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# CANONICAL DECOMPOSITIONS OF CERTAIN GENERALIZED BROWNIAN BRIDGES

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*Abstract*

*We define a generalized Brownian bridge and we provide some information about its filtration. Two decompositions of this process as a semi-martingale are given. The first one is a Volterra decomposition and the second one is its canonical decomposition in its own filtration.*

## 1 Introduction

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Fix a time horizon $T > 0$ and a couple $(x, y)$ of reals. Informally, the Brownian bridge over the interval $[0, T]$ in between $x$ and $y$ is defined as

$$(B_u^{br,y}, u \leq T) = (B_u, u \leq T | B_T = y). \tag{1}$$

Since the probability of the event we are conditioning on is zero, some care is required in the way the latter conditioning should be understood. The standard Brownian excursion as well as the Brownian meander might be defined using a similar rigorous way that can be found, for instance, in Salminen [13]; but we also refer to Pitman[11] for a recent study where a new characterization using sampling at independent uniform times are given. In (1) and in the remainder of this paper, by conditioning a Brownian motion on a Gaussian random vector $\overline{Z} = (Z^{(1)}, Z^{(2)}, ..., Z^{(N)})$ we mean disintegrating the Wiener measure along the linear Gaussian subspace generated by the random variables $Z^{(1)}, Z^{(2)}, ..., Z^{(N)}$, i.e. the regular conditional process which law is specified as follows. For any bounded functional $F$ and $\overline{z} \in \mathbb{R}^N$ and $T > 0$, we have

$$\mathbb{E}(F(B_u, u \leq T) | \overline{Z} = \overline{z}) P(\overline{Z} \in d\overline{z}) = \mathbb{E}(F(B_u, u \leq T), \overline{Z} \in d\overline{z}). \tag{2}$$

Thus, some natural generalizations of the Brownian bridge come out by conditioning on a vector $(Z_T^{(1)}, Z_T^{(2)}, ..., Z_T^{(N)})$, where $N$ is a finite positive integer and $Z_T^{(j)}$, $j \leq N$, are $\mathcal{F}_T^B$-adapted functionals, instead of the terminal value of $B$ at time $T$. This kind of processes appear in insider trading financial models where the insider has in advance access to some information about the price of a stock in the future. The information is represented by the values of the latter functionals at the horizon time $T$. In this setting, provided that the functional $\overline{Z}$ is differentiable in Malliavin's sense, Baudoin [2] gave a general formula involving Malliavin's derivatives and the Clark-Ocone formula for the canonical decomposition of the conditioned Brownian motion as a semi-martingale in its own filtration. Explicit calculations seem to be a hard task in the general case. In this paper, we treat the case where $(Z_t^{(j)}, t \geq 0, j \leq N)$ is a vector of Gaussian local martingales. To be more precise, we set

$$Z_T^{(j)} = \int_0^T f_j(s) dB_s, j \leq N$$

where $\overline{f} = (f_1, f_2, ..., f_N)$ is a given vector of $L^2_{loc}(\mathbb{R}_+)$ functions . In the particular case of indicator functions, the conditioned process is nothing but a multiple Brownian bridge, that corresponds to sticking together independent copies of Brownian bridges. For instance, such processes appeared in [7] where they were defined using iteration method by conditioning on $B_{t_n}$ at the step $n$, for a given positive integer $N$ and a fixed sequence $0 < t_1 < t_2 < ... < t_N$. This paper is organized as follows. Section 2 is devoted to setting up the definition of a $\overline{f}$-generalized Gaussian bridge. We recall, in section 3, some basic facts about Volterra transforms and Goursat-Volterra Kernels. We then provide a non-canonical decomposition of the studied processes. In section 4, the canonical decomposition of the $\overline{f}$-generalized bridge, as a semi-martingale in its own filtration, is given. We end this introduction by resuming the fact that results in this paper light up some explicit examples of conditioned stochastic differential equation locally equivalent to a Brownian motion, but also some explicit constructions of Brownian motions in enlarged filtrations.

