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ON NODAL LINES OF NEUMANN EIGENFUNCTIONS

RAMI ATAR [1]
Department of Electrical Engineering, Technion - Israel Institute of Technology
Haifa 32000, Israel
email: atar@ee.technion.ac.il

KRZYSZTOF BURDZY [2]
Department of Mathematics, University of Washington Box 354350
Seattle, WA 98195-4350, USA
email: burdzy@math.washington.edu

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Abstract

We present a new method for locating the nodal line of the second eigenfunction for the Neumann problem in a planar domain.

1. Introduction. This note is inspired by recent progress on the "hot spots" conjecture of J. Rauch originally proposed in 1974. The conjecture states that the maximum of the second Neumann eigenfunction in a Euclidean domain is attained on the boundary. This formulation is somewhat vague but the counterexamples given by Burdzy and Werner (1999) and Bass and Burdzy (2000) show that even the weakest version of the conjecture fails in some planar domains. The first positive results on the "hot spots" conjecture appeared in Kawohl (1985) and Bañuelos and Burdzy (1999). The last article contains a proof that the conjecture holds in all triangles with an obtuse angle. The theorem was generalized to "lip domains", i.e., planar domains between two graphs of Lipschitz functions with the Lipschitz constant 1; this more general result was hinted at in Bañuelos and Burdzy (1999) and proved in Atar and Burdzy (2002). See Atar (2001), Jerison and Nadirashvili (2000) and Pascu (2002) for some other positive results.

There is currently a significant gap between the best positive results and the strongest counterexamples to the "hot spots" conjecture. It is widely believed that the conjecture holds in all convex domains in $\mathbf{R}^n$ for all $n \geq 2$. But the problem remains open even for triangles whose all angles are acute.

One of the key assumptions in some theorems of Bañuelos and Burdzy (1999), and all results of Jerison and Nadirashvili (2000) and Pascu (2002), is the symmetry of the domain with respect to a straight line. The assumption is only needed to show that the nodal line of the

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second Neumann eigenfunction lies on the line of symmetry. Then one can replace the original Neumann problem on the whole domain with the mixed Dirichlet-Neumann problem on a nodal subdomain. As it often happens in analysis, the Dirichlet problem (in this case, the mixed Dirichlet-Neumann problem) turns out to be considerably easier to deal with than the original Neumann problem. Hence, one would like to be able to find the location of the nodal line for the second Neumann eigenfunction in domains which are not necessarily symmetric. If this goal is achieved, some further progress on the "hot spots" conjecture may be expected. Nodal lines also seem to be of interest of their own; see, for example, a paper of Melas (1992) on the nodal lines in the Dirichlet case.
Very few methods for finding the nodal line are known; the following short list is probably complete.

(a) If a domain has a line of symmetry, the nodal line lies on that line, subject to some natural extra assumptions.

(b) There are a handful of classes of domains, such as rectangles and ellipses, for which explicit formulae for eigenfunctions are known.

(c) One can find an approximate location of the nodal line in domains which are "long and thin" in the asymptotic sense; see Jerison (2000).

The purpose of this note is to add the following item to this list.

(d) In some domains, one can use the probabilistic method of "couplings" to delineate a region which the nodal line must intersect.

It should be emphasized that one cannot expect to find the location of the nodal line of the second Neumann eigenfunction in the sense of an explicit formula, except in some trivial cases. What one can realistically hope to achieve is to obtain sufficiently accurate information about the nodal line so that this information can in turn be used to prove some other results of interest; the combination of Lemmas 2 and 3 of Burdzy and Werner (1999) is an example of such a result on nodal lines.
We are grateful to the referee for many useful suggestions.

