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FKG INEQUALITY FOR BROWNIAN MOTION AND STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

The purpose of this work is to study some possible application of FKG inequality to the Brownian motion and to Stochastic Differential Equations. We introduce a special ordering on the Wiener space and prove the FKG inequality with respect to this ordering. Then we apply this result on the solutions X_t of a stochastic differential equation with a positive coefficient σ, we prove that these solutions X_t are increasing with respect to the ordering, and finally we deduce a correlation inequality between the solution of different stochastic equations.

Introduction

The FKG inequality is a correlation inequality for monotone functions. It is named after Fortuine, Ginibre and Kasteleyn, who gave a rigorous formulation and established sufficient condition for its validity [3]. In the following years a lot of works were inspired by this inequality, new inequalities generalizing FKG were discovered and a great deal of applications were found, like in the field of statistical mechanics, which it was born for, or in different fields: for example the FKG inequality for the optimal transportation problems [2], or the applications of FKG inequality to cellular automata [7]. In [3] Fortuine, Kasteleyn and Ginibre found some sufficient condition to build spaces where the inequality (1) holds . This immediately permitted the application of this inequality to the rigorous analysis of percolation and ferromagnetic models. Many generalizations followed: Holley [5] introduced an inequality on convex dominations of measures, Preston [8] passed from discrete to continuous spin models, Kamae, Krengel and O'Brien [6] made a work on partially ordered Polish spaces and Ahlswede and Daykin [1] found a brilliant generalization of Holley's work [5] introducing a combinative inequality. This work aims at proving the FKG inequality for the Wiener space with a special ordering on the increments. Then we apply this FKG inequality to prove (theorem 7) a correlation inequality between the solutions of two stochastic equations.

1 Preliminaries

Let us start recalling the definition of FKG space. Let from now on $(\Omega, \mathcal{F}, \mathbb{P}, \geq)$ be a partially ordered probability space, where $\geq$ is an order relation on the set Ω. A function $f : \Omega \to \mathbb{R}$ is said to be increasing or $\geq$*increasing* if for all $\omega_2 \geq \omega_1$, we have $f(\omega_2) \geq f(\omega_1)$. A set $A \in \mathcal{F}$ is said to be increasing if for all $\omega_2 \geq \omega_1$, $\omega_1 \in A$ implies $\omega_2 \in A$. That is the same of saying an event is increasing if its indicator function I_A is so.

Definition 1.1 *$(\Omega, \mathcal{F}, \mathbb{P}, \geq)$ is said to satisfy the FKG inequality if for all increasing functions f g in $L^2(\Omega)$, the inequality*

$$\mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g] \tag{1}$$

holds.

Proposition 1 *Let $(\Omega, \mathcal{F}, \mathbb{P}, \geq)$ be a partially ordered probability space, then it satisfies the FKG inequality if and only if for all $A, B \in \mathcal{F}$ increasing sets, the inequality*

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B) \tag{2}$$

is verified.

Definition 1.2 *Let $(E, \mathcal{E}, \geq)$ be a partially ordered measurable space, let $X : \Omega \to E$ be a r.v. and let $\mu = X(\mathbb{P})$ be its law. The r.v. X is said to satisfy FKG if the space $(E, \mathcal{E}, \mu, \geq)$ does.*

1.1 FKG and the increasing functions

A first way to prove that the space satisfy FKG is to show that it is the increasing image of an FKG space. Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1, \geq_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2, \geq_2)$ be two partially ordered spaces.

Proposition 2 *Let $f : \Omega_1 \to \Omega_2$ be measurable and increasing, and let $\mathbb{P}_2 = f(\mathbb{P}_1)$. If FKG holds for $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1, \geq_1)$ then FKG holds for*
$(\Omega_2, \mathcal{F}_2, \mathbb{P}_2, \geq_2)$.

