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ON AN EXTENSION OF JUMP-TYPE SYMMETRIC DIRICHLET FORMS

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Abstract
We show that any element from the (L^2-)maximal domain of a jump-type symmetric Dirichlet form can be approximated by test functions under some conditions. This gives us a direct proof of the fact that the test functions is dense in Bessel potential spaces.

1 Introduction

In this note, we are concerned with the following symmetric quadratic form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ defined on $L^2(\mathbb{R}^d)$:

$$\begin{cases} \mathcal{E}(u,v) := \dfrac{1}{2}\displaystyle\iint_{x\neq y} (u(x)-u(y))(v(x)-v(y))\, n(x,y)\, dx\, dy, \\ \mathcal{D}(\mathcal{E}) := \left\{u \in L^2(\mathbb{R}^d) \,:\, \mathcal{E}(u,u) < \infty\right\}, \end{cases} \tag{1}$$

where $n(x,y)$ is a positive measurable function on $x \neq y$.

In order that the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ makes sense, we assume that the set $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, n(x,y) = \infty\}$ is a Lebesgue null set. In fact, under this condition, we have already shown that the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(\mathbb{R}^d)$ in the wide sense (see [11] and [6]). Moreover if we set $C_0^{0,1}(\mathbb{R}^d)$ the totality of all uniformly Lipschitz continuous functions defined on $\mathbb{R}^d$ with compact support, then $\mathcal{D}(\mathcal{E}) \supset C_0^{0,1}(\mathbb{R}^d)$ if and only if the following conditions are satisfied(see [12], [6] and also [3, Example 1.2.4]): For some $\varepsilon > 0$,

$$\Phi_\varepsilon(\bullet) := \int_{|h|\leq\varepsilon} |h|^2 j(\bullet, \bullet + h)dh \;\in\; L^1_{\rm loc}(\mathbb{R}^d), \tag{A}$$

$$\Psi_\varepsilon(\bullet) := \int_{|h|>\varepsilon} j(\bullet, \bullet + h)dh \;\in\; L^1_{\rm loc}(\mathbb{R}^d), \tag{B}$$

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where $j(x,y) = n(x,y) + n(y,x)$. Then under (A) and (B), the quadratic form $(\mathcal{E},\mathcal{F})$ becomes a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d)$, where $\mathcal{F}$ is the closure of $C_0^{0,1}(\mathbb{R}^d)$ with respect to the norm $\sqrt{\mathcal{E}(\bullet,\bullet) + ||\bullet||_{L^2}^2}$. Note that, from the integral representation of the form $\mathcal{E}$, we can adopt the test functions, $C_0^\infty(\mathbb{R}^d)$, as a core instead of $C_0^{0,1}(\mathbb{R}^d)$ under the conditions (A) and (B). We now give some examples(see *e.g.,* [11, 12]):

Example 1

(1) (symmetric α-stable process) Let

$$n(x,y) = c|x-y|^{-\alpha-d}, \quad x \neq y.$$

Then (A) and (B) hold if and only if $0 < \alpha < 2$ and $c > 0$. This is nothing but the Dirichlet form corresponding to a symmetric α-stable process on $\mathbb{R}^d$.

(2) (symmetric stable-like process) For a measurable function $\alpha(x)$ defined on $\mathbb{R}^d$, set

$$n(x,y) = |x-y|^{-\alpha(x)-d}, \quad x \neq y.$$

Then (A) and (B) hold if and only if the following three conditions are satisfied:

(i) $0 < \alpha(x) < 2$ a.e.,

(ii) $1/\alpha,\ 1/(2-\alpha) \in L^1_{\text{loc}}(\mathbb{R}^d)$,

(iii) for some compact set K, $\int_{K^c} |x|^{-d-\alpha(x)} dx < \infty$.

(3) (symmetric Lévy process) For a positive measurable function $\tilde{n}$ defined on $\mathbb{R}^d - \{0\}$ satisfying $\tilde{n}(x) = \tilde{n}(-x)$ for any $x \neq 0$, set

$$n(x,y) = \tilde{n}(x-y), \quad x \neq y.$$

(A) and (B) are satisfied if and only if $\int_{h \neq 0} (1 \wedge |h|^2) \tilde{n}(h) dh < \infty$.