## 2 Generalized Gaussian bridges

In the following lines we give a natural generalization of the Brownian bridge. Let $\overline{f} = (f_i, 1 \leq i \leq N)$ be a fixed vector of linearly independent functions of $L^2_{loc}(\mathbb{R}_+)$. Consider the local martingales $Z_t^{(j)} = \int_0^t f_j(s) dB_s, t \geq 0, j \leq N$, and the $N$-dimensional Gaussian process $(\overline{Z}_t, t \geq 0) = (Z_t^{(1)}, ..., Z_t^{(N)}, t \geq 0)$. The associated covariance matrix $Q(s,t)$ is defined by

$$Q(s,t) = \mathbb{E}\big[(\overline{Z}_t - \overline{Z}_s)^*(\overline{Z}_t - \overline{Z}_s)\big] = \Big(\int_s^t dr f_i(r) f_j(r)\Big)_{i,j \leq N} \tag{3}$$

where $*$ denotes the transpose operator, for $s < t$ positive reals. In order to simplify the study, in the remainder of this paper, we suppose that $T > 0$ is chosen such that the matrix $Q(s,T)$ is invertible for any $s < T$. For $\overline{x}$ and $\overline{y} \in \mathbb{R}^N$ we consider the associated bridge over the interval $[0,T]$ in between $\overline{x}$ and $\overline{y}$, which is the conditioned process $(\overline{Z}_u, u \leq T | \overline{Z}_T = \overline{y}, \overline{Z}_0 = \overline{x})$. Then clearly each of its components is adapted to the underlying generalized bridge $(B_u, u \leq T | \overline{Z}_T = \overline{y})$. The distribution of the latter process is clearly well-defined by relation (2) which however does not provide a realization of such a process. Now, we show how to construct a canonical realization of a process having the latter conditioned distribution. Provided that the matrix $Q(0,T)$ is invertible, it is easy to check that the orthogonal decomposition of $(B_u, u \leq T)$ with respect to $(Z_T^1, Z_T^2, ..., Z_T^N)$, is

$$B_u = (B_u - \sum_1^N \psi_j(u) Z_T^{(j)}) + \sum_1^N \psi_j(u) Z_T^{(j)} \tag{4}$$

where $(\psi_j(u), j \leq N)$ is the unique sequence of functions that solves the linear system

$$\int_0^u f_i(s)ds = \sum_j \psi_j(u) \int_0^T f_i(s)f_j(s)ds, \qquad i = 1, \cdots, N. \tag{5}$$

Therefor, conditioning on $\sigma(\overline{Z})$, we get that for any fixed real vector $\overline{y}$ the continuous Gaussian process $(B_u^{\overline{y}}, u \leq T)$ defined for a fixed $u \leq T$ by

$$B_u^{\overline{y}} = B_u - \sum_1^N \psi_j(u) Z_T^{(j)} + \sum_1^N \psi_j(u) y_j \tag{6}$$

is independent of $\overline{Z}_T$. Its law, likewise the Brownian bridge, is characterized by its mean and covariance function that can be computed from (6). This conditional distribution is also characterized by the corresponding finite dimensional distributions. Following [10], there is no ambiguity in introducing the

**Definition 2.1.** A $\overline{f}$-generalized bridge over the interval $[0, T]$ with end-point $\overline{y} = (y_1, y_2, ..., y_N)$ is any almost surely continuous process on $[0, T]$ having the same finite dimensional distributions as $(B_u^{\overline{y}}, u \leq T)$. When $\overline{y} = \overline{0}$, such a process is called a standard $\overline{f}$-generalized bridge.

Notice that by the "end-point $\overline{y}$" we mean that $\int_0^T f_j(s)dB_s^{\overline{y}} = y_j, 1 \leq j \leq N$, which refers to the terminal value of a vector of functionals of the whole path of the process over the interval $[0, T]$. Additionally and by construction $(B_u^{\overline{y}}, u \leq T)$ is independent of $\sigma(\int_0^T f_i(s)dB_s, 1 \leq j \leq N)$. So that, in comparison with the Brownian bridge, unless 1 is a linear combination of the functions $f_1, f_2, ...f_N,$, the terminal value $B_T^{\overline{y}}$ is random. This is the reason why we prefer to not use the name "Brownian bridge" in the above definition.