Proof. Let A_2, B_2 be two increasing events in $\mathcal{F}_2$ and let us define $A_1 := f^{-1}(A_2)$, $B_1 := f^{-1}(B_2)$. A_1 and B_1 are increasing events in $\mathcal{F}_1$. Let us check that A_1 is increasing, if $\omega_1, \omega_2 \in \Omega_1$, $\omega_2 \geq \omega_1$, $\omega_1 \in A_1$, then $f(\omega_1) \in A_2$, $f(\omega_2) \geq f(\omega_1)$; since A_2 is increasing, $f(\omega_2) \in A_2$, and therefore $\omega_2 \in A_1$. A_1 and B_1 increasing implies that they satisfy the inequality $\mathbb{P}_1(A_1 \cap B_1) \geq \mathbb{P}_1(A_1)\mathbb{P}_1(B_1)$ and since $\mathbb{P}_2 = f(\mathbb{P}_1)$ then it also holds: $\mathbb{P}_2(A_2 \cap B_2) \geq \mathbb{P}_2(A_2)\mathbb{P}_2(B_2)$. □

1.2 FKG inequality on product spaces

Another way of building FKG spaces from FKG spaces is that of making their product. Let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i, \geq_i)_{i \in \Gamma}$ be a family of partially ordered probability spaces, and let us define the space $(\Omega, \mathcal{F}, \mathbb{P}, \geq)$ in this way:

$$\Omega := \prod_{i \in \Gamma} (\Omega_i) \qquad \mathcal{F} := \bigotimes_{i \in \Gamma} (\mathcal{F}_i) \qquad \mathbb{P} := \bigotimes_{i \in \Gamma} (\mathbb{P}_i) \qquad \geq := \bigotimes_{i \in \Gamma} (\geq_i).$$

With the last definition we mean that for all $u, v \in \Omega$

$$u \geq v \qquad \Longleftrightarrow \qquad (\forall i \in \Gamma \quad u_i \geq_i v_i).$$

From now on we will denote this product in this way:

$$(\Omega, \mathcal{F}, \mathbb{P}, \geq) = \prod_{i \in \Gamma} (\Omega_i, \mathcal{F}_i, \mathbb{P}_i, \geq_i)$$

Theorem 3 *Let* $(\Omega, \mathcal{F}, \mathbb{P}, \geq) = \prod_{i \in \Gamma} (\Omega_i, \mathcal{F}_i, \mathbb{P}_i, \geq_i)$. *Then* $(\Omega, \mathcal{F}, \mathbb{P}, \geq)$ *satisfies FKG if and only if every space* $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i, \geq_i)$ *does so.*

PROOF. See appendix. □

This theorem allows to tie the FKG spaces of classic literature together. In this way we easily originate examples of FKG spaces as it is showed in the next section.

Examples

Remark 1.1 *Every totally ordered probability space satisfies FKG.*

PROOF. Let $(\Omega, \mathcal{A}, \mathbb{P}, \geq)$ be totally ordered and let $A, B \in \mathcal{A}$ be two increasing events, then either $A \subseteq B$ or $B \subseteq A$; therefore $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$. □

Example 1 *Every probability measures* μ *on* $\mathbb{R}$ *satisfies the FKG inequality. Therefore if* f *and* g *are increasing and integrable the following inequality holds:*

$$\int_{\mathbb{R}} fg \; d\mu \geq \int_{\mathbb{R}} f \; d\mu \int_{\mathbb{R}} g \; d\mu$$

Example 2 *Every real random variable satisfies FKG and, by theorem 3, so does their product.*

2 FKG inequality and Brownian motion

In this section the FKG inequality is proved for the standard Brownian motion $(W_t)_{t \in [0,T]}$ on the canonical Wiener space $(\Omega, \mathcal{A}, \mathbb{P})$ endowed with a special partial order. Here $\Omega = C_0([0, T]; \mathbb{R})$ is the space of continuous function vanishing at zero, $\mathcal{A} = \mathcal{B}(\Omega)$ is the Borel $\sigma - algebra$ induced by the uniform convergence topology and $\mathbb{P}$ is the Wiener measure. Moreover, W_t denotes the canonical process, $W_t(\omega) = \omega(t)$. The first thing we must do now is to introduce an ordering on $(\Omega, \mathcal{A}, \mathbb{P})$. The right choice of such an ordering is very important and it can be useful to recall the following remark.