In general, we do not know whether the set $\mathcal{F}$ coincides with $\mathcal{D}(\mathcal{E})$. Determining the domains of the Dirichlet form corresponds, in some sense, to solve the boundary problem of the associated Markov processes. This analytic structure was investigated first by Silverstein in [7] and [8], and then by Chen [1] and Kuwae [5].

2 Identification of the domains

In order to classify the domains of the forms, we will consider the following conditions: there exists a positive constant $C > 0$ such that

$$\Phi_1 \in L^1_{\text{loc}}(\mathbb{R}^d), \quad j(x+z, y+z) \leq Cj(x,y), \quad |x-y| \leq 1,\ |z| \leq 1 \tag{A'}$$

or

$$\Phi_1 \in L^\infty(\mathbb{R}^d), \quad j(x+z, y+z) \leq Cj(x,y), \quad |x-y| \leq 1,\ |z| \leq 1, \tag{A''}$$

and

$$\Psi_1(\bullet) = \int_{|h|>1} j(\bullet, \bullet + h) dh \in L^\infty(\mathbb{R}^d). \tag{B'}$$

Note that (A")⇒(A')⇒(A) and (B')⇒(B).

Theorem 1 *Assume that* (A”) *and* (B’) *hold. Then we can show*

$$\mathcal{D}(\mathcal{E}) = \{u \in L^2(\mathbb{R}^d) : \ \mathcal{E}(u,u) < \infty\} = \mathcal{F},$$

that is, any element in $\mathcal{D}(\mathcal{E})$ *can be approximated from elements of* $C_0^\infty(\mathbb{R}^d)$ *with respect to* $\mathcal{E}_1$.

Proof: Take $\rho \in C_0^\infty(\mathbb{R}^d)$ satisfying

$$\rho(x) \geq 0, \quad \rho(x) = \rho(-x), \quad x \in \mathbb{R}^d, \quad \mathrm{supp}[\,\rho\,] = \overline{B_0(1)}, \quad \int_{\mathbb{R}^d} \rho(x)dx = 1.$$

For any $\varepsilon > 0$, define $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ so that $\int_{\mathbb{R}^d} \rho_\varepsilon dx = 1$.
For $u \in \mathcal{D}(\mathcal{E})$, set the convolution of u and $\rho_{1/n}$:

$$w_n(x) := J_{1/n}(u)(x) := \rho_{1/n} * u(x) = \int_{\mathbb{R}^d} \rho_{1/n}(x-z)u(z)dz, \quad x \in \mathbb{R}^d.$$

Since $u \in L^2(\mathbb{R}^d)$, $w_n \in C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and

$$||w_n||_{L^2} \leq ||\rho_{1/n}||_{L^1}||u||_{L^2} = ||u||_{L^2} \quad \text{and} \quad ||w_n - u||_{L^2} \to 0 \text{ as } n \to \infty.$$

Let $\psi_n(t),\ t \geq 0$, be non-negative C^∞-functions such that

$$\psi_n(t) = 1,\ 0 \leq t \leq n, \quad \psi_n(t) = 0, \quad t \geq n+2, \quad -1 \leq \psi_n'(t) \leq 0, \quad t \leq n+2.$$

We put $v_n(x) = \psi_n(|x|),\ x \in \mathbb{R}^d$. Then $v_n \in C_0^\infty(\mathbb{R}^d)$ and

$$V_n(x) := \int_{|x-y|<1} \big(v_n(x) - v_n(y)\big)^2 j(x,y)dy, \quad x \in \mathbb{R}^d$$

satisfies the following inequality:

$$V_n(x) \leq d\int_{|x-y|<1} |x-y|^2 j(x,y)dy = d\,\Phi_1(x), \quad x \in \mathbb{R}^d. \tag{2}$$

Then we see that

$$v_n(x) \nearrow 1, \quad x \in \mathbb{R}^d \quad \text{and} \quad M := \sup_n \sup_{x\in\mathbb{R}^d} V_n(x) \leq d||\Phi_1||_\infty < \infty$$

and

$$\begin{aligned} ||w_nv_n - u||_{L^2} &\leq ||w_nv_n - uv_n||_{L^2} + ||uv_n - u||_{L^2} \\ &\leq ||w_n - u||_{L^2} + ||uv_n - u||_{L^2} \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Now we estimate $\mathcal{E}(w_nv_n, w_nv_n)$:

$$\begin{aligned} \mathcal{E}(w_nv_n, w_nv_n) &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \big(w_n(x)v_n(x) - w_n(y)v_n(y)\big)^2 j(x,y)dxdy \\ &= \left(\iint_{|x-y|<1} + \iint_{|x-y|\geq 1}\right)\big(w_n(x)v_n(x) - w_n(y)v_n(y)\big)^2 j(x,y)dxdy \\ &=: (I) + (II). \end{aligned}$$

$$
\begin{aligned}
(II) &= \iint_{|x-y|\geq 1} \big(w_n(x)v_n(x) - w_n(y)v_n(y)\big)^2 j(x,y)dxdy \\
&\leq 2\iint_{|x-y|\geq 1} \Big(\big(w_n(x)\big)^2 v_n(x)^2 + \big(w_n(y)\big)^2 v_n(y)^2\Big) j(x,y)dxdy
\end{aligned}
$$

Since $j(x,y) = j(y,x)$, we see

$$
\begin{aligned}
(II) &\leq 4\iint_{|x-y|\geq 1} \big(w_n(x)\big)^2 j(x,y)dxdy \\
&= 4\int_{\mathbb{R}^d} \big(J_{1/n}(u)(x)\big)^2 dx \int_{|x-y|\geq 1} j(x,y)dy \\
&= 4\int_{\mathbb{R}^d} \big(J_{1/n}(u)(x)\big)^2 \,\Psi_1(x)dx \\
&\leq 4\,||\Psi_1||_{L^\infty} \int_{\mathbb{R}^d} \big(J_{1/n}(u)(x)\big)^2 dx \leq 4\,||\Psi_1||_{L^\infty}\,||u||_{L^2}^2.
\end{aligned}
$$

Now we estimate (I).

$$
\begin{aligned}
(I) &= \iint_{|x-y|<1} \big(w_n(x)v_n(x) - w_n(y)v_n(y)\big)^2 j(x,y)dxdy \\
&\leq 2\iint_{|x-y|<1} \big(w_n(x) - w_n(y)\big)^2 (v_n(x))^2 j(x,y)dxdy \\
&\quad +2\iint_{|x-y|<1} \big(v_n(x) - v_n(y)\big)^2 (w_n(y))^2 j(x,y)dxdy \\
&\leq 2\iint_{|x-y|<1} \Big(\int_{\mathbb{R}^d} \rho_{1/n}(z)\big(u(x-z) - u(y-z)\big)dz\Big)^2 j(x,y)dxdy \\
&\quad +2\int_{\mathbb{R}^d} \big(w_n(y)\big)^2 \int_{|x-y|<1} \big(v_n(x) - v_n(y)\big)^2 j(x,y)dxdy \\
&=: 2(\text{I-1}) + 2(\text{I-2}).
\end{aligned}
$$

Since $\mathrm{supp}[\rho_{1/n}] \subset \overline{B_{1/n}(0)} \subset \overline{B_1(0)}$ for $n \in \mathbb{N}$, we see

$$
\begin{aligned}
(\text{I-1}) &\leq \iint_{|x-y|<1} \Big(\int_{\mathbb{R}^d} \big(u(x-z) - u(y-z)\big)^2 \rho_{1/n}(z)dz\Big) j(x,y)dxdy \\
&= \int_{\mathbb{R}^d} \Big(\iint_{|x-y|<1} \big(u(x-z) - u(y-z)\big)^2 j(x,y)dxdy\Big) \rho_{1/n}(z)dz \\
&= \int_{\mathbb{R}^d} \Big(\iint_{|x-y|<1} \big(u(x) - u(y)\big)^2 j(x+z,y+z)dxdy\Big) \rho_{1/n}(z)dz \\
&\leq \int_{\mathbb{R}^d} \Big(\iint_{|x-y|<1} \big(u(x) - u(y)\big)^2 Cj(x,y)dxdy\Big) \rho_{1/n}(z)dz \\
&= C\iint_{|x-y|<1} \big(u(x) - u(y)\big)^2 j(x,y)dxdy \leq C\mathcal{E}(u,u) < \infty.
\end{aligned}
$$

In the first inequality, we used the Jensen inequality for the measure $\rho_{1/n}(z)dz$, while the second is from the Fubini theorem, the third is by translation and the fourth is obtained by the assumption (A").