2. Couplings and nodal lines. Suppose that D is a planar domain (open and connected set) with a piecewise smooth boundary, i.e., the boundary consists of finitely many C^2 parts, and the uniform exterior sphere condition and the uniform interior cone condition are satisfied. Under these assumptions on D, one can prove strong existence and pathwise uniqueness for reflecting Brownian motion in $\overline{D}$ (see Lions and Sznitman (1984) or Bass, Burdzy and Chen (2002) for the new developments and a historical review). A *coupling* is a pair of reflecting Brownian motions X_t and Y_t in $\overline{D}$, defined on a common probability space.
Our first theorem is inspired by "mirror couplings." Informally speaking, a mirror coupling is a pair (X_t, Y_t) of reflecting Brownian motions in $\overline{D}$, such that the line of symmetry K_t between X_t and Y_t does not change on any interval (s,u) such that $X_t \notin \partial D$ and $Y_t \notin \partial D$ for all $t \in (s,u)$. A rigorous construction of the mirror coupling was given in Atar and Burdzy (2002) although mirror couplings had been informally used in the past; see Atar and Burdzy (2002) for a short review of the history of this concept. Although mirror couplings can be used to prove theorems about nodal lines (by applying Theorem 2.1 below with K_t being the

mirror), we will not use them in this paper. A different coupling, presented below, seems to be better suited to this problem.

Recall that the first eigenvalue for the Laplacian in D with Neumann boundary conditions is equal to 0 and the corresponding eigenfunction is constant. The second eigenvalue need not be simple but its multiplicity can be only 1 or 2 (Nadirashvili (1986, 1988)). The nodal set is the set of points in D where the second eigenfunction vanishes. If D is not simply connected, the nodal set need not be a connected curve. The nodal set of any second Neumann eigenfunction divides the domain D into exactly two nodal domains (connected and open sets).

Let x, y be points and K be a straight line, and denote by A_1 and A_2 the two connected componenets of $\mathbf{R}^2 \setminus K$. We say that x and y lie on the opposite sides of K, if either $x \in \overline{A_1}$ and $y \in \overline{A_2}$, or $x \in \overline{A_2}$ and $y \in \overline{A_1}$.

Theorem 2.1. *Suppose that $A \subset \overline{D}$ is closed, $x \in D \setminus A$, D_1 is a connected component of $D \setminus A$ which contains x, and $y \in D \setminus D_1$. Assume that there exists a coupling (X_t, Y_t) of reflecting Brownian motions in $\overline{D}$ and a line-valued stochastic process K_t with the following properties. For every $t \geq 0$, X_t and Y_t lie on the opposite sides of K_t. Moreover, $(X_0, Y_0) = (x, y)$ and $K_t \cap D \subset A$ for all $t \geq 0$ a.s. Then for any second Neumann eigenfunction in D, none if its nodal domains can have a closure which is a subset of $\overline{D} \setminus A$ containing x.*

Proof. Suppose that B is one of the nodal domains for a second eigenfunction and $x \in \overline{B} \subset \overline{D} \setminus A$. We will show that this assumption leads to a contradiction. Let C_1 be a non-empty open disc centered at a boundary point of B, but not on the boundary of D, so small that $B_* = B \cup C_1$ satisfies $x \in \overline{B}_* \subset \overline{D} \setminus A$, and there exists a non-empty open disc $C_2 \subset D \setminus B_*$. Assume that y, X_t, Y_t and K_t satisfy the assumptions of the theorem. Since the stationary measure for X_t is the uniform distribution in D, X_t will hit C_2 with probability one, and so it will leave B_* with probability 1. Recall that there are exactly two nodal domains—B is one of them; let B_1 denote the other one. Let $\tau_X(B_*) = \inf\{t \geq 0 : X_t \notin B_*\}$ and $\tau_Y(B_1) = \inf\{t \geq 0 : Y_t \notin B_1\}$. We will argue that $\tau_X(B_*) < \tau_Y(B_1)$ a.s. Suppose otherwise. By the continuity of $t \to Y_t$, $Y_{\tau_Y(B_1)}$ belongs to $\overline{\partial B_1 \setminus \partial D}$, and so $Y_{\tau_Y(B_1)} \in \overline{B} \subset \overline{B}_*$. We have assumed that $\tau_X(B_*) \geq \tau_Y(B_1)$, so $X_{\tau_Y(B_1)} \in \overline{B}_*$. Since the points $X_{\tau_Y(B_1)}$ and $Y_{\tau_Y(B_1)}$ belong to the connected set $\overline{B}_*$, the line $K_{\tau_Y(B_1)}$ must also intersect $\overline{B}_*$. We have $K_t \cap D \subset A$ for all t so $A \cap \overline{B}_* \neq \emptyset$, which is a contradiction. We conclude that $\tau_X(B_*) < \tau_Y(B_1)$ a.s.