Remark 2.1 *Let* $\geq_1$, $\geq_2$ *be two ordering relation on* Ω *and let* $\geq_2$ *be finer than* $\geq_1$, $\big($*that is* $(\omega_1 \leq_1 \omega_2)$ *implies* $(\omega_1 \leq_2 \omega_2)$ $\forall \omega_1, \omega_2 \in \Omega\big)$. *Then, if FKG works for* $(\Omega, \mathcal{A}, \mathbb{P}, \geq_1)$ *it also works for* $(\Omega, \mathcal{A}, \mathbb{P}, \geq_2)$. *Moreover, if* f *is increasing for* $\geq_2$ ($_{\geq_2}$*increasing*)*, then it is increasing also for* $\geq_1$. *Summarizing, on the space* $(\Omega, \mathcal{A}, \mathbb{P}, \geq_1)$ *there are more increasing functions and it is more difficult to prove the FKG inequality.*

This means that a too fine ordering could lead to a too weak statement, and a too weak ordering could lead to a space that doesn't satisfy FKG anymore. In this work we choose the natural order for additive processes based on path increments.

Definition 2.1 *Given* $\omega_1, \omega_2 \in \Omega$, *we say that* $\omega_1 \leq \omega_2$ *if and only if for all* $0 \leq t_1 \leq t_2 \leq T$ *we have* $W_{t_2}(\omega_1) - W_{t_1}(\omega_1) \leq W_{t_2}(\omega_2) - W_{t_1}(\omega_2)$.

This ordering is less fine than the ordering induced by the direct comparison of the trajectories (namely $\omega_1 \le \omega_2$ if $W_{t_1}(\omega_1) \le W_{t_2}(\omega_2)$ for every $t \in [0,T]$), and for remark 2.1 we obtain a stronger statement. Moreover the choice of this ordering is necessary for the purposes of section 3.

Theorem 4 *For the Wiener space $(\Omega, \mathcal{A}, \mathbb{P}, \ge)$ with the ordering $\ge$ of definition 2.1, the FKG inequality holds.*

Proof. The idea of the proof is to gradually proceed towards the σ-algebra $\mathcal{A}$. We shall verify (2) at first on a suitable sub-algebra $\mathcal{B}$ and then we will proceed with density arguments on all $\mathcal{A}$.
Let n be an integer $n \ge 2$. Let $H = \{t_1, t_2, \dots t_n\}$ be a partition with $t_0 = 0 < t_1 < t_2 < \dots < t_n = T$. Let $X_i := W_{t_i} - W_{t_{i-1}}$ be the i_th increment, and $X_H := (X_i)_{i \in \{1,\dots,n\}}$. Let finally $\mathcal{A}_H \subset \mathcal{A}$ the sub-σ-algebra generated by X_H and let $\ge_H$ be the ordering induced by X_H(that is $\omega_1 \le_H \omega_2$ if and only if $X_i(\omega_1) \le X_i(\omega_2)$ for all $i \in \{1,\dots,n\}$). Then, as it was showed in example 2, X_H satisfies FKG and also for $(\Omega, \mathcal{A}_H, \mathbb{P}, \ge_H)$ the FKG inequality holds. We also have $\omega_1 \le \omega_2$ implies $\omega_1 \le_H \omega_2$ and then, $\forall A \in \mathcal{A}$, if A is $_{\ge_H}$*increasing* then A is $_{\ge}$*increasing*. Generally the converse is not true. But as it is showed in proposition 11 the following lemma holds:

Lemma 5 *The following conditions are equivalent:*

(i) $A \in \mathcal{A}_H$ *and* A *is* $_{\ge}$*increasing.*

(ii) $A \in \mathcal{A}$ *and* A *is* $_{\ge_H}$*increasing.*

Let $\mathcal{B} = \bigcup_H \mathcal{A}_H$.
Let $A, B \in \mathcal{B}$ be increasing for the ordering $\ge$. Then $A, B \in \mathcal{B}$ implies the existence of a finite set $H \subset [0,T]$ such that $A, B \in \mathcal{A}_H$.
For lemma 5 A, B are $_{\ge_H}$*increasing* and then the inequality $\mathbb{P}(A \bigcap B) \ge \mathbb{P}(A)\mathbb{P}(B)$ holds. In this way we proved the inequality for the elements of $\mathcal{B}$.
In order to complete the proof we have to show that it is possible to approximate increasing events of $\mathcal{A}$ with increasing events of $\mathcal{B}$. This is a less trivial fact and it is done by the following lemma:

Lemma 6 $\forall \varepsilon > 0$ *and* $\forall A \in \mathcal{A}$ *increasing* $\exists B \in \mathcal{B}$ *increasing such that* $\mathbb{P}(A \triangle B) \le \varepsilon$.

Proof of Lemma 6 :
$\mathcal{B}$ is an algebra and a basis of $\mathcal{A}$, then $\mathcal{B}$ is dense in $\mathcal{A}$ that means: $\forall \varepsilon > 0 \quad \forall A \in \mathcal{A} \quad \exists B \in \mathcal{B}$ such that $\mathbb{P}(A \triangle B) \le \varepsilon$. Now we want to show that if A is increasing then it is possible to approximate A with increasing events of $\mathcal{B}$. Let us fix now $\varepsilon > 0$ and $A \in \mathcal{A}$ increasing: then, from what we said before, we can choose a partition $0 = t_0 < t_1 < \dots < t_n = T$ with $H = \{t_0, t_1, \dots, t_n\}$ such that $\exists C \in \mathcal{A}_H$ with $\mathbb{P}(A \triangle C) \le \varepsilon$.
Let $E \subset \Omega$ be the set of applications from $[0,T]$ to $\mathbb{R}$ that are continuous, vanishing at zero and linear on every interval $[t_{i-1}, t_i]$.
Let $F \subset \Omega$ be the set of applications from $[0,T]$ to $\mathbb{R}$ continuous and vanishing at t_i for all $i \in \{0,1,\dots,n\}$.
The set E and F are two linear subspaces of Ω , $\Omega = E \oplus F$ and every element of Ω can be written in an unique way as the sum of an element of E and one of F. We can define two maps $L : \Omega \to E$ and $Y : \Omega \to F$ such that $\forall \omega \in \Omega$ we have $W(\omega) = L(\omega) + Y(\omega)$.
Let now $\mathcal{E}$ and $\mathcal{F}$ be the traces $\mathcal{A}$ on E and F: we can easily verify that the maps L and Y are

random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in $(E, \mathcal{E})$ and $(F, \mathcal{F})$, with $L^{-1}(\mathcal{E}) = \mathcal{A}_H$ and L ,Y independent random variables [9]. Let $\mathbb{P}_1 = L(\mathbb{P})$ and $\mathbb{P}_2 = Y(\mathbb{P})$. Then the application

$$(L, Y) : (\Omega, \mathcal{A}, \mathbb{P}) \to (E \times F, \mathcal{E} \otimes \mathcal{F}, \mathbb{P}_1 \otimes \mathbb{P}_2)$$

is bijective, bi-measurable and it preserves the measure.
Let now $\varphi : \Omega \to [0, 1]$ be the application $\varphi(\omega) = \mathbb{P}_2\Big(Y\big(A \cap L^{-1}(L(\omega))\big)\Big)$.
If $\omega = e + f$ with $e \in E$ and $f \in F$ then

$$\varphi(\omega) = \int_F I_A(e, g) d\mathbb{P}_2(g) \ .$$

This expression shows that φ is a version of the conditional expectation of I_A respect to $\mathcal{A}_H$. Let $B := \{\varphi \geq \frac{1}{2}\}$. Then $B \in \mathcal{A}_H$ and B is the best approximation of A in $\mathcal{A}_H$. That is, $\forall \tilde{B} \in \mathcal{A}_H$ we have $\mathbb{P}(A \vartriangle B) \leq \mathbb{P}(A \vartriangle \tilde{B})$. Indeed,

$$\mathbb{P}(A \vartriangle B) = \mathbb{E}\big[\ |I_A - I_B|\ \big] = \mathbb{E}\big[\ |\varphi - I_B|\ \big]$$