$$\begin{aligned}\text{(I-2)} &= \int_{\mathbb{R}^d}\big(w_n(y)\big)^2\int_{|x-y|<1}\big(v_n(x)-v_n(y)\big)^2 j(x,y)dxdy \\ &= \int_{\mathbb{R}^d}\big(w_n(y)\big)^2 V_n(y)dy \le M\int_{\mathbb{R}^d}\big(w_n(y)\big)^2 dy \le M||u||_{L^2}^2.\end{aligned}$$

Summarizing the calculus done above, we see

$$\begin{aligned}\mathcal{E}(w_nv_n,w_nv_n) &= \iint_{\mathbb{R}^d\times\mathbb{R}^d}\big(w_n(x)v_n(x)-w_n(y)v_n(y)\big)^2 j(x,y)dxdy \\ &\le 4||\Psi_1||_{L^\infty}||u||_{L^2}^2 + 2C\,\mathcal{E}(u,u) + 2M||u||_{L^2}^2 \\ &= 2\Big(C\,\mathcal{E}(u,u) + \big(2||\Psi_1||_{L^\infty}+M\big)||u||_{L^2}^2\Big) < \infty.\end{aligned}$$

That is, $\mathcal{E}(w_nv_n,w_nv_n)$ are uniformly bounded. Moreover we have seen that $||w_nv_n||_{L^2}$ are also uniform bounded and w_nv_n converges to u in $L^2(\mathbb{R}^d)$. Thus the Cesàro means of a subsequence of $\{w_nv_n\}$ are $\mathcal{E}_1$-Cauchy and convergent to u a.e. Hence $u\in\mathcal{F}$. Thus

$$\{u\in L^2(\mathbb{R}^d):\ \mathcal{E}(u,u)<\infty\} = \mathcal{F} := \overline{C_0^\infty(\mathbb{R}^d)}^{(\mathcal{E}(\bullet,\bullet)+||\bullet||_{L^2}^2)^{1/2}}. \qquad \square$$

Example 2

(1) Let $n(x,y)=c|x-y|^{-d-\alpha},\ x\neq y$ for some $0<\alpha<2$ and $c>0$. For this n, we can easily see that the conditions (A") and (B') hold. In this case, the L^2-maximal domain $\mathcal{D}(\mathcal{E})$ is nothing but the "Bessel potential space" $\mathcal{L}^2_{\alpha/2}(\mathbb{R}^d)$ (see Proposition V. 4 in [9]).

(2) For $0<\alpha<2$ and $c_i>0\,(i=1,2)$, we assume

$$c_1|x-y|^{-d-\alpha}\le n(x,y)\le c_2|x-y|^{-d-\alpha},\quad 0<|x-y|\le 1.$$

and

$$\sup_x\int_{|x-y|\ge 1}\big(n(x,y)+n(y,x)\big)dy<\infty.$$

Then this satisfies the conditions (A") and (B'). A Markov process corresponding to the Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is called "stable-like process" by Chen-Kumagai[2].

For a subclass $\mathcal{B}$ of all measurable functions on $\mathbb{R}^d$, we denote by $\mathcal{B}_b$ the bounded functions in $\mathcal{B}$. In the following, we always assume that (A) and (B) hold. Then a symmetric Dirichlet form $(\eta,\mathcal{D}(\eta))$ on $L^2(\mathbb{R}^d)$ is said to be an *extension* of the Dirichlet form $(\mathcal{E},\mathcal{F})$ if $\mathcal{D}(\eta)\supset\mathcal{F}$ and $\eta(u,u)=\mathcal{E}(u,u)$ whenever $u\in\mathcal{F}$. Denote by $\mathcal{A}(\mathcal{E},\mathcal{F})$ the totality of the extensions of $(\mathcal{E},\mathcal{F})$. By this definition, $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is an element of $\mathcal{A}(\mathcal{E},\mathcal{F})$. An element $(\eta,\mathcal{D}(\eta))$ of $\mathcal{A}(\mathcal{E},\mathcal{F})$ is called a Silverstein extension if $\mathcal{F}_b$ is an algebraic ideal in $\mathcal{D}(\eta)_b$. For the probabilistic counterpart or an application of Silverstein extensions, *see*, for example, [8], [10] and [4].

