*Elect. Comm. in Probab.* **11** (2006), 118–128

**ELECTRONIC COMMUNICATIONS in PROBABILITY**

# RECURRENCE AND TRANSIENCE OF EXCITED RANDOM WALKS ON $\mathbb{Z}^d$ AND STRIPS

Dedicated to the memory of Prof. Hans G. Kellerer (1934 - 2005)

MARTIN P. W. ZERNER
*Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany*
email: `martin.zerner@uni-tuebingen.de`

*Submitted* December 30, 2005, *accepted in final form* June 22, 2006

AMS 2000 Subject classification: 60K35, 60K37, 60J10
Keywords: Excited Random Walk, Recurrence, Self-Interacting Random Walk, Transience

*Abstract*
We investigate excited random walks on $\mathbb{Z}^d, d \geq 1$, and on planar strips $\mathbb{Z} \times \{0, 1, \ldots, L-1\}$ which have a drift in a given direction. The strength of the drift may depend on a random i.i.d. environment and on the local time of the walk. We give exact criteria for recurrence and transience, thus generalizing results by Benjamini and Wilson for once-excited random walk on $\mathbb{Z}^d$ and by the author for multi-excited random walk on $\mathbb{Z}$.

## 1 Introduction

We consider excited random walks (ERWs), precisely to be defined below, which move on either $\mathbb{Z}^d$ or strips, i.e. which have state space

$$Y = \mathbb{Z}^d \quad (d \geq 1) \quad \text{or} \quad Y = \mathbb{Z} \times \{0, 1, \ldots, L-1\} \subset \mathbb{Z}^2 \quad (L \geq 2).$$

In general, ERWs are not Markovian. Instead, the transition probabilities may depend on how often the walk has previously visited its present location and additionally on the environment at this location.

To be more precise, let us first fix two quantities for the rest of the paper: A direction $\ell$ and the so-called ellipticity constant $\kappa$. In the case $Y = \mathbb{Z}$ or $Y = \mathbb{Z} \times \{0, 1, \ldots, L-1\}$ we always choose $\ell = e_1 \in Y$ to be the first standard unit vector. In the case $Y = \mathbb{Z}^d$, $d \geq 2$, we let $\ell \in \mathbb{R}^d$ be any direction with $|\ell|_1 = 1$. The ellipticity constant $\kappa \in (0, 1/(2d)]$ will be a uniform lower bound for the probability of the walk to jump from $x$ to any nearest neighbor of $x$. Then an environment $\omega$ for an ERW is an element of

$$\Omega := \Big\{ \Big( \big( (\omega(x,e,i))_{|e|=1} \big)_{i\geq 1} \Big)_{x\in Y} \in [\kappa, 1-\kappa]^{2d\times\mathbb{N}\times Y} \ \Big| \ \forall x\in Y\ \forall i\geq 1 \sum_{e\in\mathbb{Z}^d,|e|=1} \omega(x,e,i) = 1, \sum_{e\in\mathbb{Z}^d,|e|=1} \omega(x,e,i)\, e\cdot\ell \geq 0 \Big\}.$$

Here in the case of $Y$ being a strip, $d = 2$ and $x + e$ is modulo $L$ in the second coordinate.
An ERW starting at $x \in Y$ in an environment $\omega \in \Omega$ is a $Y$-valued process $(X_n)_{n\geq 0}$ on some suitable probability space $(\Omega', \mathcal{F}, P_{x,\omega})$ for which the history process $(H_n)_{n\geq 0}$ defined by $H_n := (X_m)_{0\leq m\leq n} \in Y^{n+1}$ is a Markov chain which satisfies $P_{x,\omega}$-a.s.

$$\begin{aligned} P_{x,\omega}[X_0 = x] &= 1, \\ P_{x,\omega}[X_{n+1} = X_n + e \mid H_n] &= \omega(X_n, e, \#\{m \leq n \mid X_m = X_n\}). \end{aligned}$$

Thus $\omega(x,e,i)$ is the probability to jump upon the $i$-th visit to $x$ from $x$ to $x+e$. In the language introduced in [Ze05], an environment $\omega \in \Omega$ consists of infinite sequences of cookies attached to each site $x \in Y$. The $i$-th cookie at $x$ is the transition vector $(\omega(x,e,i))_{|e|=1}$ to the neighbors $x+e$ of $x$. Each time the walk visits $x$ it removes the first cookie from the sequence of cookies at $x$ and then jumps according to this cookie to a neighbor of $x$. Note that the assumption $\sum_e \omega(x,e,i)e\cdot\ell \geq 0$ means that we allow only cookies which create a non-negative drift in direction $\ell$. A model in which different sites may induce drift into opposite directions has been studied in [ABK05].
The model described above generalizes ERW as introduced by Benjamini and Wilson [BW03]. Their walk, which we will call BW-ERW, is an ERW on $\mathbb{Z}^d$, $d \geq 1$, in the environment $\omega$ given by $\omega(x,e,i) = 1/(2d)$ for all $(x,e,i)$ with the only exception that $\omega(x,\pm e_1, 1) = 1/(2d) \pm \varepsilon$, where $0 < \varepsilon < 1/(2d)$ is fixed. Thus on the first visit to any site $x$, BW-ERW steps to $x \pm e_1$ with probability $1/(2d) \pm \varepsilon$ and to all the other neighboring sites $x+e$ with probability $1/(2d)$, while on any subsequent visit to $x$ a neighbor is chosen uniformly at random. A main result of [BW03] is the following.