# 3 The non-canonical decomposition

Since the decomposition is strongly connected to Volterra linear transforms, we shall start this paragraph by recalling their definition and some of their elementary properties. A Volterra transform is a mapping $\Sigma$ acting on the Brownian paths in the following way

$$\Sigma(B)_t = B_t - \int_0^t \int_0^u l(u, v)dB_v du. \tag{7}$$

We assume that $l$ is a continuous Volterra kernel. That is, $l$ satisfies

$$l(u, v) = 0, \quad \text{for} \quad 0 \leq u \leq v \leq T,$$

and the function

$$\tilde{l}(u, v) = \begin{cases} l(u, v) & \text{if } u \geq v; \\ l(v, u) & \text{if } u < v \end{cases}$$

is continuous on $[0, T] \times [0, T]$. Additionally, we assume the following integrability condition

$$\int_0^t \left( \int_0^u l^2(u, v)dv \right)^{\frac{1}{2}} du < \infty, \qquad \text{for all } t \geq 0. \tag{8}$$

From Föllmer-Wu-Yor [7] we know that the transformed process $(\Sigma(B)_t)_{t\geq 0}$ is again a Brownian motion if and only if $l$ is self-reproducing. That is, $l$ satisfies the integral equation

$$l(t,s) = \int_0^s l(t,u)l(s,u)du, \qquad \text{for all } t \geq 0 \text{ and for all } s \leq t. \tag{9}$$

To a continuous Gaussian process $(X_u, u \geq 0)$ and a fixed $t \geq 0$ we associate the closed linear span $\Gamma_t(X)$ of the Gaussian family $(X_u, u \leq t)$. It is easily seen that the orthogonal complement of $\Gamma_t(\Sigma(B))$ in $\Gamma_t(B)$ is

$$\Gamma_t^{(l)} = \text{Span}\Big(\int_0^t f(r)dB_r : f \in L^2([0,t]),\, f(s) = \int_0^s l(s,u)f(u)du \text{ for a.e. all } s \in [0,t]\Big). \tag{10}$$

We can now fix the reproducing Gaussian space to be

$$\Gamma_t^{(l)} = \text{Span}\big(\int_0^t f_j(r)dB_r, j \leq N\big)$$

and seek for the corresponding Volterra transform and its kernel that will be denoted by $\Sigma$ and $\bar{l}$ respectively. We keep the notation $\Sigma$ instead of introducing $\overline{\Sigma}$ in order to avoid additional subscripts. Following [1], such a transform exists and it is unique. More specifically, the corresponding Volterra kernel, denoted by $\bar{l}$, takes the form

$$\bar{l}(t,s) = \sum_1^N \phi_j(t) f_j(s)$$

for $s < t$, where the sequence of functions $(\phi_j, j \leq N)$ is identified to be the unique solution to the linear system of equations

$$f_i(t) = \sum_j \phi_j(t) \int_0^t f_i(s)f_j(s)ds, \qquad i = 1, \cdots, N. \tag{11}$$

Because of the expanded form of such kernels these are called Goursat-Volterra kernels. The latter factorization is important since it implies the following two properties of the transform $\Sigma$:

*(1) $\Sigma$ preserves the Wiener measure*

*(2) For any $u > 0$, $\mathcal{F}_u^{\Sigma(B)}$ is independent of $\sigma(\overline{Z}_u)$.*

The following result is an easy consequence of the preceding observations and the known properties of Volterra linear transforms. In particular, it shows that relation (7), with $B$ replaced by $B^{\overline{y}}$, is not the canonical decomposition of the semi-martingale $(B^{\overline{y}}, u \leq T)$ in its own filtration.