Theorem 2 *Suppose that* (A') *and* (B) *hold. Then the Dirichlet form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is a Silverstein extension of the form* $(\mathcal{E}, \mathcal{F})$*. That is,* $\mathcal{F}_b$ *is an ideal of* $\mathcal{D}(\mathcal{E})_b$*.*

Proof: It is enough to show that $u \cdot f \in \mathcal{F}_b$ whenever $u \in \mathcal{D}(\mathcal{E})_b$ and $f \in C_0^\infty(\mathbb{R}^d)$. Let ρ and ρ_ε be the same functions in the proof of the preceding theorem. Take the convolution of functions uf and $\rho_{1/n}$: $w_n = \rho_{1/n} * (uf)$. Then $w_n \in C_0^\infty(\mathbb{R}^d)$, w_n converges to uf in the L^2-space and the inequality $||w_n||_{L^\infty} \le ||uf||_{L^\infty}$ holds.
Denote by K the support of the function f. As in the proof of the preceding theorem, we estimate $\mathcal{E}(w_n, w_n)$ as follows:

$$\begin{aligned}
\mathcal{E}(w_n, w_n) &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \big(w_n(x) - w_n(y)\big)^2 j(x,y)dxdy \\
&= \left(\iint_{|x-y|<1} + \iint_{|x-y|\ge 1}\right) \big(w_n(x) - w_n(y)\big)^2 j(x,y)dxdy \\
&=: (I) + (II). \\
(II) &= \iint_{|x-y|\ge 1} \big(w_n(x) - w_n(y)\big)^2 j(x,y)dxdy \\
&\le 2\iint_{|x-y|\ge 1} \Big(\big(w_n(x)\big)^2 + \big(w_n(y)\big)^2\Big) j(x,y)dxdy.
\end{aligned}$$

Since $j(x,y) = j(y,x)$, we see

$$\begin{aligned}
(II) &\le 4\iint_{|x-y|\ge 1} \big(w_n(x)\big)^2 j(x,y)dxdy \\
&= 4\int_{\mathbb{R}^d} \big(w_n(x)\big)^2 dx \int_{|x-y|\ge 1} j(x,y)dy \\
&= 4\int_{K_n} \big(w_n(x)\big)^2 \Psi_1(x)dx \\
&\le 4\,||w_n||_{L^\infty}^2 \int_{K_1} \Psi_1(x)dx \le 4\,||uf||_{L^\infty}^2 ||\Psi_1 \mathbf{1}_{K_1}||_{L^1},
\end{aligned}$$

where $K_n = \{x + y \in \mathbb{R}^d : \ x \in K, \ y \in B(0, 1/n)\}$.
Now we estimate (I).

$$\begin{aligned}
(I) &= \iint_{|x-y|<1} \big(w_n(x) - w_n(y)\big)^2 j(x,y)dxdy \\
&= \iint_{|x-y|<1} \Big(\int_{\mathbb{R}^d} \rho_{1/n}(z)\big((uf)(x-z) - (uf)(y-z)\big)dz\Big)^2 j(x,y)dxdy \\
&\le \iint_{|x-y|<1} \Big(\int_{\mathbb{R}^d} \big((uf)(x-z) - (uf)(y)\big)^2 \rho_{1/n}(z)dz\Big) j(x,y)dxdy \\
&\le \int_{\mathbb{R}^d} \Big(\iint_{|x-y|<1} \big((uf)(x) - (uf)(y)\big)^2 j(x+z, y+z)dxdy\Big)\rho_{1/n}(z)dz
\end{aligned}$$

$$\leq \;\; C \iint_{|x-y|<1} \big((uf)(x) - (uf)(y)\big)^2 j(x,y)dxdy \int_{\mathbb{R}^d} \rho_{1/n}(z)dz$$
$$\leq \;\; 2\,C\,\big(||u||^2_{L^\infty}\mathcal{E}(f,f) + ||f||^2_{L^\infty}\mathcal{E}(u,u)\big).$$

Combining the estimates (II) and (I), we have

$$\mathcal{E}(w_n, w_n) \leq 2\,C\,\big(||u||^2_{L^\infty}\mathcal{E}(f,f) + ||f||^2_{L^\infty}\mathcal{E}(u,u)\big) + 4||uf||^2_{L^\infty}||\Psi_1 \mathbf{1}_{K_1}||_{L^1} < \infty.$$