**Theorem A** (see [BW03]) *BW-ERW on $\mathbb{Z}^d$, $d \geq 2$, is transient in direction $e_1$, i.e. $X_n \cdot e_1 \to \infty$ almost surely as $n \to \infty$.*

Besides this it is also shown in [BW03] that BW-ERW has positive liminf speed if $d \geq 4$. Kozma extended this result to $\mathbb{Z}^3$ in [K03] and very recently even to $\mathbb{Z}^2$ in [K05].
The proof of Theorem A presented in [BW03] uses the following strategy. Firstly, BW-ERW $(X_n)_n$ is coupled in the canonical way to a simple symmetric random walk $(Y_n)_n$ such that $0 \leq (X_n - Y_n)\cdot e_1$ is non-decreasing in $n$ and $X_n \cdot e_i = Y_n \cdot e_i$ for all $n$ and $i \geq 2$. Then so-called tan points are considered, which are points $x$ to the right of which no other point has been visited prior to $x$. Any time $n$ at which the walk reaches a new tan point is called a tan time. It is easy to see that any tan time for $(Y_n)_n$ is also a tan time for $(X_n)_n$. Moreover, at any tan time, $(X_n)_n$ consumes a cookie and thus gets a drift to the right, i.e. into direction $\ell = e_1$. Then, roughly speaking, using a lower bound on the number of tan points for $(Y_n)_n$, one gets a lower bound on the number of cookies consumed by $(X_n)_n$, which Benjamini and Wilson show to be sufficient to ensure transience to the right.
We do not see how this line of proof could be adapted to other settings, in which for instance the excitement occurs not on the first but only on the second visit to a site and points into a direction other than a coordinate direction. For this reason we suggest in the present paper an

alternative method of proof. It is based on martingales and on the environment viewed from the particle and applies to BW-ERW as well as to other more general environments $\omega$ which are sampled from $\Omega$ according to a probability measure $\mathbb{P}$ on $\Omega$ such that the family

$$(\omega(x,\cdot,\cdot))_{x\in Y} \quad \text{is i.i.d. under } \mathbb{P}. \tag{1}$$

Throughout the paper we will assume (1) and denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$. Note that we do not assume independence between different cookies at the same site nor between transition probabilities to different neighbors of the same site, but only between cookies at different sites. An important quantity will be the total drift $\delta^x$ in direction $\ell$ of all the cookies stored at site $x \in Y$, i.e.

$$\delta^x(\omega) := \sum_{i\geq 1, |e|=1} \omega(x,e,i)\, e\cdot\ell\,.$$

Note that by definition of $\Omega$, $\delta^x(\omega) \geq 0$ for all $x \in Y$ and $\omega \in \Omega$. We shall generalize Theorem A as follows.

**Theorem 1** *Let $d \geq 2$, $Y = \mathbb{Z}^d$ and $\mathbb{E}[\delta^0] > 0$. Then the walk is for $\mathbb{P}$-almost all $\omega$ transient in direction $\ell$, i.e. $P_{0,\omega}$-a.s. $X_n \cdot \ell \to \infty$ as $n \to \infty$.*

The technique of proof improves methods used in [Ze05] to show the following result for $d = 1$. Some simulation studies for $Y = \mathbb{Z}$ can be found in [AR05].

**Theorem B** (see [Ze05, Theorem 12]) *Let $Y = \mathbb{Z}$. Then for $\mathbb{P}$-almost all environments $\omega \in \Omega$, $(X_n)_n$ is recurrent, i.e. returns $P_{0,\omega}$-a.s. (infinitely often) to its starting point, if and only if $\mathbb{E}[\delta^0] \leq 1$.*

In fact, [Ze05, Theorem 12] is more general since it does not need any ellipticity condition and allows the environment to be stationary and ergodic only instead of i.i.d.. In the present paper we shall generalize Theorem B to strips as follows.

**Theorem 2** *Let $Y = \mathbb{Z}$ and $L = 1$ or $Y = \mathbb{Z} \times \{0, \ldots, L-1\}$ for some $L \geq 2$. If $\mathbb{E}[\delta^0] > 1/L$ then the walk is for $\mathbb{P}$-almost all $\omega$ transient in direction $e_1$. If $\mathbb{E}[\delta^0] \leq 1/L$ then the walk is for $\mathbb{P}$-almost all $\omega$ recurrent, and moreover $P_{0,\omega}$-a.s. $\limsup_{n\to\infty} X_n \cdot e_1 = \infty$ and $\liminf_{n\to\infty} X_n \cdot e_1 = -\infty$.*

So if the strip is made wider and wider while the distribution of $\omega(x,\cdot,\cdot)$ is kept fixed, the walk will eventually become transient if $\mathbb{E}[\delta^0] > 0$. This provides some additional support for Theorem 1.

# 2 Preliminaries

For $z \in \mathbb{R}, n \in \mathbb{N} \cup \{\infty\}$ we let

$$D_n^z := \sum_{x\in\mathcal{S}_z} \sum_{i=1}^{\#\{m<n \mid X_m = x\}} \sum_{|e|=1} \omega(x,e,i) e\cdot\ell$$

denote the drift absorbed by the walk by time $n$ while visiting the slab $\mathcal{S}_z := \{x \in Y \mid z \leq x\cdot\ell < z+1\}$. Then $D_n := \sum_{z\in\mathbb{Z}} D_n^z$ is the total drift encountered by the walk up to time $n$. Observe that $D_n^z \geq 0$ and therefore also $D_n \geq 0$ for all $\omega \in \Omega$ and all paths $(X_m)_m$.

By standard arguments, for any $\omega \in \Omega$ the process $(M_n)_{n\geq 0}$ defined by

$$M_n := X_n \cdot \ell - D_n \tag{2}$$

is a martingale under $P_{0,\omega}$ with respect to the filtration generated by $(X_n)_{n\geq 0}$. Indeed, (2) is just the Doob-Meyer decomposition of the submartingale $(X_n \cdot \ell)_n$.
In the setting considered in [BW03] and [Ze05] part of the following fact was achieved by coupling the ERW to a simple symmetric random walk staying always to the left of the ERW. For the present more general setting we need a different argument.