**Theorem 3.1.** *For any fixed $u < T$, we have the orthogonal decomposition*

$$\Gamma_u(B^{\overline{y}}) = \Gamma_u(\Sigma(B)) \oplus \mathit{Span}\,\big(\int_0^u f_j(s)dB_s^{\overline{y}}, j \leq N\big). \tag{12}$$

*That implies the strict inclusion $\mathcal{F}_u^{\Sigma(B)} \subsetneq \mathcal{F}_u^{B^{\overline{f}}}$. At the terminal time $T$, we have $\mathcal{F}_T^{B^{\overline{y}}} = \mathcal{F}_T^{\Sigma(B)}$.*

*Proof.* Let us fix $u \leq T$. Because $\Sigma(B^{\overline{y}}) = \Sigma(B)$ the inclusion $\Gamma_u(\Sigma(B)) \subset \Gamma_u(B^{\overline{y}})$ holds. Since we deal with Gaussian spaces, we can compute the orthogonal complement and get (12). The assertions on filtrations follow from $\mathcal{F}_u^X = \sigma(\Gamma_u(X))$ for any continuous Gaussian process $X$. □

**Remark 3.1.** Assertion (2) reads also

$$\Gamma_u(B) = \Gamma_u(\Sigma(B)) \oplus \text{Span}\,(\int_0^u f_j(s)dB_s, j \leq N), \text{ for } u \leq T. \tag{13}$$

Thus (12) is a conditional version of (13). Assertion (2) remains valid with $B$ replaced by $B^{\overline{y}}$. The transform $\Sigma$ is not injective since, on the canonical space, the Wiener measure and the associated $\overline{f}$-generalized bridge probability measure have the same image.

In terms of stochastic differential equations, the above assertion (1) means that $(B_u^{\overline{y}}, u \leq T)$ solves the linear equation

$$X_u = \Sigma(B)_u + \int_0^u ds \int_0^s l(s,v)dX_v, \tag{14}$$

with $X_0 = 0$. It is a fact that there are infinitely many other solutions. Indeed, as was shown in Alili-Wu [1], the general solution takes the form

$$X_u = B_u + \sum_{j=1}^N Y_j \int_0^u f_j(s)ds, \tag{15}$$

where $(Y_1, Y_2, ..., Y_N)$ is any sequence of random variables. However for our case and our needs, we can state the following

**Corollary 3.1.** *The unique continuous solution of (14) satisfying the end-point condition*

$$\int_0^T f_j(s)dX_s = y_j, 1 \leq j \leq N \tag{16}$$

*is the process $(B_u^{\overline{y}}, u \leq T)$ given by formula (6).*

*Proof.* The solution has to be of the form (15). For any $1 \leq j \leq N$, condition (16) implies that

$$\begin{aligned} y_j &= \int_0^T f_j(s)dX_s \\ &= \int_0^T f_j(s)dB_s + \sum_{k=1}^N Y_k \int_0^T f_j(r)f_k(r)dr. \end{aligned}$$

Thus, thanks to linear algebra, we have

$$Y_k = \sum_{i=1}^N (y_i - \int_0^T f_i(s)dB_s)\alpha_{ik}, k \leq N, \tag{17}$$

where $\alpha_{ik}$ stands for the $ik^{th}$ element of the inverse matrix of $Q(0,T)$. Hence

$$\begin{aligned}\sum_{j=1}^{N} Y_j \int_0^u f_j(s)ds &= \sum_{i=1}^{N}(y_i - \int_0^T f_i(s)dB_s)\sum_{j=1}^{N}\alpha_{ij}\int_0^u f_j(s) \\ &= \sum_{i=1}^{N}(y_i - \int_0^T f_i(s)dB_s)\psi_i(u)\end{aligned}$$

which is the desired result. □

To a continuous process $(X_s, s \geq 0)$ we may associate, like in the Brownian case, an $\overline{f}$-generalized bridge given by (6) where we replace $B$ by $X$. It is then a natural problem to find all continuous processes on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having the same $\overline{f}$-generalized bridge as that associated to a Brownian motion. For interested readers, we refer to [5] for a related study in a general Markovian setting. In our situation, the following result provides a partial answer.

**Corollary 3.2.** *Let $(X, t \geq 0)$ be a continuous process of the form (15). Then for any $T > 0$ and $\overline{y} \in \mathbb{R}$ the $\overline{f}$-generalized bridges over the interval $[0,T]$ with end-point $\overline{y}$ associated to $X$ and $B$ respectively have identical laws if and only if $(Y_1, Y_2, ..., Y_N)$ is independent of $\mathcal{F}^B_\infty$.*