$$\mathbb{P}(A \vartriangle \tilde{B}) = \mathbb{E}\big[\ |I_A - I_{\tilde{B}}|\ \big] = \mathbb{E}\big[\ |\varphi - I_{\tilde{B}}|\ \big]$$

and by the definition of φ we have $|\varphi - I_B| \leq |\varphi - I_{\tilde{B}}|$ and finally

$$\mathbb{P}(A \vartriangle B) \leq \mathbb{P}(A \vartriangle \tilde{B})$$

Now we only have to show that if A is increasing then φ is increasing. Let $\omega_1 \leq \omega_2$ be two trajectory with $\omega_1 = e_1 + f_1$, $\omega_2 = e_2 + f_2$, $e_i \in E$, $f_i \in F$.

$$\varphi(\omega_1) = \int_F I_A(e_1, g) d\mathbb{P}_2(g) \qquad \varphi(\omega_2) = \int_F I_A(e_2, g) d\mathbb{P}_2(g)$$

$\omega_1 \leq \omega_2$ implies $e_1 \leq e_2$ and $e_1 + g \leq e_2 + g$ for all $g \in F$. This means $I_A(e_1, g) \leq I_A(e_2, g)$ for all $g \in F$ and finally

$$\varphi(\omega_1) \leq \varphi(\omega_2)$$

φ increasing implies B increasing and this finishes the proof of the lemma and of the theorem.
□

3 FKG inequality for stochastic differential equations.

In this section we prove that the solution X_t of a stochastic equation with quite general coefficients is a random increasing variable on the space $(\Omega, \mathcal{A}, \mathbb{P}, \geq)$. In this way we may use the FKG inequality for Brownian motion to show a correlation inequality for the random variables X_t and Y_s, solutions of two different equation, evaluated at different times.
Let $(\Omega, \mathcal{A}, \mathbb{P}, \geq)$ be the *Wiener* space, with the ordering of definition 2.1. Let X_t, Y_t be the solution of the stochastic equations

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0 \end{cases} \tag{3}$$

$$\begin{cases} dY_t = \tilde{b}(Y_t)dt + \tilde{\sigma}(Y_t)dW_t \\ Y_0 = x_0 \end{cases}$$

Theorem 7 *Let $b, \tilde{b}$ be Lipschitz continuous.Let $\sigma, \tilde{\sigma}$ be differentiable with Lipschitz derivative. Assume there exist two positive constants ϵ, M such that $\epsilon \le \sigma \le M$, $\epsilon \le \tilde{\sigma} \le M$. Then, $\forall$ $t, s \ge 0$, the following inequality holds:*

$$\mathbb{E}[X_t \cdot Y_s] \ge \mathbb{E}[X_t]\mathbb{E}[Y_s] \tag{4}$$

This result is a direct conseguence of the FKG inequality for the space $(\Omega, \mathcal{A}, \mathbb{P}, \ge)$ and the following lemma, which we believe is of conceptual interest in itself and may find other applications.

Lemma 8 *Let b be Lipschitz continuous, let σ be differentiable with Lipschitz derivative. Assume there exist two positive constants ϵ, M such that $\epsilon \le \sigma \le M$. Then there exists a stochastic process $(X_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, with $X_t(\omega)$ defined for all $\omega \in \Omega$, such that;*

(i) X_t is a solution of (3)

(ii) for every $t \in [0,T]$, X_t is an increasing function on $(\Omega, \mathcal{A}, \mathbb{P}, \ge)$.