So $\mathcal{E}(w_n, w_n)$ are uniformly bounded. We have already known that $w_n \in C_0^\infty(\mathbb{R}^d)$ converges to uf in L^2. Then by making use of the Banach-Saks theorem, the Cesàro means of a subsequence of $\{w_n\}$ are $\mathcal{E}_1$-Cauchy and converges to uf a.e. Hence $uf \in \mathcal{F}$. This shows that $\mathcal{F}_b$ is an ideal of $\mathcal{D}(\mathcal{E})_b$, whence $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Silverstein extension of $(\mathcal{E}, \mathcal{F})$. □

Remark 1 *If the form* $(\mathcal{E}, \mathcal{F})$ *is moreover conservative, then, using a theorem from [5], we can show that the Silverstein extension is unique. Hence this implies that* $\mathcal{F} = \mathcal{D}(\mathcal{E})$*. In [6], we showed that under some conditions (which includes the condition* (B')*), the form* $(\mathcal{E}, \mathcal{F})$ *is conservative. So, we have an alternative proof of Theorem 1 under* (A") *and* (B')*.*

In the following, we consider 'the homogeneous' Dirichlet space:

$$\mathcal{D}_0(\mathcal{E}) = \{u \in L^0(\mathbb{R}^d) : \; \mathcal{E}(u,u) < \infty\},$$

where $\mathcal{E}$ is defined in §1 and $L^0(\mathbb{R}^d)$ is the family of all measurable functions on $\mathbb{R}^d$. We assume (A) and (B) hold. Since $\mathcal{E}$ is defined as an integral form, we can easily see that $\mathcal{D}_0(\mathcal{E}) \cap L^\infty(\mathbb{R}^d) =: \mathcal{D}_\infty(\mathcal{E})$ is dense in $\mathcal{D}_0(\mathcal{E})$ with respect to quasi-norm $\mathcal{E}$.
We now want to consider when any function in $\mathcal{D}_\infty(\mathcal{E})$ (hence, in $\mathcal{D}_0(\mathcal{E})$) can be approximated from a sequence of the test functions with respect to $\mathcal{E}$. Of couse, this relates the notion of 'the extended Dirichlet space' $\mathcal{F}_e$. In general,

$$\mathcal{D}_0(\mathcal{E}) \supset \mathcal{F}_e \supset \mathcal{F} := \overline{C_0^{0,1}(\mathbb{R}^d)}^{\mathcal{E}_1}.$$

If the form $(\mathcal{E}, \mathcal{F})$ is transient, then $\mathcal{F} = \mathcal{F}_e \cap L^2(\mathbb{R}^d)$ (see Theorem 1.5.2(iii) in [3]). It is not easy to see whether the 'homogeneous' domain $\mathcal{D}_0(\mathcal{E})$ coincides with $\mathcal{F}_e$ except the special cases. In order to consider this, we introduce a little bit stronger condition as follows: there exists a positive function $\tilde{n}(x)$ defined on $\mathbb{R}^d - \{0\}$ satisfying the condition in Example 1 (3) so that for some constants $c_i > 0$ $(i = 1, 2)$,

$$c_1\,\tilde{n}(x-y) \leq n(x,y) \leq c_2\,\tilde{n}(x-y), \quad x \neq y. \tag{C}$$

Proposition 1 Suppose that (C) holds. Moreover, we assume the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is recurrent. Then any element in $\mathcal{D}_\infty(\mathbb{R}^d)$ (hence, in $\mathcal{D}_0(\mathcal{E})$) can be approximated from the test functions with respect to $\mathcal{E}$. That is, $\mathcal{D}_0(\mathcal{E}) = \mathcal{F}_e$.

Proof: First note that a similar argument developed in the proof of Theorem 2 gives us that $\varphi \cdot u \in \mathcal{D}_0(\mathcal{E})$ provided that $u \in \mathcal{D}_0(\mathcal{E})$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. Take the test function ρ defined in the proof of Theorem 1. And also consider the function $\rho_{1/n}$ for each n. Then considering the convolution u_n of u and $\rho_{1/n}$, we have the following estimate:

$$\begin{aligned}
\mathcal{E}(u_n,u_n) &= \iint_{x\neq y}\big(u_n(x)-u_n(y)\big)^2 j(x,y)dxdy \\
&= \iint_{x\neq y}\left(\int_{\mathbb{R}^d}\big(\varphi(x-z)u(x-z)-\varphi(y-z)u(y-z)\big)\rho_n(z)dz\right)^2 j(x,y)dxdy \\
&\leq \int_{\mathbb{R}^d}\left(\iint_{x\neq y}\big(\varphi(x-z)u(x-z)-\varphi(y-z)u(y-z)\big)^2 j(x,y)dxdy\right)\rho_n(z)dz \\
&= \int_{\mathbb{R}^d}\left(\iint_{x\neq y}\big(\varphi(x)u(x)-\varphi(y)u(y)\big)^2 j(x+z,y+z)dxdy\right)\rho_n(z)dz \\
&\leq C\int_{\mathbb{R}^d}\left(\iint_{x\neq y}\big(\varphi(x)u(x)-\varphi(y)u(y)\big)^2 j(x,y)dxdy\right)\rho_n(z)dz \\
&\leq C\Bigg(||\varphi||^2_{L^\infty}\iint_{x\neq y}\big(u(x)-u(y)\big)^2 j(x,y)dxdy \\
&\qquad\qquad +||u||^2_{L^\infty}\int_{x\neq y}\big(\varphi_n(x)-\varphi_n(y)\big)^2 j(x,y)dxdy\Bigg) \\
&\leq C\big(||\varphi||^2_{L^\infty}\mathcal{E}(u,u)+||u||^2_{L^\infty}\mathcal{E}(\varphi,\varphi)\big).
\end{aligned}$$

In the first inequality, we used the Schwarz inequality, and the second follows from (C). Accordingly, we see that the sequence $\{u_n\}$ is $\mathcal{E}$-bounded. Since $||u_n-\varphi u||_{L^2}$ converges to 0, a subsequence of u_n converges to φu *almost everywhere.* So we can find the Casaro mean $\{\tilde{u}_{n_k}\}$ of some subsequence from $\{u_n\}_n$ so that $\mathcal{E}(\tilde{u}_{n_k}-u,\tilde{u}_{n_k}-u)$ converges to 0 and $\tilde{u}_{n_k}\to\varphi\, u$ a.e. This means that there exists a sequence from test functions which conveges to $\varphi\, u$ with respect to $\mathcal{E}$ and with respect to *almost everywhere* convergence.
On the other hand, the Dirichlet form $(\mathcal{E},\mathcal{F})$ is recurrent, we can construct a sequence $\{\varphi_k\}\subset C_0^\infty(\mathbb{R}^d)$ satisfying

$$0\leq\varphi_k\to 1\quad\text{a.e.},\qquad ||\varphi_k||_{L^\infty}\leq 1\quad\text{and}\quad\mathcal{E}(\varphi_k,\varphi_k)\to 0.$$

Note that $\varphi_k\cdot u\in\mathcal{D}(\mathcal{E})\cap L^2(\mathbb{R}^d)$ for each k because $\varphi_k\in C_0^\infty(\mathbb{R}^d)$. Similarly, noting the following estimates and the property of φ_k, we can see that the cesaro means $\tilde{\varphi}_{n_k}u$ of some subsequence of $\{\varphi_k u\}$ converges to u with respect to $\mathcal{E}$ and with respect to *almost everywhere* convergence:

$$\mathcal{E}(\varphi_k u,\varphi_k u)\leq 2\,\mathcal{E}(u,u)+2\,||u||^2_{L^\infty}\mathcal{E}(\varphi_k,\varphi_k).$$

Now for each k, take $f_k\in C_0^\infty(\mathbb{R}^d)$ so that $\mathcal{E}(\tilde{\varphi}_{n_k}u-f_k,\tilde{\varphi}_{n_k}u-f_k)<1/k$, Then we see

$$\begin{aligned}
\mathcal{E}(f_k-u,f_k-u)^{1/2} &\leq \mathcal{E}(f_k-\tilde{\varphi}_{n_k}u,f_k-\tilde{\varphi}_{n_k}u)^{1/2}+\mathcal{E}(\tilde{\varphi}_{n_k}u-u,\tilde{\varphi}_{n_k}u-u)^{1/2} \\
&\leq 1/k+\mathcal{E}(\tilde{\varphi}_{n_k}u-u,\tilde{\varphi}_{n_k}u-u)^{1/2}.
\end{aligned}$$

So, taking $k\to\infty$, we see that f_k converges to u with respect to the quasi-norm $\mathcal{E}$. This concludes the proof. □

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