*Proof.* Let $X$ be a continuous process of the form (15) and fix $T > 0$ and $\overline{y} \in \mathbb{R}$. Conditioning on $\int_0^T f_k(s)dX_s = y_k, k \leq N$ is the same as conditioning on the event given by formula (17). Thus, repeating the same argument as at the end of the proof of corollary 3.1, we see that a necessary and sufficient condition for the generalized bridges to be equal in law is that $(X^{\overline{y}}_u, u \leq T)$ has the same law as $(B^{\overline{y}}_u, u \leq T)$ conditional on the events given in (17). This is true for any $T > 0$ and $\overline{y} \in \mathbb{R}$ if and only if $(Y_1, Y_2, ..., Y_N)$ is independent of $\mathcal{F}^B_\infty$. □

## 4 The canonical decomposition

Introduce the Gaussian bridge $(\overline{x} + \overline{Z}^{\overline{y}}_u, u \leq T)$ over the interval $[0,T]$, in between $\overline{x}$ and $\overline{y}$ associated to the Gaussian process $(\overline{Z})$ defined in the beginning of section 2. We denote its law by $\mathbb{P}^{(T)}_{\overline{x}\to\overline{y}}$. We shall start with studying the equivalence in between $\mathbb{P}^{(T)}_{\overline{x}\to\overline{y}}$ and $\mathbb{P}_{\overline{x}}$ the law of $(\overline{Z}_u, u \leq T)$ started at $\overline{x}$. We need the density $P_{s,t}(\overline{x}, \overline{y})$ of the distribution of the random vector $\overline{x} + \overline{Z}_t - \overline{Z}_s$, for $0 < s < t < \infty$. This is well-known and specified by

$$P_{s,t}(\overline{x}, \overline{y}) = (2\pi \det Q(s,t))^{-N/2} \exp -\frac{1}{2}(\overline{y} - \overline{x})Q^{-1}(s,t)(\overline{y} - \overline{x})^*$$

where $*$ denotes the transpose operator.

**Theorem 4.1.** *For any fixed $s < T$, $\mathbb{P}^{(T)}_{\overline{x}\to\overline{y}|\mathcal{F}_s}$ is equivalent to $\mathbb{P}_{\overline{x}|\mathcal{F}_s}$. The Radon-Nikodym density is given by*

$$d\,\mathbb{P}^{(T)}_{\overline{x}\to\overline{y}\,|\mathcal{F}_s} = \frac{P_{s,T}(\overline{Z}_s, \overline{y})}{P_{s,T}(\overline{x}, \overline{y})}\, d\,\mathbb{P}_{\overline{x}\,|\mathcal{F}_s}. \tag{18}$$

*Proof.* For any $\mathcal{F}_s^{\overline{Z}}$-measurable functional $F$ and any bounded function $f:\mathbb{R}^N\to\mathbb{R}$, we have on one hand

$$\mathbb{E}_{\overline{x}}\big[F_s f(\overline{Z}_T)\big]=\int_R dy_1\int_R dy_2...\int_R dy_N f(\overline{y})P_{0,T}(\overline{x},\overline{y})\mathbb{E}^{(T)}_{\overline{x}\to\overline{y}}\big[F_s\big], \tag{19}$$

where $\mathbb{E}^{(T)}_{\overline{x}\to\overline{y}}$ stands for the expectation under the probability measure $\mathbb{P}^{(T)}_{\overline{x}\to\overline{y}}$. On the other hand, since the Brownian motion $B$ has independent increments, we get

$$\mathbb{E}_{\overline{x}}\big[F_s f(\overline{Z}_T)\big]=\int_R dy_1\int_R dy_2...\int_R dy_N f(\overline{y})\mathbb{E}_{\overline{x}}\big[F_s P_{s,T}(\overline{Z}_s,\overline{y})\big]. \tag{20}$$

The assertion follows by disintegrating the above formulas. □

The following result is a straightforward consequence of theorem 4.1 and Girsanov Theorem.