Let $\mu > 0$ denote the second Neumann eigenvalue for D (recall that the first eigenvalue is equal to 0). Then μ is the first eigenvalue for the mixed problem in the nodal domain B, with the Neumann boundary conditions on ∂D and the Dirichlet boundary conditions on the nodal line. The fact that B_* is strictly larger than B easily implies that $\mu > \mu^*$, where μ^* is the analogous mixed eigenvalue for B_*. Using the well known identification of Brownian motion density with the heat equation solution, we obtain from Proposition 2.1 of Bañuelos and Burdzy (1999) that

$$\lim_{t \to \infty} P(\tau_Y(B_1) > t) e^{\mu t} \in (0, \infty).$$

Since $\tau_X(B_*) < \tau_Y(B_1)$ a.s.,

$$\lim_{t \to \infty} P(\tau_X(B_*) > t) e^{\mu t} < \infty,$$

and so $\mu^* \geq \mu$, but this contradicts the fact that $\mu > \mu^*$. Our initial assumption that $x \in \overline{B} \subset \overline{D} \setminus A$ for some nodal domain B must be false. $\square$

As we said before, Theorem 2.1 can be applied to mirror couplings to yield some results on nodal lines. We will now introduce a new coupling which gives stronger results than the mirror coupling for some families of domains. First, we will define a family of domains.

Let β denote the mapping which assigns to each point in $\mathbf{R}^2$ its symmetric image about the vertical axis. Let (e_1, e_2) be the usual orthonormal basis for $\mathbf{R}^2$ and let $\mathbf{U}_+$ be the right half plane, i.e., $\mathbf{U}_+ = \{x \in \mathbf{R}^2 : x \cdot e_1 \geq 0\}$. We will denote the left half plane $\mathbf{U}_- = \beta \mathbf{U}_+$. A convex set $C \subset \mathbf{R}^2$ will be called a cone if $\alpha x \in C$ for any $x \in C$ and $\alpha \geq 0$.

Recall that D is a planar piecewise smooth domain. We will define a family of domains via conditions (A1)-(A4) below, chosen to satisfy the demands of the proof of Theorem 2.2. Conditions (A3)-(A4) are the most important of the four conditions because they impose considerable restrictions on the shape of D. Assume that D intersects the vertical axis. Let

$$D_1 = D \cap \mathbf{U}_+, \quad D_2 = D \cap \mathbf{U}_-,$$

$$\widetilde{D} = \beta D, \quad \widetilde{D}_1 = \beta D_1, \quad \widetilde{D}_2 = \beta D_2.$$

For $i = 1, 2$, let ∂D_i denote the set of all boundary points of D_i, where the boundary is smooth, except those that lie on the vertical axis. Let $\partial \widetilde{D}_i$ be defined similarly. We will write $n(x)$ for the inward normal to ∂D_i at $x \in \partial D_i$, for $i = 1, 2$. With an abuse of notation, the same symbol $n(x)$ will be used to denote the inward normal to $\partial \widetilde{D}_i$ at $x \in \partial \widetilde{D}_i$.

(A1) $D_1 \subset \widetilde{D}_2$.

(A2) For all $x \in \partial D \cap \partial \widetilde{D}$ except for a finite set of such x, the sets ∂D and $\partial \widetilde{D}$ agree in some neighborhood of x.

(A3) There exists a closed cone $C \subset \mathbf{U}_-$ such that for any $x \in \partial D_1 \setminus \partial \widetilde{D}_2$, one has $n(x) \in C$.

In the case when C is a proper subset of $\mathbf{U}_-$, let v_1 and v_2 denote the two unit vectors that generate the cone C, i.e., all vectors in C are linear combinations of v_1 and v_2 with non-negative coefficients. For $i = 1, 2$, let $v_i^\perp$ denote the unit vector orthogonal to v_i, such that $v_i^\perp \cdot v_{3-i} > 0$. In the case when $C = \mathbf{U}_-$, let $v_1 = -e_2$, $v_2 = e_2$, and $v_1^\perp = v_2^\perp = -e_1$. Note that in both cases,

$$v_i + \varepsilon v_i^\perp \in C, \tag{2.1}$$

provided that $\varepsilon > 0$ is small enough.
For $x \in \mathbf{R}^2$, let the ray through x be denoted as $R(x) = \{\alpha x : \alpha > 0\}$.