Remark 3.1 *Under the imposed assumptions, equation(3) have a unique solution. However, solutions are usually defined up to null sets. Here we need a solution defined on the whole space Ω.*

3.1 Proof of lemma 8

The idea of the proof is to obtain a simpler differential equation with an increasing change of variable from X_t to Z_t. Let

$$F(x) = \int_0^x \frac{1}{\sigma(t)}\, dt,$$

and $G = F^{-1}$. By the above assumption on σ, the function F and G are $C^2(\mathbb{R}, \mathbb{R})$,increasing, bijective and Lipschitz continuous, $G'(z) = \sigma(G(z))$, $G''(z) = \sigma(G(z)) \cdot \sigma'(G(z))$. Let us define:

$$\widehat{b}(Z) := \frac{b(G(Z))}{\sigma(G(Z))} - \frac{1}{2}\sigma'(G(Z))$$

By the assumptions on b and σ we obtain $\widehat{b}$ locally Lipschitz continuous. Let $z_0 = F(x_0)$ and let Z be the solution of the following equation:

$$Z(t) = z_0 + \int_0^t \widehat{b}(Z(s))ds + W_t \tag{5}$$

If L is a common Lipschitz constant for b, σ' and G. Then

$$|\widehat{b}(z)| \le \frac{|b(0)|}{\epsilon} + \frac{1}{2}|\sigma'(0)| + \frac{L^2}{\epsilon}z + \frac{1}{2}L^2 z$$

Let $k_1 = \frac{|b(0)|}{\epsilon} + \frac{1}{2}|\sigma'(0)|$ and $k_2 = \frac{L^2}{\epsilon} + \frac{1}{2}L^2$, then

$$|\widehat{b}(z)| \le k_1 + k_2 z \qquad \forall z \in \mathbb{R}$$

By proposition 10 the integral equation (5) have one and only one solution through $\mathbb{R}$. Moreover by the comparison theorem 9 we can deduce that the random variable Z(t) (fixed t) is increasing in the means of definition 2.1. If we define $X_t : \Omega \to \mathbb{R}$, $X_t = G(Z_t)$ then X_t is

increasing on $(\Omega, \mathcal{A}, \mathbb{P}, \geq)$. It remain to be proof that the random variable X_t satisfies the stochastic equation (3).
By the Itô's formula

$$dG(Z_t) = G'(Z_t)d(Z_t) + \frac{1}{2}G''(Z_t)dt$$

$$dX_t = \sigma(X_t)d(Z_t) + \frac{1}{2}\sigma(X_t)\sigma'(X_t)dt$$

$$dX_t = b(X_t)dt - \frac{1}{2}\sigma(X_t)\sigma'(X_t)dt + \sigma(X_t)dW_t + \frac{1}{2}\sigma(X_t)\sigma'(X_t)dt$$

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

this completes the proof of the lemma.

Theorem 9 *Let us consider the integral equations:*

$$(Z_1)(t) := z_0 + \int_0^t b_1(Z_1(s))ds + W_t^1$$

$$(Z_2)(t) := z_0 + \int_0^t b_2(Z_2(s))ds + W_t^2$$

Where W^1, W^2 are continuous functions and b_1, b_2 are locally Lipschitz functions. Let us suppose the existence of the solutions in an interval $[0,T]$. If we assume the "comparison" hypotheses:
- $b_2(Z) \geq b_1(Z) \quad \forall Z \in \mathbb{R}$
- $W_2 - W_1$ *weakly increasing*

Then for all $t \in [0,T]$ we have the inequality

$$(Z^2)(t) \geq (Z^1)(t)$$

Proof. See appendix. □

Proposition 10 *Let W_t be continuous.Let $\widehat{b}$ be locally Lipschitz. If there exist two positive constant k_1 and k_2 such that $|\widehat{b}(z)| \leq k_1 + k_2 z \quad \forall z \in \mathbb{R}$, then the integral equation:*

$$(Z)(t) := z_0 + \int_0^t \widehat{b}(Z(s))ds + W_t$$

has one and only one solution through $\mathbb{R}$.