**Corollary 4.1.** *The canonical decomposition of the generalized bridge $(\overline{Z}^{\overline{y}}_u, u\le T)$ is given by*

$$Z_s^{(k)}=x_k+W_s^{(k)}+\sum_{i=1}^N\sum_{j=1}^N\int_0^s dr\alpha_{i,j}(r)f_j(r)f_k(r)\big(y_i-Z^{(i)}(r)\big) \tag{21}$$

*for $s\le T$, where $\alpha_{ij}(r)$ denotes the $ij^{th}$ element of the inverse of the covariance matrix $Q(r,T)$, $W_s^{(k)}=\int_0^s f_k(u)dW_u$ for $1\le k\le N$ and $(W_u,u\le T)$ is a standard Brownian motion.*

The essence of the above corollary is the following translation in terms of the underlying bridge.

**Proposition 4.1.** *The canonical decomposition of the $\overline{f}$-generalized bridge $(B^{\overline{y}}_u, u\le T)$, as a semi-martingale in its own filtration, is given by*

$$X_s=W_s+\sum_{i=1}^N\sum_{j=1}^N\int_0^s dr\alpha_{i,j}(r)f_j(r)\big(y_i-\int_0^r f_i(u)dX_u\big), s\le T. \tag{22}$$

Now, we provide an explicit construction of a Brownian motion that is adapted to the $\overline{f}$-generalized bridge's natural filtration. This gives a better interpretation and understanding of the latter decompositions.

**Theorem 4.2.** *The process $(W_s, s\le T)$ defined by*

$$W_s=B_s-\sum_{i=1}^N\sum_{j=1}^N\int_0^s dr\alpha_{i,j}(r)f_j(r)\int_r^T f_i(u)dB_u, \tag{23}$$

*is a standard Brownian motion independent of $\sigma(\overline{Z}_T)$. It is in fact a $\mathcal{F}^{B^{\overline{y}}}$-Brownian motion. Conditioning on $\overline{Z}_T=\overline{y}$, we get corollary 4.1.*

*Proof.* First, we show that $(W_s, s\le T)$ is a standard Brownian motion. Since it is a continuous Gaussian process we only need to check that for any $0\le s\le t\le T$ we have $E(W_sW_t)=s$. To simplify notations, set

$$H_s=\sum_{i=1}^N\sum_{j=1}^N\int_0^s dr\alpha_{i,j}(r)f_j(r)\int_r^T f_i(u)dB_u.$$

Then we see that $E(W_sW_t) = s - E(B_sH_t) - E(B_tH_s) + E(H_sH_t)$.
Now, we have

$$E(B_sH_t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^s dr\alpha_{i,j}(r)f_j(r)\int_r^s f_i(u)du,$$

and

$$E(B_tH_s) = \sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^s dr\alpha_{i,j}(r)f_j(r)\int_r^t f_i(u)du.$$

For the last term, we recall that by the definition of the matrix $(\alpha_{i,j}(r))_{i,j\leq N}$, we have

$$\sum_{i=1}^{N}\alpha_{i,j}(r)\int_r^T f_i(u)f_k(u)du = \delta_{jk} \tag{24}$$

where $\delta$ is the Kronecker function. Hence

$$\begin{aligned} E(H_sH_t) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{p=1}^{N}\sum_{k=1}^{N}\int_0^s dr\alpha_{i,j}(r)f_j(r)\int_0^t dx\alpha_{p,k}(x)f_k(x))\int_{r\vee x}^T f_i(u)f_p(u)du \\ &= \sum_{j=1}^{N}\sum_{p=1}^{N}\sum_{k=1}^{N}\int_0^s drf_j(r)\int_0^r dx\alpha_{p,k}(x)f_k(x))\delta_{pj} \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\int_0^s dr\alpha_{i,j}(r)f_j(r)\int_r^t dxf_k(x))\delta_{ik} \\ &= E(B_sH_t) + E(B_tH_s), \end{aligned}$$

which shows that $E(W_sW_t) = s$. We conclude that $W$ is a Brownian motion. For any $s \leq T$ and $k \leq N$, we have

$$E(W_sZ^{(k)}) = \int_0^s f_k(r)dr - \sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^s dr\alpha_{i,j}(r)f_j(r)\int_r^T f_i(u)f_k(u)du$$

which vanishes by virtue of (24). Thus $(W_s, s \leq T)$ is independent of $\sigma(\overline{Z}_T)$. Let us determine the Gaussian random variables of the form $\int_0^T g(u)dB_u$, where $g \in L^2_{loc}([0,T])$, that are orthogonal to $\Gamma_T(W)$. For this to be true it necessary and sufficient that the function $g$ satisfies the integral equation

$$\int_0^s g(r)dr = \sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^s dr\alpha_{i,j}(r)f_j(r)\int_r^T f_i(u)g(u)du. \tag{25}$$