(A4) (i) For $y \in \partial \widetilde{D}_2 \cup (\partial \widetilde{D}_1 \cap \partial D_2)$ and $i = 1, 2$, if $y + R(v_i)$ intersects $\overline{D}$ then $n(x) \cdot v_i^\perp \leq 0$ for all $x \in \partial \widetilde{D}_2$ in a neighborhood of y. (ii) For $y \in \partial D_2 \cup (\partial D_1 \cap \partial \widetilde{D}_2)$ and $i = 1, 2$, if $y - R(v_i)$ intersects $\overline{\widetilde{D}}$ then $n(x) \cdot v_i^\perp \geq 0$ for all $x \in \partial D_2$ in a neighborhood of y.

Geometric conditions (A1)-(A4) (especially (A3)-(A4)) might not be easy to visualize. To help the reader, we present an "extreme" domain satisfying all these conditions in Example 3.4 and the corresponding Figure 4 in the next section. The example is our attempt to show the "most irregular" or "unusual" shape for which (A1)-(A4) hold.

Suppose that $x \in D \cap \widetilde{D}$ and W_t is a planar Brownian motion. Then there exist a reflecting Brownian motion X_t in D, and a reflecting Brownian motion Y_t in $\widetilde{D}$, driven by W_t, and starting from x:

$$X_t = x + W_t + \int_0^t n(X_s) dL_s, \quad Y_t = x + W_t + \int_0^t n(Y_s) dM_s. \tag{2.2}$$

Here L_t and M_t are local times on the boundary for X_t and Y_t; see Lions and Sznitman (1984) for more details on boundary local times.

Theorem 2.2. *Assume that D satisfies conditions (A1)-(A4) and X_t and Y_t satisfy (2.2). Then with probability one, for all $t \geq 0$, $X_t - Y_t \in C$.*

Proof: We have assumed that ∂D is smooth, except for a finite number of points—it is well known that the set of these points is polar for the reflected Brownian motion in D. For all $x \in \partial D$ outside this set, $n(x)$ is well defined and $n(y)$ depends continuously on $y \in \partial D$ in a neighborhood of x. Assume that the assertion of the theorem does not hold, and let $\sigma = \inf\{t : X_t - Y_t \notin C\}$. Clearly, $X_\sigma - Y_\sigma \in \partial C$.

If $X_\sigma \in D$ and $Y_\sigma \in \widetilde{D}$, then there is $\varepsilon > 0$ such that $X_t \in D$ and $Y_t \in \widetilde{D}$ for all $t \in [\sigma, \sigma + \varepsilon]$. By (2.2), $X_t - Y_t = X_\sigma - Y_\sigma \in \partial C$ for all such t, contradicting the definition of σ. Hence either $X_\sigma \in \partial D$ or $Y_\sigma \in \partial\widetilde{D}$.

If $X_\sigma = Y_\sigma$, then using condition (A1), either $X_\sigma \in \partial D_1$ or $Y_\sigma \in \partial\widetilde{D}_1$. Since the cases are similar, we will discuss only the former. Consider the following two cases.
(a) If $X_\sigma \in \partial D \cap \partial\widetilde{D}$ then, by (A2), ∂D and $\partial\widetilde{D}$ agree in a small neighborhood of X_σ. Using (2.2) and strong uniqueness, we see that $X_t = Y_t$ for $t \in [\sigma, \sigma + \delta]$, if $\delta > 0$ is sufficiently small. This contradicts the definition of σ.
(b) The other case is when $X_\sigma \in \partial D_1 \setminus \partial\widetilde{D}_2$ and $Y_\sigma \in \widetilde{D}$. Then for small $\varepsilon > 0$ and all $t \in [\sigma, \sigma + \varepsilon]$, $X_t - Y_t = \int_\sigma^t n(X_s) dL_s$. Condition (A3) implies that $\int_\sigma^t n(X_s) dL_s \in C$ for all such t, provided that ε is small enough, contradicting the definition of σ.
We conclude that $X_\sigma \neq Y_\sigma$ and so $X_\sigma - Y_\sigma \in \partial C \setminus \{0\}$. In other words,

$$X_\sigma - Y_\sigma = \alpha v_i, \tag{2.3}$$

where $\alpha > 0$ and either $i = 1$ or $i = 2$.