**Lemma 3** *Let $\omega \in \Omega$. Then $P_{0,\omega}$-a.s.*

$$\liminf_{n\to\infty} X_n \cdot \ell \in \{-\infty, +\infty\} \quad \textit{and} \quad \limsup_{n\to\infty} X_n \cdot \ell = +\infty.$$

*In particular, for all $x \geq 0$,*

$$T_x := \inf\{n \geq 0 \mid X_n \cdot \ell \geq x\} < \infty \quad P_{0,\omega}\text{-a.s.}.$$

*Proof.* Let $c \in \mathbb{R}$. Then on the event $\{X_n \cdot \ell \in [c, c+1]$ infinitely often$\}$, due to ellipticity and the Borel-Cantelli lemma, almost surely $X_n \cdot \ell < c$ infinitely often. This implies the first statement.
For the statement about lim sup, let $x \geq 0$. Since $D_n \geq 0$ for all $n$, the martingale $(M_{n\wedge T_x})_n$ is bounded from above by $x$ and hence converges $P_{0,\omega}$-a.s. to a finite limit as $n \to \infty$. Therefore, it suffices to show that $(M_n)_n$ itself $P_{0,\omega}$-a.s. does not converge, because then the convergence of $(M_{n\wedge T_x})_n$ can only be due to $T_x$ being $P_{0,\omega}$-a.s. finite.
So if $(M_n)_n$ did converge, then $|(X_{n+1} - X_n) \cdot \ell - (D_{n+1} - D_n)| \to 0$ as $n \to \infty$. However, this is impossible. Indeed, let $e_0 \in Y$ be a unit vector which maximizes $e_0 \cdot \ell$. Then due to ellipticity and the Borel-Cantelli lemma, $|(X_{n+1} - X_n) \cdot \ell| = e_0 \cdot \ell$ infinitely often, whereas, again by ellipticity, for all $n$ and some random $i = i(n) \in \mathbb{N}$,

$$\begin{aligned}
|D_{n+1} - D_n| &= \left| \sum_e \omega(X_n, e, i) e \cdot \ell \right| \\
&\leq \left( |\omega(X_n, e_0, i) - \omega(X_n, -e_0, i)| + \sum_{e \neq \pm e_0} \omega(X_n, e, i) \right) e_0 \cdot \ell \\
&= (1 - 2\left(\omega(X_n, e_0, i) \wedge \omega(X_n, -e_0, i)\right))\ e_0 \cdot \ell,
\end{aligned}$$

which is at most $(1 - 2\kappa) e_0 \cdot \ell$. □

The next result bounds the number of cookies consumed by the walk before it reaches a certain level.

**Lemma 4** *For all $\omega \in \Omega$ and all $x \geq 0$, $E_{0,\omega}[D_{T_x}] \leq x + 1$.*

*Proof.* By the Optional Stopping Theorem for all $n \in \mathbb{N}$, $0 = E_{0,\omega}[M_{T_x \wedge n}]$ and consequently by (2), $E_{0,\omega}[D_{T_x\wedge n}] = E_{0,\omega}[X_{T_x\wedge n} \cdot \ell] \leq x + \max_e e \cdot \ell \leq x + 1$. The statement now follows from monotone convergence. □

Now we introduce some notation taken from [Ze05] for the cookie environment left over by the walk. For $\omega \in \Omega$ and any finite sequence $(x_n)_{n\leq m}$ in $Y$ we define $\psi(\omega, (x_n)_{n\leq m}) \in \Omega$ by

$$\psi(\omega, (x_n)_{n\leq m})(x, e, i) := \omega\left(x, e, i + \#\{n < m \mid x_n = x\}\right).$$

This is the environment created by the ERW by following the path $(x_n)_{n\le m}$ and removing all the first cookies encountered, except for the last visit to $x_m$. Finiteness of $T_1$, guaranteed by Lemma 3, implies that the Markov transition kernel

$$R(\omega,\omega') := P_{0,\omega}\left[\theta^{X_{T_1}}\left(\psi\left(\omega, H_{T_1}\right)\right) = \omega'\right]$$

for $\omega,\omega' \in \Omega$ is well-defined. Here $\theta^z$ denotes the spatial shift of the environment by $z$, i.e. $\theta^z(\omega(x,\cdot,\cdot)) := \omega(x+z,\cdot,\cdot)$. The probability measure $R(\omega,\cdot)$ is the distribution of the modified environment $\omega$ viewed from the particle at time $T_1$. Note that it is supported on those countably many $\omega' \in \Omega$, which are obtained from $\omega$ by removing finitely many cookies from $\omega$.

**Lemma 5** *$R$ is weak Feller, i.e. convergence w.r.t. the product topology on $\Omega$ of $\omega_n \in \Omega$ towards $\omega \in \Omega$ as $n \to \infty$ implies*

$$\left|\sum_{\omega'\in\Omega} R(\omega_n,\omega')f(\omega') - \sum_{\omega'\in\Omega} R(\omega,\omega')f(\omega')\right| \longrightarrow 0 \quad \text{as } n\to\infty \tag{3}$$

*for any bounded continuous function $f:\Omega\to\mathbb{R}$.*

Note that the assumption of boundedness of $f$ is redundant since $\Omega$ is compact. Also note that due to the discreteness of $R(\omega,\cdot)$ only countably many terms in the sums in (3) do not vanish.