One can show that relation (25) is equivalent to an $N^{th}$-order linear differential equation. But we have already checked that $f_1, f_2, ..., f_N$ are $N$ linearly independent solutions. Thus the orthogonal complement of $\Gamma_T(W)$ in $\Gamma_T(B)$ is precisely $Span(\overline{Z}_T)$, which ends the proof of our result. $\square$

# 5 Application to conditioned stochastic differential equations

We will now apply the above canonical decomposition to a construction of some conditioned stochastic differential equations which are locally equivalent to the Brownian motion. Following the terminology of Baudoin [2], a conditioning on the Wiener space is a triplet $(T, Y, \nu)$ where $T$ corresponds to the horizon time required in the conditioning, $Y$ is a functional of the paths and $\nu$ is a probability measure. A probability measure $\mathbb{P}^\nu$ associated with the conditioning $(T, Y, \nu)$ is a probability having the following two properties

**(i)** $\mathbb{P}^\nu$ and $\mathbb{P}$ coincide on the events which are independent of $Y$.

**(ii)** The law of $Y$ under $\mathbb{P}^\nu$ is $\nu$.

The semi-martingale decomposition of the canonical process under the probability $\mathbb{P}^\nu$ provides then a conditioned stochastic differential equation (in abbreviate CSDE). Formally, a CSDE with conditioning $(T, Y, \nu)$, is a stochastic differential equation of the form

$$X_t = \beta_t + \int_0^t F(r; X_r, r \leq s)ds, \;\; t < T \tag{26}$$

where $(\beta_u, u \geq 0)$ is a standard Brownian motion, $F$ is an adapted functional sufficiently smooth in order to ensure the existence of a weak solution to (26). In comparison to the above discussion, the triplet $(T, Y, \nu)$ is a conditioning i.e. the functional $Y \circ X$ is $\mathcal{F}_T^X$-measurable and its law is $\nu$. In this context, let us fix $T > 0$ and $\overline{Y} = \overline{Z}_T$, that is $Y_k = \int_0^T f_k(s)dX_s, k \leq N$, and denote by $\nu(d\overline{y})$ the distribution of $\overline{Y}$. It is clear that the law of the canonical process under $\mathbb{P}^\nu$ is identical to its associated $\overline{f}$-generalized bridge over the interval $[0, T]$ with endpoint $\overline{Y}$ (which is now random) under $\mathbb{P}$. A realization of its law is provided by (6) whereas a non-canonical decomposition of the canonical process under $\mathbb{P}^\nu$ is given by proposition (4.1), of course with $\overline{y}$ replaced by $\overline{Y}$.
The canonical decomposition might be computed from (18) in the following way. To start with, we replace $\overline{y}$ by $\overline{Y}$ in the absolute continuity relationship (18), we then condition on $\overline{Y}$ and finally we apply Girsanov Theorem. The resulting CSDE takes the form (26) with

$$F(s; X_r, r \leq s) = \sum_{i=1}^N \sum_{j=1}^N \alpha_{i,j}(s) f_j(s) G(s; X_r, r \leq s), \tag{27}$$

where

$$G(s; X_r, r \leq s) = \frac{\int_{\mathbb{R}^N} \nu(d\overline{y}) \kappa_{s,T}^{\overline{x}}(\overline{Z}_s^X, \overline{y}) \big(y_i - \int_0^s f_i(u)dX_u\big)}{\int_{\mathbb{R}^N} \nu(d\overline{y}) \kappa_{s,T}^{\overline{x}}(\overline{Z}_s^X, \overline{y})}$$

with

$$\overline{Z}_s^X = \Big(\int_o^s f_i(r)dX_r, i \leq N\Big)$$

and

$$\kappa_{s,t}^{\overline{x}}(\overline{z}, \overline{y}) = \frac{P_{s,t}(\overline{z}, \overline{y})}{P_{s,t}(\overline{x}, \overline{y})}.$$

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