The rest of the argument is split into three cases with subcases.

Case 1: $Y_\sigma \in \widetilde{D}$.
(a) $X_\sigma \in \partial D_1 \setminus \partial\widetilde{D}_2$. In this case, $X_t - Y_t = \alpha v_i + \int_\sigma^t n(X_s) dL_s$ for $t \in [\sigma, \sigma + \varepsilon]$ provided that $\varepsilon > 0$ is small enough. By condition (A3), $X_t - Y_t \in C$ for all such t, taking, if necessary, $\varepsilon > 0$ smaller. This ontradicts the definition of σ.
(b) $X_\sigma \in \partial D_2 \cup (\partial D_1 \cap \partial\widetilde{D}_2)$. This is similar to Case 2(a) considered below, except that we should use condition (A4)(ii) in place of (A4)(i).

Case 2: $Y_\sigma \in \partial\widetilde{D}_2 \cup (\partial\widetilde{D}_1 \cap \partial D_2)$.
A single argument will apply to subcases (a) and (b):
(a) $X_\sigma \in D$.
(b) $X_\sigma \in \partial D_1 \setminus \partial\widetilde{D}_2$.
In either case we have

$$X_t - Y_t = \alpha v_i + \int_\sigma^t n(X_s) dL_s - \int_\sigma^t n(Y_s) dM_s. \tag{2.4}$$

Since $X_\sigma \in \overline{D}$, (2.3) implies that $Y_\sigma + R(v_i)$ intersects $\overline{D}$. Hence, by condition (A4)(i), $\int_\sigma^t n(Y_s) dM_s \cdot v_i^\perp \leq 0$ for $t \in [\sigma, \sigma + \varepsilon]$, provided $\varepsilon > 0$ is small. As before, condition (A3) implies that $\int_\sigma^t n(X_s)dL_s \in C$. As a result, using (2.1), $X_t - Y_t \in C$ for $t \in [\sigma, \sigma+\varepsilon]$ if $\varepsilon > 0$ is small enough. This contradicts the definition of σ.

(c) $X_\sigma \in \partial D_2 \cup (\partial D_1 \cap \partial \widetilde{D}_2)$. In this case (2.4) again holds, and using condition (A4)(i), $\int_\sigma^t n(Y_s)dM_s \cdot v_i^\perp \leq 0$. Moreover, by condition (A4)(ii), $\int_\sigma^t n(X_s)dL_s \cdot v_i^\perp \geq 0$. An application of (2.1) again shows that $X_t - Y_t \in C$ for $t \in [\sigma, \sigma+\varepsilon]$ and $\varepsilon > 0$ small. This is impossible, in view of the definition of σ.

Case 3: $Y_\sigma \in \partial \widetilde{D}_1 \setminus \partial D_2$.

(a) $X_\sigma \in D$. Note that condition (A3) implies that for $y \in \partial \widetilde{D}_1 \setminus \partial D_2$, $n(y) \in -C$. Hence this case is similar to Case 1(a), with the difference that now we have $X_t - Y_t = \alpha v_i - \int_\sigma^t n(Y_s)dM_s$ for $t \in [\sigma, \sigma+\varepsilon]$.

(b) $X \in \partial D_1 \setminus \partial \widetilde{D}_2$. Since $\partial \widetilde{D}_1 + C$ does not intersect ∂D_1, this case is ruled out by (2.3).

(c) $X_\sigma \in \partial D_2 \cup (\partial D_1 \cap \partial \widetilde{D}_2)$. This case is similar to Case 2(b) considered above.

We arrived at a contradiction in all cases. As a result, $X_t - Y_t \in C$ must hold for all $t \geq 0$, with probability one. □

Recall that e_1 is the first base vector of $\mathbf{R}^2$. Under the assumptions of Theorem 2.2 we have $e_1 \cdot (X_t - Y_t) \leq 0$ for all t. Let $\widetilde{Y} = \beta Y$ and note that X_t and $\widetilde{Y}_t$ are reflecting Brownian motions in D. Then $e_1 \cdot (X_t + \widetilde{Y}_t) \leq 0$ for all t. This implies that if K_t denotes the vertical line passing through the midpoint of X_t and $\widetilde{Y}_t$ then $K_t \subset \mathbf{U}_-$ for all $t \geq 0$. This and Theorem 2.1, applied with $A = \mathbf{U}_- \cap \overline{D}$, yield the following.