4 Appendix

Product spaces

Proof. of theorem 3
We will proceed by induction on the cardinality of Γ, proving firstly the finite case, then the enumerable one and finally the generic one.
Finite case:
Let n be the cardinality of Γ. Let us begin with $n = 2$.
Let A,B be increasing events, let $\Omega = \Omega_1 \times \Omega_2$, $f := I_A$, $g := I_B$. Let us define $f_1(x) :=$

$\mathbb{E}_2[f(x,\cdot)]$ and $g_1(x) := \mathbb{E}_2[g(x,\cdot)]$. It is easy to check that f_1 and g_1 are increasing with respect to $\geq_1$ .Then

$$\mathbb{E}[f]\mathbb{E}[g] = \mathbb{E}_1[f_1]\mathbb{E}_1[g_1] \leq \mathbb{E}_1[f_1 g_1] = \mathbb{E}_1\big[\mathbb{E}_2[f]\mathbb{E}_2[g]\big] \leq \mathbb{E}_1\big[\mathbb{E}_2[fg]\big] = \mathbb{E}[fg]$$

Therefore $\mathbb{P}(A)\mathbb{P}(B) \leq \mathbb{P}(A \cap B)$ and this verifies the hypotheses of proposition 1. For $n \geq 2$ we have

$$\Omega = \prod_{1\leq i\leq n} (\Omega_i) \quad = \prod_{1\leq i\leq n-1} (\Omega_i) \quad \times \Omega_n$$

This completes the proof in the finite case.
Enumerable case:
Let us suppose $\Gamma = \mathbb{N}$. Let $A, B \in \mathcal{F}$ be two increasing events. Let $f = I_A$, $g = I_B$ be the indicator functions. Let us define $(\Omega, \mathcal{F}, \mathbb{P}, \geq) = \prod_{i\in\mathbb{N}}(\Omega_i, \mathcal{F}_i, \mathbb{P}_i, \geq_i)$, $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{\mathbb{P}}_n, \tilde{\geq}_n) := \prod_{i=1}^{n}(\Omega_i, \mathcal{F}_i, \mathbb{P}_i, \geq_i)$ and
$(\overline{\overline{\Omega}}_n, \overline{\overline{\mathcal{F}}}_n, \overline{\overline{\mathbb{P}}}_n, \overline{\overline{\geq}}_n) := \prod_{i>n}(\Omega_i, \mathcal{F}_i, \mathbb{P}_i, \geq_i)$.
Let $p_n : \Omega \to \tilde{\Omega}_n$ be the projection on the first n coordinates, and $\overline{\overline{p}}_n : \Omega \to \overline{\overline{\Omega}}_n$ the projection on the other ones. If we define $f_n, g_n : \tilde{\Omega}_n \to \mathbb{R}$ in this way: $f_n(\tilde{x}) := \overline{\overline{\mathbb{P}}}_n(\overline{\overline{p}}_n(p_n^{-1}(\tilde{x}) \cap A))$, $g_n(\tilde{x}) := \overline{\overline{\mathbb{P}}}_n(\overline{\overline{p}}_n(p_n^{-1}(\tilde{x}) \cap B))$ then it easily follows that f_n and g_n are increasing, bounded and they satisfy the correlation inequality

$$\mathbb{E}[f_n g_n] \geq \mathbb{E}[f_n]\mathbb{E}[g_n] \tag{6}$$

By the Lévy's Upward Theorem we have $f_n \to I_A$ and $g_n \to I_B$ in $\mathcal{L}^1$ and a.s.

$$\mathbb{E}[I_A I_B] \geq \mathbb{E}[I_A]\mathbb{E}[I_B]$$

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$$

This verifies the hypothesis of proposition 1 and completes the proof in the enumerable case.
Generic case:
For all $J \subset \Gamma$, let $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J, \geq_J) := \prod_{i\in J}(\Omega_i, \mathcal{F}_i, \mathbb{P}_i, \geq_i)$ and let p_J be the natural projection from Ω to Ω_J. For all $A, B \in \mathcal{F}$ increasing events, it exists $J \subset \Gamma$ such that J is enumerable, $A, B \in \mathcal{F}_J$. It follows that $\mathbb{P}_J(p_J(A) \cap p_J(B)) \geq \mathbb{P}_J(p_J(A))\mathbb{P}_J(p_J(B))$ and finally

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$$

□

4.1 Lemma 5

With the notation of Lemma 5 the following proposition holds.