*Proof.* Let $\varepsilon > 0$. Since $T_1$ is $P_{0,\omega}$-a.s. finite due to Lemma 3, there is some finite $t$ such that

$$\varepsilon \;>\; P_{0,\omega}[T_1 > t] \;=\; 1 - \sum_{\pi\in\Pi_t} P_{0,\omega}\left[(X_m)_m \text{ follows } \pi\right], \tag{4}$$

where $\Pi_t$ denotes the set of nearest-neighbor paths $\pi$ starting at the origin and ending at time $T_1(\pi)$ with $T_1 \le t$. Since $\omega_n \to \omega$,

$$P_{0,\omega_n}\left[(X_m)_m \text{ follows } \pi\right] \longrightarrow P_{0,\omega}\left[(X_m)_m \text{ follows } \pi\right] \qquad \text{as } n\to\infty \tag{5}$$

for all $\pi \in \Pi_t$. Therefore, by (4),

$$P_{0,\omega_n}[T_1 > t] < \varepsilon \quad \text{for } n \text{ large.} \tag{6}$$

Now partition $\Pi_t$ into sets $\Pi_t^z$ according to the final point $z$ of the paths. Then the left-hand side of (3) can be bounded from above by

$$\begin{aligned} \sum_{z\in Y}\sum_{\pi\in\Pi_t^z} \Big| & P_{0,\omega_n}[(X_m)_m \text{ follows } \pi] f\left(\theta^z(\psi(\omega_n,\pi))\right) \\ & - P_{0,\omega}[(X_m)_m \text{ follows } \pi] f\left(\theta^z(\psi(\omega,\pi))\right)\Big| \\ & + cP_{0,\omega}[T_1 > t] + cP_{0,\omega_n}[T_1 > t], \end{aligned} \tag{7}$$

where $c$ is a bound on $|f|$. Since $f$ is continuous, $f\left(\theta^z(\psi(\omega_n,\pi))\right)$ converges to $f\left(\theta^z(\psi(\omega,\pi))\right)$ as $n\to\infty$. Together with (4), (5) and (6) this shows that the whole expression in (7) is less than $2c\varepsilon$ for $n$ large. $\square$

**Lemma 6** *There is a probability measure $\widetilde{\mathbb{P}}$ on $\Omega$ which is invariant under $R$ and under which*

$$(\omega(x,\cdot,\cdot))_{x\in Y, x\cdot\ell\geq 0} \quad \text{has the same distribution as under } \mathbb{P}. \tag{8}$$

*Proof.* Being a closed subset of the compact set $[\kappa, 1-\kappa]^{2d\times\mathbb{N}\times Y}$, $\Omega$ is compact, too. Consequently, the set of all probability measures on $\Omega$ is compact with respect to weak convergence as well. Since the set $\mathcal{M}$ of all probability measures on $\Omega$ under which (8) holds is a closed subset of this compact set, $\mathcal{M}$ is compact, too. Moreover, observe that $MR \in \mathcal{M}$ for all $M \in \mathcal{M}$ since the part of the environment $\psi(\omega, H_{T_1})$ which is to the right of $X_{T_1}$ has by time $T_1$ not been touched by the walk yet and is therefore still i.i.d.. Hence, since $R$ is weak Feller due to Lemma 5 the statement follows from standard arguments, see e.g. [MT96, Theorem 12.0.1 (i)]. □

For the remainder of this paper we fix $\widetilde{\mathbb{P}}$ according to Lemma 6 and let $\widetilde{\mathbb{E}}$ be its expectation operator. We also introduce the annealed probability measures $P_0 = \mathbb{P}\times P_{0,\omega}$ and $\widetilde{P}_0 = \widetilde{\mathbb{P}}\times P_{0,\omega}$ with expectation operators $E_0$ and $\widetilde{E}_0$, respectively, which one gets by averaging the so-called quenched measure $P_{0,\omega}$ over $\mathbb{E}$ and $\widetilde{\mathbb{E}}$, respectively, i.e. $P_0[\cdot] = \mathbb{E}[P_{0,\omega}[\cdot]]$ and $\widetilde{P}_0[\cdot] = \widetilde{\mathbb{E}}[P_{0,\omega}[\cdot]]$. The following statement is similar to [Ze05, Lemma 11].

**Lemma 7** *If $Y$ is a strip or $\mathbb{Z}$ then $\widetilde{E}_0[D^0_\infty] \leq 1$. If $Y = \mathbb{Z}^d$, $d\geq 2$, then $\widetilde{E}_0[D^0_\infty] \leq 2$.*

*Proof.* Consider the stopping times defined by $\tau_0 := 0$ and $\tau_{n+1} := \inf\{n > \tau_n \ :\ X_n\cdot\ell \geq X_{\tau_n}\cdot\ell + 1\}$ for $n\geq 0$. Note that

$$\tau_n = T_n \quad \text{if } Y \text{ is a strip or } \mathbb{Z} \text{ and} \quad \tau_n \leq T_{2n} \quad \text{if } Y = \mathbb{Z}^d,\ d\geq 2, \tag{9}$$

because in the second case, due to $|\ell|_1 = 1$, $X_{\tau_{n+1}}\cdot\ell \leq X_{\tau_n}\cdot\ell + 2$. Since the slabs $\mathcal{S}_{X_{\tau_n}\cdot\ell}$, $n\geq 0$, are disjoint, we have $D_{T_K} \geq \sum_{n\geq 0} D^{X_{\tau_n}\cdot\ell}_{T_K}$ for all $K\geq 0$. Therefore, for all $0\leq k < K/2$,

$$D_{T_K} \geq \sum_{n=0}^{m} D^{X_{\tau_n}\cdot\ell}_{\tau_{n+k}}, \tag{10}$$

where $m = m(K,k) := K-k$ for $Y$ being a strip or $\mathbb{Z}$ and $m(K,k) := \lfloor K/2\rfloor - k$ for $Y = \mathbb{Z}^d$, $d\geq 2$. Indeed, in both cases $\tau_{n+k} \leq T_K$ for all $n\leq m$ due to (9). Consequently, by Lemma 4 and (10),

$$K+1 \geq \widetilde{E}_0[D_{T_K}] \geq \sum_{n=0}^{m} \widetilde{E}_0\left[D^{X_{\tau_n}\cdot\ell}_{\tau_{n+k}}\right]. \tag{11}$$

