for the specific choice $\nu = (x, 0, 0)$. Fix a constant $u \in (0, 1)$ and let ξ_1, ξ_2, ξ_3 be i.i.d. standard Gaussian random variables. Then, for $t := uT$,

$$R_{uT} \stackrel{d}{=} \sqrt{\left(\sqrt{Tu(1-u)}\xi_1 + x(1-u)\right)^2 + Tu(1-u)\xi_2^2 + Tu(1-u)\xi_3^2}.$$

The random variable $\theta := \xi_2^2 + \xi_3^2$ has χ_2^2 distribution, which is the same as the exponential distribution with parameter $1/2$ and hence

$$R_{uT} \stackrel{d}{=} b\sqrt{\left(\xi + a\right)^2 + \theta}, \tag{3.4}$$

where ξ is written for ξ_1 and $a := x\sqrt{(1-u)/Tu}$ and $b := \sqrt{(1-u)Tu}$ are defined for brevity.

The density of $(\xi+a)^2$ is given by:

$$f_1(z) := \frac{d}{dz}P\Big((\xi+a)^2 \le z\Big) = \frac{d}{dz}\int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-(x-a)^2/2} dx = \frac{1}{2\sqrt{2\pi}\sqrt{z}}\Big(e^{-(\sqrt{z}-a)^2/2} + e^{-(\sqrt{z}+a)^2/2}\Big).$$

The density of $(\xi+a)^2+\theta$ is the convolution of f_1 and the exponential density with parameter $1/2$:

$$\begin{aligned}
f_3(y) &:= \int_0^y f_1(z) f_2(y-z)dz = \int_0^y \frac{1}{2\sqrt{2\pi}\sqrt{z}}\Big(e^{-(\sqrt{z}-a)^2/2} + e^{-(\sqrt{z}+a)^2/2}\Big)\frac{1}{2}e^{-1/2(y-z)}dz \\
&= \int_0^{\sqrt{y}} \frac{1}{2\sqrt{2\pi}}\Big(e^{-(\tilde z-a)^2/2} + e^{-(\tilde z+a)^2/2}\Big)e^{-1/2(y-\tilde z^2)}d\tilde z \\
&= \frac{e^{-a^2/2-y/2}}{2\sqrt{2\pi}}\int_0^{\sqrt{y}} \Big(e^{\tilde z a} + e^{-\tilde z a}\Big)d\tilde z = \frac{e^{-a^2/2-y/2}}{\sqrt{2\pi}}\int_0^{\sqrt{y}} \cosh(\tilde z a)d\tilde z \\
&= \frac{e^{-a^2/2-y/2}}{\sqrt{2\pi}a}\sinh(\sqrt{y}a).
\end{aligned}$$

Consequently, the density of $\sqrt{(\xi+a)^2+\theta}$ is given by

$$f_4(z) := 2z f_3(z^2) = \frac{\sqrt{2}e^{-a^2/2}}{\sqrt{\pi}a} z e^{-z^2/2}\sinh(za),$$

and, finally, the density of $b\sqrt{(\xi+a)^2+\theta}$ is

$$f_5(z) := \frac{1}{b}f_4(z/b) = \frac{\sqrt{2}e^{-a^2/2}}{\sqrt{\pi}ab^2} z e^{-z^2/2b^2}\sinh(za/b) = \frac{z}{\sqrt{2\pi}ab^2}\Big\{e^{-(a-z/b)^2/2} - e^{-(a+z/b)^2/2}\Big\}.$$

Hence by (3.4), R_{uT} has the density

$$q_{uT}(x;y) = \frac{y}{\sqrt{2\pi}x(1-u)}\frac{1}{\sqrt{Tu(1-u)}}\Bigg\{\exp\Bigg(-\frac{1}{T}\frac{\big(x(1-u)-y\big)^2}{2u(1-u)}\Bigg) - \exp\Bigg(-\frac{1}{T}\frac{\big(x(1-u)+y\big)^2}{2u(1-u)}\Bigg)\Bigg\},$$

We shall abbreviate by writing

$$C_1 = \frac{\big(x(1-u)-y\big)^2}{2u(1-u)},\ C_2 = \frac{\big(x(1-u)+y\big)^2}{2u(1-u)},\ C_3 = \frac{y}{\sqrt{2\pi}xu^{1/2}(1-u)^{3/2}}.$$

Then

$$\begin{aligned}
\int_0^\infty \int_E \phi\big(\omega(ut)\big)\nu_x(t,d\omega)f(x;t)dt &= \int_0^\infty \phi(y)\int_0^\infty q_{ut}(x;y)f(x;t)dt \\
&= \int_0^\infty \phi(y)\int_0^\infty C_3\frac{1}{t^{1/2}}\Big\{e^{-C_1/t} - e^{-C_2/t}\Big\}\frac{x}{\sqrt{2\pi}t^{3/2}}e^{-x^2/2t}dtdy =: \int_0^\infty \phi(y)p(u,x;y)dy
\end{aligned}$$

and by a change of variables,

$$p(u,x;y) = C_3 \frac{x}{\sqrt{2\pi}} \int_0^\infty \left\{ e^{-(C_1+x^2/2)t} - e^{-(C_2+x^2/2)t} \right\} t^{-2} dt = C_3 \frac{x}{\sqrt{2\pi}} \left\{ \frac{1}{C_1+x^2/2} - \frac{1}{C_2+x^2/2} \right\}.$$

A calculation yields,

$$C_1 + x^2/2 = \frac{(1-u)(x-y)^2 + y^2u}{2u(1-u)} \quad \text{and} \quad C_2 + x^2/2 = \frac{(1-u)(x+y)^2 + y^2u}{2u(1-u)},$$

and, consequently,

$$\begin{aligned} p(u,x;y) &= \frac{y}{\sqrt{2\pi} x u^{1/2}(1-u)^{3/2}} \frac{x}{\sqrt{2\pi}} \left\{ \frac{2u(1-u)}{(1-u)(x-y)^2 + y^2u} - \frac{2u(1-u)}{(1-u)(x+y)^2 + y^2u} \right\} \\ &= \frac{4xy^2\sqrt{u(1-u)}}{\pi\{(1-u)(x-y)^2 + y^2u\}\{(1-u)(x+y)^2 + y^2u\}}, \end{aligned}$$

which in view of (3.2) and (1.2), imply (3.3).

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