Proposition 11 *Let $A \in \mathcal{A}_H$. Then A is $_{\geq}$increasing if and only if A is $_{\geq_H}$increasing.*

PROOF.
To prove that A $_{\geq_H}$*increasing* implies A $_{\geq}$*increasing* it is sufficient to remark that $\geq_H$ is finer than $\geq$.
The proof of A $_{\geq}$*increasing* implies A $_{\geq_H}$*increasing* is a bit more difficult. We know that $\big(\omega_1 \leq \omega_2\ ,\ \omega_1 \in A\big)$ implies $\omega_2 \in A$ and we have to show $\big(\omega_1 \leq_H \omega_2\ ,\ \omega_1 \in A\big)$ implies

$\omega_2 \in A$.
Let us suppose $\omega_1 \leq_H \omega_2$ and $\omega_2 \in A$. Let us define $p_i := W_{t_i}(\omega_2) - W_{t_i}(\omega_1)$ for all $i \in \{0, 1, \dots, n\}$. By the definition 2.1 we have $W_{t_{i-1}}(\omega_2) - W_{t_{i-1}}(\omega_1) \leq W_{t_i}(\omega_2) - W_{t_i}(\omega_1)$ and so $0 = p_0 \leq p_1 \leq \cdots \leq p_n$. Let now f from $[0, T]$ to $\mathbb{R}$ weakly increasing, continuous and such that $f(t_i) = p_i$ for all $i \in \{0, \dots, n\}$ then $f(t) + W_t(\omega_1)$ is continuous in t. Therefore it exists $\omega_3 \in \Omega$ such that $W_t(\omega_3) = f(t) + W_t(\omega_1)$ for all $t \in [0, T]$. From the definition of f it results $\omega_3 \geq \omega_1$ and then $\omega_3 \in A$. ω_3 coincides with ω_2 on H (that is $W_{t_i}(\omega_3) = W_{t_i}(\omega_2)$ for all $t_i \in H$) and then also $\omega_2 \in A$. □

Remark 4.1 *Let $A \in \mathcal{A}$ and A be $_{\geq_H}$increasing. Then, $A \in \mathcal{A}_H$.*

PROOF. It is sufficient to notice that $A = P_n^{-1}(P_n(A))$. With the projection. $P_n := (W_{t_1}, \dots, W_{t_n})$.
□

Comparison Theorem

PROOF. of theorem 9
Suppose by absurd that there exists some $\bar{t} > 0$ such that $(Z^2)(\bar{t}) < (Z^1)(\bar{t})$. Let $t_0 = \sup\{t < \bar{t} \mid (Z^2)(t) = (Z^1)(t)\}$. And so $(Z^1)(t) > (Z^2)(t)\ \forall t \in (t_0, \bar{t}]$. Let L be a Lipschitz constant for $b_2(Z_t^2)$ in $[t_0, \bar{t}]$. And let τ a constant such that $t_0 < \tau < \bar{t}$ and $\tau < \frac{1}{L}$.
Making the difference $Z^1 - Z^2$, for all $t \in (t_0, \tau]$ we have:

$$Z^1(t) - Z^2(t) = \int_{t_0}^{t} \Big[b_1(Z_1(s)) - b_2(Z_2(s)) \Big] ds + (W_t^1 - W_{t_0}^1) - (W_t^2 - W_{t_0}^2)$$

By comparison hypothesis we have:

$$Z^1(t) - Z^2(t) \leq \int_{t_0}^{t} \Big[b_2(Z_1(s)) - b_2(Z_2(s)) \Big] ds$$

By the lipschitz condition on b_2,

$$Z^1(t) - Z^2(t) \leq \tau L \sup_{s \in [t_0, \tau]} \Big\{ \big| Z_1(s) - Z_2(s) \big| \Big\}$$

then

$$\sup_{s \in [t_0, \tau]} \Big\{ \big| Z_1(s) - Z_2(s) \big| \Big\} \leq \tau L \sup_{s \in [t_0, \tau]} \Big\{ \big| Z_1(s) - Z_2(s) \big| \Big\}$$

but $\tau L < 1$ implies $\sup_{s \in [t_0, \tau]} \Big\{ \big| Z_1(s) - Z_2(s) \big| \Big\} = 0$ that contradicts the hypothesis $(Z^1)(t) > (Z^2)(t)\ \forall t \in (0, S]$. □

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