By conditioning on the history up to time $\tau_n$ and using the strong Markov property we get

$$\begin{aligned} \widetilde{E}_0\left[D^{X_{\tau_n}\cdot\ell}_{\tau_{n+k}}\right] &= \widetilde{\mathbb{E}}\left[E_{0,\omega}\left[E_{0,\theta^{X_{\tau_n}}(\psi(\omega,H_{\tau_n}))}[D^0_{\tau_k}]\right]\right] \\ &= \widetilde{\mathbb{E}}\left[\sum_{\omega'\in\Omega} E_{0,\omega}\left[E_{0,\omega'}[D^0_{\tau_k}], \theta^{X_{\tau_n}}\left(\psi(\omega,H_{\tau_n})\right) = \omega'\right]\right] \\ &= \widetilde{\mathbb{E}}\left[\sum_{\omega'\in\Omega} E_{0,\omega'}[D^0_{\tau_k}]R^n(\omega,\omega')\right] \ = \ \widetilde{\mathbb{E}}\left[E_{0,\omega}[D^0_{\tau_k}]\right], \end{aligned} \tag{12}$$

where $R^n$ denotes the $n$-th iteration of $R$ and the last identity holds due to $\widetilde{\mathbb{P}}R^n = \widetilde{\mathbb{P}}$. Consequently, we obtain from (11) that $\widetilde{E}_0[D^0_{\tau_k}] \leq (K+1)/m(K,k)$. Letting $K\to\infty$ gives, for all $k\geq 0$, $\widetilde{E}_0[D^0_{\tau_k}] \leq 1$ for the strip and $\mathbb{Z}$ and $\widetilde{E}_0[D^0_{\tau_k}] \leq 2$ for $\mathbb{Z}^d$, $d\geq 2$. Monotone convergence as $k\to\infty$ then yields the claim. □

# 3 Transience on $\mathbb{Z}^d$ and strips

We denote by

$$A_\ell := \left\{ \lim_{n\to\infty} X_n \cdot \ell = +\infty \right\} \quad \text{and} \quad B_\ell := \{\forall n \geq 1 \ X_n \cdot \ell > X_0 \cdot \ell\}$$

the event that the walk tends to the right and the event that it stays forever strictly to the right of its initial point, respectively. As a preliminary result, we are now going to prove Theorem 1 with $\widetilde{\mathbb{P}}$ instead of $\mathbb{P}$.

**Lemma 8** *Let $d \geq 2$, $Y = \mathbb{Z}^d$ and $\mathbb{E}[\delta^0] > 0$. Then $\widetilde{P}_0[A_\ell] = 1$.*

*Proof.* On $A_\ell^c$, $X_n \cdot \ell$ changes sign $\widetilde{P}_0$-a.s. infinitely often due to Lemma 3. Therefore, because of ellipticity, on $A_\ell^c$ it also visits $\widetilde{P}_0$-a.s. infinitely many sites in the slab $\mathcal{S}_0$. Among these sites $x$ there are $\widetilde{P}_0$-a.s. infinitely many ones with $\sum_{|e|=1, i\leq I} \omega(x,e,i)e \cdot \ell > \varepsilon$ for some $\varepsilon > 0$ and some finite $I$ due to the assumption of independence in the environment and $\mathbb{E}[\delta^0] > 0$. Again by ellipticity, on $A_\ell^c$, $\widetilde{P}_0$-a.s. infinitely many of those sites will be visited at least $I$ times. Indeed, after the first visit to any site $x$ the walk has a chance of at least $\kappa^{2I}$ to visit that site $I$ times by just jumping $2I$ times back and forth between $x$ and one of its neighbors. This yields that on $A_\ell^c$, $\widetilde{P}_0$-a.s. $D_\infty^0 = \infty$, which would contradict Lemma 7 unless $\widetilde{P}_0[A_\ell^c] = 0$. $\square$

The following type of result is standard, see e.g. [Se94, Lemma 1], [SzZe99, Proposition 1.2] and [Ze05, Lemma 8].

**Lemma 9** *Let $\omega \in \Omega$ such that $P_{0,\omega}[A_\ell] > 0$. Then $P_{0,\omega}[A_\ell \cap B_\ell] > 0$.*

*Proof.* By assumption there is a finite nearest-neighbor path $\pi_1$ starting at 0 and ending at some $a$ with $a \cdot \ell > d$ such that with positive $P_{0,\omega}$-probability the walk first follows $\pi_1$ and then stays to the right of $a$, while tending to the right, i.e.

$$P_{0,\omega}[(X_n)_n \text{ follows } \pi_1] \ P_{a,\psi(\omega,\pi_1)}[A_\ell \cap B_\ell] > 0, \tag{13}$$

see Figure 1. In particular, the second factor in (13) is positive. Now on $A_\ell$, the walk can visit sites on the path $\pi_1$ only finitely often. Therefore, there is another path $\pi_2$ of length $m_2$ entirely to the right of $a$ which starts at $a$ and ends at some $b$ such that

$$\begin{aligned} 0 &< P_{a,\psi(\omega,\pi_1)}[\{(X_n)_n \text{ follows } \pi_2\} \cap \{\forall n \geq m_2 \ X_n \notin \pi_1\} \cap A_\ell \cap B_\ell] \\ &\leq P_{b,\psi(\omega,(\pi_1,\pi_2))}[\{\forall n > 0 \ X_n \cdot \ell > a \cdot \ell, X_n \notin \pi_1\} \cap A_\ell]. \end{aligned}$$