Corollary 2.3. *Assume that D satisfies conditions (A1)-(A4). Then for any second Neumann eigenfunction in D, its nodal line must intersect $\overline{D_2}$.* □

3. Examples. We will present two examples illustrating Corollary 2.3. Then we will indicate how Theorem 2.1 can be combined with mirror couplings. But first we would like to point out that in the one-dimensional case, i.e., in the case when D is a line segment, one can use the "mirror" coupling of Brownian motions reflected at the endpoints of D to prove that the nodal point lies in the middle of D—we leave the details of the proof as an elementary excercise for the reader.

Example 3.1. Suppose that D is an obtuse triangle with vertices C_1, C_2 and C_3. Let C_3 be the vertex with an angle greater than $\pi/2$, and suppose that the angle at C_2 is not smaller than that at C_1 (see Figure 1). Let $\overline{C_jC_k}$ denote the line segment with endpoints C_j and C_k. The points C_k, $k = 4, \dots, 10$, are chosen so that $C_5, C_{10} \in \overline{C_1C_3}$, $C_8 \in \overline{C_2C_3}$, $C_9, C_6, C_4, C_7 \in \overline{C_1C_2}$, and the following pairs of line segments are perpendicular: $\overline{C_1C_2}$ and $\overline{C_3C_4}$, $\overline{C_1C_2}$ and $\overline{C_5C_6}$, $\overline{C_2C_3}$ and $\overline{C_5C_9}$, $\overline{C_2C_3}$ and $\overline{C_4C_8}$, $\overline{C_1C_3}$ and $\overline{C_7C_3}$, $\overline{C_1C_3}$ and $\overline{C_6C_{10}}$. The point C_6 lies half way between C_1 and C_4.

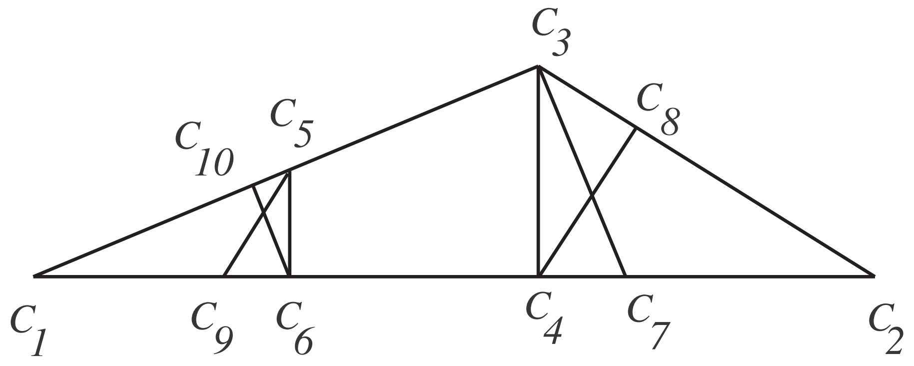

Figure 1.

Let A be the closed subset of $\overline{D}$ (trapezoid) with vertices C_3, C_4, C_6 and C_5. Let A_1 be the closure of the union of the quadrilateral with vertices $C_3C_7C_9C_5$ and pentagon $C_3C_8C_4C_6C_{10}$. The second Neumann eigenvalue is simple in obtuse triangles by a theorem of Atar and Burdzy (2002).

We will show that

(i) The nodal line for the second Neumann eigenfunction must intersect A.

(ii) The nodal line lies within A_1.