However, on the event that the walk never visits $\pi_1$ the walk does not feel whether it moves in the environment $\psi(\omega,(\pi_1,\pi_2))$ or $\psi(\omega,\pi_2)$. Therefore,

$$0 < P_{b,\psi(\omega,\pi_2)}[\{\forall n > 0 \ X_n \cdot \ell > a \cdot \ell\} \cap A_\ell].$$

Since $P_{a,\omega}[(X_n)_n \text{ follows } \pi_2] > 0$ due to ellipticity, we get from this

$$\begin{aligned} 0 &< P_{a,\omega}[(X_n)_n \text{ follows } \pi_2] \ P_{b,\psi(\omega,\pi_2)}[\{\forall n > 0 \ X_n \cdot \ell > a \cdot \ell\} \cap A_\ell] \\ &= P_{a,\omega}[\{(X_n)_n \text{ follows } \pi_2\} \cap A_\ell \cap B_\ell] \ \leq \ P_{a,\omega}[A_\ell \cap B_\ell]. \end{aligned} \tag{14}$$

Now because of $a \cdot \ell > d$ there is a nearest-neighbor path $\pi_0$ from 0 to $a$ with $0 < x \cdot \ell < a \cdot \ell$ for all sites $x$ on $\pi_0$ except for its starting and its end point. By ellipticity, the walk will follow $\pi_0$ with positive $P_{0,\omega}$-probability. Therefore, due to (14) and since $P_{a,\omega}[A_\ell \cap B_\ell] = P_{a,\psi(\omega,\pi_0)}[A_\ell \cap B_\ell]$,

$$0 < P_{0,\omega}[(X_n)_n \text{ follows } \pi_0] \ P_{a,\omega}[A_\ell \cap B_\ell] = P_{0,\omega}[\{(X_n)_n \text{ follows } \pi_0\} \cap A_\ell \cap B_\ell]$$

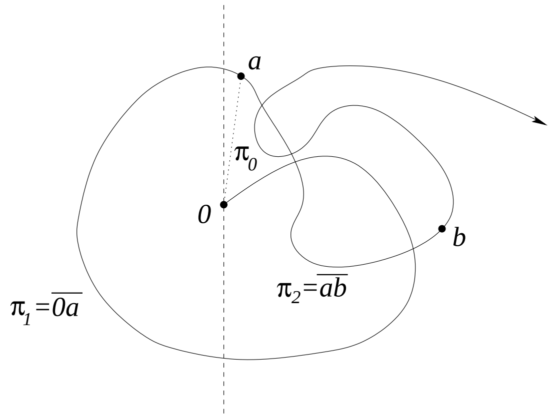


Figure 1: For the proof of Lemma 9. The path $\pi_1$ from 0 to $a$ is cut out and replaced by the dotted path $\pi_0$.

by the strong Markov property. Hence $P_{0,\omega}[A_\ell \cap B_\ell] > 0$. $\square$

We are now ready to prove a 0-1-law. We shall apply this result to $\bar{\mathbb{P}} \in \{\mathbb{P}, \widetilde{\mathbb{P}}\}$.

**Proposition 10** *Let $\bar{\mathbb{P}}$ be a probability measure on $\Omega$ and let $(\omega(x,\cdot,\cdot))_{x\cdot\ell\geq 0}$ be i.i.d. under $\bar{\mathbb{P}}$. Then $(\bar{\mathbb{P}} \times P_{0,\omega})[A_\ell] \in \{0,1\}$.*

*Proof.* For short set $\bar{P}_0 = \bar{\mathbb{P}} \times P_{0,\omega}$. Let us assume $\bar{P}_0[A_\ell] > 0$. We need to show $\bar{P}_0[A_\ell] = 1$. By Lemma 9, $\bar{P}_0[B_\ell] > 0$. The following argument is well-known, see e.g. [SzZe99, Lemma 1.1] and [ZeM01, Proposition 3]. Fix $M \in \mathbb{N}$. We define recursively possibly infinite stopping times $(S_k)_{k\geq 0}$ and $(R_k)_{k\geq 0}$ by $S_0 := T_M$,

$$\begin{aligned} R_k &:= \inf\{n \geq S_k \mid X_n \cdot \ell < M\} \quad \text{and} \\ S_{k+1} &:= \inf\left\{n \geq R_k \mid X_n \cdot \ell > \max_{m<n} X_m \cdot \ell\right\}. \end{aligned}$$

Due to Lemma 3, $S_0$ is $\bar{P}_0$-a.s. finite and any subsequent $S_{k+1}$ is $\bar{P}_0$-a.s. finite as well provided $R_k$ is finite. Moreover, at each finite time $S_k$ the walk has reached a half space it has never touched before. The environment $(\omega(x + X_{S_k},\cdot,\cdot))_{x\cdot\ell\geq 0}$ in this half space is independent of the environment visited so far and has the same distribution as $(\omega(x,\cdot,\cdot))_{x\cdot\ell\geq 0}$. Hence the walk has probability $\bar{P}_0[B_\ell]$ never to leave this half space again. Therefore, by induction, $\bar{P}_0[R_k < \infty] \leq \bar{P}_0[B_\ell^c]^k$, which goes to 0 as $k \to \infty$. Consequently, there is a random integer $K$ with $R_K = \infty$. This means that $X_n \cdot \ell \geq M$ for all $n \geq S_K$. Since this holds for all $M$, $\bar{P}_0[A_\ell] = 1$. $\square$

The following is the counterpart of Lemma 8 for $\mathbb{Z}$ and strips.

**Lemma 11** *Let $Y = \mathbb{Z}$ and $L = 1$ or $Y = \mathbb{Z} \times \{0, \ldots, L-1\}$ for some $L \geq 2$ and let $\mathbb{E}[\delta^0] > 1/L$. Then $\widetilde{P}_0[A_\ell] = 1$.*

*Proof.* Assume that $\widetilde{P}_0[A_\ell] < 1$. Then by Proposition 10, $\widetilde{P}_0[A_\ell] = 0$. Therefore, $X_n \cdot \ell$ changes sign $\widetilde{P}_0$-a.s. infinitely often due to Lemma 3. However, if the walk crosses the finite set $\mathcal{S}_0$ infinitely often then by ellipticity it will eventually eat all the cookies in $\mathcal{S}_0$, i.e. $\widetilde{P}_0$-a.s. $D^0_\infty = \sum_{x\in\mathcal{S}_0}\delta^x$. Hence $\widetilde{E}_0[D^0_\infty] = L\widetilde{E}_0[\delta^0] > 1$, which contradicts Lemma 7. □

*Proof of Theorem 1 and of transience in Theorem 2.* By Lemma 8 and Lemma 11, respectively, $\widetilde{P}_0[A_\ell] = 1$. Therefore, due to Lemma 9, $\widetilde{P}_0[A_\ell\cap B_\ell] > 0$. However, since (8) holds under $\widetilde{\mathbb{P}}$, $P_0[A_\ell\cap B_\ell] = \widetilde{P}_0[A_\ell\cap B_\ell] > 0$. Consequently, by Proposition 10, $P_0[A_\ell] = 1$. □

## 4 Recurrence on strips

*Proof of recurrence in Theorem 2.* Let $L\mathbb{E}[\delta^0] \le 1$. We need to show that $P_0$-a.s. $\liminf_n X_n\cdot e_1 \le 0$, since then, by ellipticity, $X_n = 0$ infinitely often. Assume the contrary. Then by Lemma 3, $P_0[A_\ell] > 0$. Consequently, by Lemma 9, even $P_0[A_\ell\cap B_\ell] > 0$. However, $P_0[A_\ell\cap B_\ell] = \widetilde{P}_0[A_\ell\cap B_\ell]$. Hence, by Proposition 10,

$$\widetilde{P}_0[A_\ell] = 1. \tag{15}$$

Now let $T_{-i} := \inf\{n \mid X_n\cdot\ell \le -i\}$ for $i > 0$. Then we have by the Optional Stopping Theorem for all $i, k, n \in \mathbb{N}$ and all $\omega\in\Omega$,

$$\begin{aligned} 0 \;=\; E_{0,\omega}[M_{T_k\wedge T_{-i}\wedge n}] \;&=\; kP_{0,\omega}[T_k < T_{-i}\wedge n] - iP_{0,\omega}[T_{-i} < T_k\wedge n] \\ &\quad + E_{0,\omega}[X_n\cdot\ell, n < T_k\wedge T_{-i}] - E_{0,\omega}[D_{T_k\wedge T_{-i}\wedge n}]. \end{aligned} \tag{16}$$

Using dominated convergence as $n\to\infty$ for both terms in (16) and $T_k < \infty$ $P_{0,\omega}$-a.s., see Lemma 3, for the first term in (16), we obtain

$$\frac{1}{k}E_{0,\omega}[D_{T_k\wedge T_{-i}}] \;=\; P_{0,\omega}[T_k < T_{-i}] - \frac{i}{k}P_{0,\omega}[T_{-i} < T_k].$$

Hence, due to (15), $\widetilde{\mathbb{P}}$-a.s. $\lim_{i\to\infty}\lim_{k\to\infty} k^{-1}E_{0,\omega}[D_{T_k\wedge T_{-i}}] = 1$. Splitting $D_n$ into $D^+_n := \sum_{k\ge 0} D^k_n$ and $D^-_n := \sum_{k<0} D^k_n$ then yields

$$\lim_{i\to\infty}\lim_{k\to\infty}\frac{1}{k}E_{0,\omega}[D^+_{T_k\wedge T_{-i}}] = 1, \tag{17}$$

since $E_{0,\omega}[D^-_{T_k\wedge T_{-i}}] \le \sum_{-i<x\cdot e_1<0}\delta^x(\omega)$, which is $\widetilde{\mathbb{P}}$-a.s. finite, does not depend on $k$ and thus vanishes when divided by $k\to\infty$. However,

$$E_{0,\omega}[D^+_{T_k\wedge T_{-i}}] \le E_{0,\omega}[D^+_{T_k}] \le k+1 \tag{18}$$

by Lemma 4. Therefore, (17) implies

$$\lim_{k\to\infty}\frac{1}{k}E_{0,\omega}[D^+_{T_k}] = 1. \tag{19}$$

By a calculation similar to the one in (12), $\widetilde{E}_0[D^k_\infty] = \widetilde{E}_0[D^0_\infty]$. Consequently, we can proceed like in the proof of [Ze05, Theorem 12] as follows and get

$$\widetilde{E}_0[D^0_\infty] \;=\; \frac{1}{K}\sum_{k=0}^{K-1}\widetilde{E}_0[D^k_\infty] \;\ge\; \widetilde{E}_0\left[\frac{1}{K}\sum_{k=0}^{K-1}D^k_{T_K}\right] \;=\; \widetilde{\mathbb{E}}\left[\frac{1}{K}E_{0,\omega}[D^+_{T_K}]\right].$$

Dominated convergence for $K \to \infty$, justified by (18), and (19) then yield

$$1 \le \widetilde{E}_0[D^0_\infty] \le \widetilde{\mathbb{E}}\Big[\sum_{x\in\mathcal{S}_0} \delta^x\Big] = L\widetilde{\mathbb{E}}[\delta^0]. \tag{20}$$