First, place the coordinate system so that the vertical axis passes through $\overline{C_3C_4}$. Then let v_1 be the vector perpendicular to $\overline{C_2C_3}$, pointing to the left, and let v_2 be its image under the map $(x_1, x_2) \mapsto (x_1, -x_2)$. It is then elementary to check that the assumptions (A1)-(A4) of Theorem 2.2 are satisfied and so Corollary 2.3 applies. We conclude that the nodal line must intersect the closed triangle $C_1C_4C_3$.
Next place the coordinate system in such a way that $\overline{C_5C_6}$ lies on the vertical axis. Then flip the triangle around the vertical axis, i.e., apply the transformation we call β. We will apply Theorem 2.2 and Corollary 2.3 to this new triangle. In this case we take $C = \mathbf{U}_-$. Again, it is completely routine to check that (A1)-(A4) are satisfied. We conclude that the nodal line must cross the closed quadrilateral $C_6C_2C_3C_5$. This completes the proof of (i).
Claim (ii) is a consequence of (i) and the bounds on the direction of the gradient of the second eigenfunction proved in Theorem 3.1 of Bañuelos and Burdzy (1999).

Remark. We will argue heuristically that it is impossible to obtain much sharper estimates for the nodal line location. It is not hard to show that if the triangle is isosceles then the nodal line is the line of symmetry. This shows that $\overline{C_3C_4}$ is a sharp "bound". To see how sharp is the other "bound", i.e., $\overline{C_5C_6}$, consider an obtuse triangle with two angles very close to $\pi/2$—one slightly less than the right angle and another one slightly larger than that (see Figure 2). Such a triangle has a shape very close to a thin circular sector. Let us assume that the diameter of the triangle is equal to 1. The nodal line in a circular sector is an arc with center at its vertex A. The nodal line distance from A is equal to $a_0/a_1 \approx 0.63$, where a_0 is the first positive zero of the Bessel function of order 0 and a_1 is the first positive zero of its derivative. This follows from known results on eigenfunctions in discs and estimates for Bessel function zeroes, see Bandle (1980), p. 92. Our methods yield 0.5 as the lower bound for the distance of the nodal line from A; we see that this estimate cannot be improved beyond 0.63.

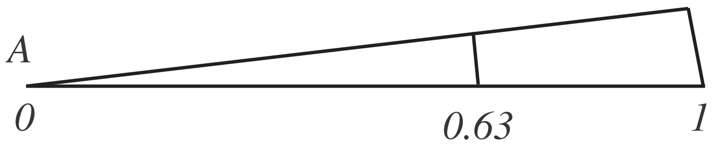

Figure 2.

We end this example with a conjecture.

Conjecture 3.2. *The nodal line for the second Neumann eigenfunction is contained in the triangle $C_1C_3C_4$.*

Little is known about eigenfunctions in triangles different from equilateral; see Pinsky (1980, 1985) for that special case.

Example 3.3. Consider a domain D in the plane whose boundary contains line segments $\{(x_1,x_2) : -a \le x_1 \le a, x_2 = 0\}$ and $\{(x_1,x_2) : -a \le x_1 \le a, x_2 = 1\}$ for some $a > 0$. Suppose that the remaining parts of ∂D are contained in $\{(x_1,x_2) : -a-b \le x_1 \le -a, 0 \le x_2 \le 1\}$ and $\{(x_1,x_2) : a \le x_1 \le a+b, 0 \le x_2 \le 1\}$, and moreover, they are graphs of functions in the coordinate system with the basis $(e_2, -e_1)$, i.e., the usual coordinate system rotated by the angle $\pi/2$. Let

$$\begin{aligned} A &= \{(x_1,x_2) \in \overline{D} : -b/2 \le x_1 \le b/2\}, \\ A_1 &= \{(x_1,x_2) \in \overline{D} : -b/2 - 1 \le x_1 \le b/2 + 1\}. \end{aligned}$$

Figure 3 shows an example of D and the corresponding rectangle A.

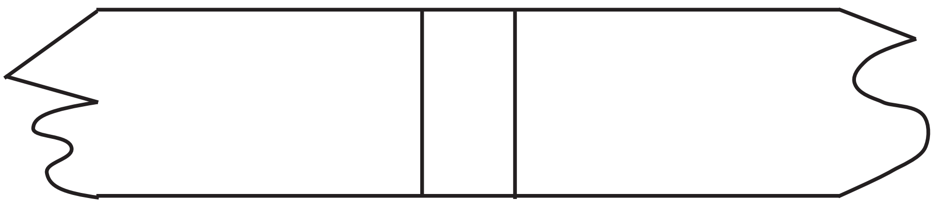

Figure 3.