Now consider the event $S := \big\{\sum_{x\in\mathcal{S}_0}\delta^x > \omega(0,e_1,1) - \omega(0,-e_1,1)\big\}$ that not all the drift contained in the slab $\mathcal{S}_0$ is stored in the first cookie at 0. Observe that $\widetilde{\mathbb{P}}[S] > 0$. Indeed, for $L \ge 2$ this follows from independence of the environment at different sites and for $L = 1$ the opposite would imply $\widetilde{\mathbb{E}}[\delta^0] \le 1 - \kappa$, contradicting (20).
Now according to (15) we have $\widetilde{\mathbb{P}}$-a.s. $P_{0,\omega}[A_\ell] = 1$. Therefore, by Lemma 9, $\widetilde{\mathbb{P}}$-a.s. $P_{0,\omega}[A_\ell \cap B_\ell] > 0$. Hence, since $\widetilde{\mathbb{P}}[S] > 0$, as shown above,

$$0 \;<\; \widetilde{\mathbb{E}}[P_{0,\omega}[B_\ell], S] \;\le\; \widetilde{P}_0[D^0_\infty = \omega(0,e_1,1) - \omega(0,-e_1,1), S] \;\le\; \widetilde{P}_0\Big[D^0_\infty < \sum_{x\in\mathcal{S}_0}\delta^x\Big].$$

Since $D^0_\infty \le \sum_{x\in\mathcal{S}_0}\delta^x$ anyway, this implies $\widetilde{E}_0[D^0_\infty] < L\widetilde{\mathbb{E}}[\delta^0] = L\mathbb{E}[\delta^0]$. Along with (20) this contradicts the assumption $L\mathbb{E}[\delta^0] \le 1$. □

We conclude with some remarks, discussing the assumption of uniform ellipticity and some relation to branching processes with immigration.

**Remark 1.** The following example shows that the assumption of uniform ellipticity in Theorems 1 and 2 for $Y \ne \mathbb{Z}$ is essential. Let $\ell = e_1$, and let $(\omega(x))_x$ be i.i.d. under $\mathbb{P}$ with $\mathbb{P}[\omega(0) = \omega_+] = 1/2 = \mathbb{P}[\omega(0) = \omega_-]$, where $\omega_+$ and $\omega_-$ are such that for all $i \ge 1$, $(\omega_\pm(e,i))_{|e|=1} \in (0,1)^{2d}$ is a probability transition vector with $\omega_\pm(\pm e_2, i) = 1 - 2^{-i}$ and $\omega_\pm(e_1,i) \ge \omega_\pm(-e_1,i)$. Then all the requirements for $\omega \in \Omega$ are fulfilled except for $\omega(x,e,i) \ge \kappa > 0$. However, by independence in the environment and the Borel-Cantelli lemma, the walk will $P_0$-a.s. eventually become periodic and get stuck on two random sites $x$ and $x + e_2$ with $\omega(x) = \omega_+$ and $\omega(x+e_2) = \omega_-$. Hence it will not be transient and might not be recurrent to its starting point.
**2.** It is well-known that recurrence of the simple symmetric random walk $(Y_n)_n$ on $\mathbb{Z}$ corresponds to extinction of the Galton-Watson process $(Z_m)_m$ with geometric(1/2) offspring distribution. Indeed, let the walk start at $Y_0 = 1$, set $Z_0 = 1$ and denote by $Z_m$, $m \ge 1$, the number of transitions of $(Y_n)_n$ from $m$ to $m+1$ before the walk hits 0. Since for ERW transitions to the right are more likely than for $(Y_n)_n$, ERW can be viewed as a Galton-Watson process with immigration. Pakes [P71, Theorem 1] and Zubkov [Zu72, Theorem 3] showed that adding to each non-empty generation of a critical Galton-Watson process an i.i.d. number of immigrants makes it supercritical if the mean number of immigrants exceeds a certain critical threshold. This is reminiscent of Theorem B. However, since the immigration component of the Galton-Watson process derived from ERW is not independent these results do not directly translate into results for ERW.

## References

[ABK05] G. Amir, I. Benjamini and G. Kozma (2005). Excited random walk against a wall. Preprint. math.PR/0509464

[AR05] T. Antal and S. Redner (2005). The excited random walk in one dimension. *J. Phys. A: Math. Gen.* **38**, 2555–2577

[BW03] I. Benjamini and D.B. Wilson (2003). Excited random walk. *Elect. Comm. Probab.* **8**, 86–92

[K03] G. Kozma (2003). Excited random walk in three dimensions has positive speed. Preprint, math.PR/0310305

[K05] G. Kozma (2005). Excited random walk in two dimensions has linear speed. Preprint, math.PR/0512535

[MT96] S. P. Meyn and R. L. Tweedie (1993). *Markov Chains and Stochastic Stability.* Springer.

[P71] A.G. Pakes (1971). On the critical Galton-Watson process with immigration. *J. Austral. Math. Soc.* **12**, 476–482.

[Se94] T. Sellke (1994). Reinforced random walk on the $d$-dimensional integer lattice. Technical report #94-26, Dept. of Statistics, Purdue University.

[SzZe99] A.-S. Sznitman and M.P.W. Zerner (1999). A law of large numbers for random walks in random environment. *Ann. Probab.* **27**, No. 4, 1851–1869

[Ze05] M.P.W. Zerner (2005). Multi-excited random walks on integers. *Probab. Theory Related Fields* **133**, 98–122

[ZeM01] M.P.W. Zerner and F. Merkl (2001). A zero-one law for planar random walks in random environment. *Ann. Probab.* **29**, 1716–1732.

[Zu72] A. Zubkov (1972). The life spans of a branching process with immigration. *Theory Prob. Appl.* **17**, 174–183.