We will prove that

(i) The nodal line for any second Neumann eigenfunction must intersect A.

We call a set D a "lip domain" if its boundary consists of graphs of two Lipschitz functions with the Lipschitz constant 1. The second eigenvalue is simple in every lip domain, by a result of Atar and Burdzy (2002).

Our second claim is

(ii) If D is a lip domain then the nodal line for the Neumann eigenfunction lies within A_1.

Place the coordinate system so that the right vertical edge of the boundary of A lies on the vertical axis. Let the cone C be equal to U_-. Then conditions (A1)-(A4) of Theorem 2.2 and Corollary 2.3 are easily verified and it follows that the nodal line has to intersect the set to the left of the right edge of A. A similar argument applies to the left edge of A and this completes the proof of (i).
The second claim follows from the first one and the bounds on the direction of the gradient of the second eigenfunction in lip domains given in Example 3.1 of Bañuelos and Burdzy (1999).

Example 3.4. A domain D and its mirror image $\widetilde{D}$ are depicted in Figure 4. If we take $C = \mathbf{U}_-$ then Corollary 2.3 can be used to show that the nodal line intersects the part of D to the left of the dotted vertical line.

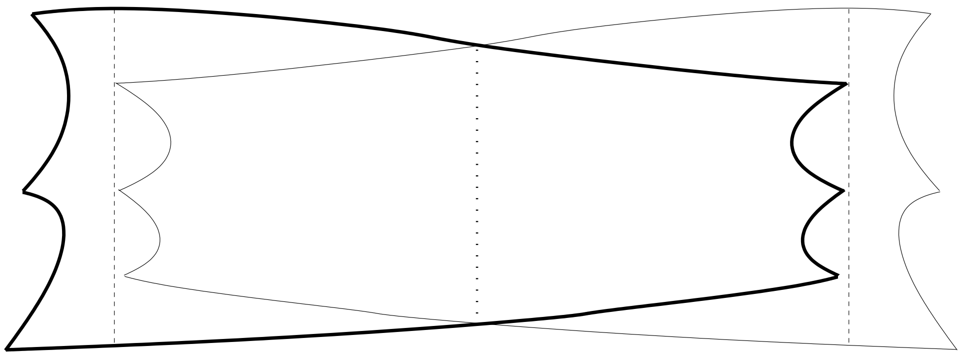

Figure 4. The thick line is the boundary of D.

We end the article with a challenge for the reader to use mirror couplings in conjuction with Theorem 2.1. Such couplings have been applied to study Neumann eigenfunctions in several papers (see, e.g., Atar and Burdzy (2002) or Bañuelos and Burdzy (1999)) so we felt that it would be more exciting to present here examples based on different couplings, described in Theorem 2.2.
We will recall a few crucial facts about mirror couplings from Burdzy and Kendall (2000) and Bañuelos and Burdzy (1999). Suppose that D is a polygonal domain and I is a line segment contained in its boundary. Let J denote the straight line containing I and recall that K_t denotes the mirror line. Let H_t denote the "hinge," i.e., the intersection of K_t and J. Note that H_t need not belong to ∂D. Suppose that for all t in $[t_1, t_2]$, the reflected Brownian motions X_t and Y_t do not reflect from any part of ∂D except I. Let α_t denote the smaller of the two angles formed by K_t and J. Then all possible movements of K_t have to satisfy the following condition.

(**M**) The hinge does not move within the time interval $[t_1, t_2]$, i.e., $H_t = H_{t_1}$ for all $t \in [t_1, t_2]$. The angle α_t is a non-decreasing function of t on $[t_1, t_2]$.

A domain D with piecewise smooth boundary can be approximated by polynomial domains D_n. Mirror couplings in the approximating domains D_n converge weakly to a mirror coupling in D. One can deduce which motions of the mirror K_t in D are possible by analyzing all allowed movements of mirrors in D_n's and then passing to the limit.

Excercise 3.5. *Use mirror couplings to show that in Example 3.1, the nodal line has to intersect the subset of D which lies to the right of the arc passing through C_5 and centered at C_1.*

Exercise 3.5 was originally a part of a theorem. We deleted the proof to keep the article short.

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