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UNIFORM UPPER BOUND FOR A STABLE MEASURE OF A SMALL BALL

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Abstract
The authors of [1] stated the following conjecture: Let μ be a symmetric α-stable measure on a separable Banach space and B a centered ball such that $\mu(B) \leq b$. Then there exists a constant $R(b)$, depending only on b, such that $\mu(tB) \leq R(b)t\mu(B)$ for all $0 < t < 1$. We prove that the above inequality holds but the constant R must depend also on α.

Recently, the authors of [1] proved the following (Theorem 6.4 in [1]):
Let μ be a symmetric α-stable measure, $0 < \alpha \leq 2$, on a separable Banach space, fix $b < 1$, and let B denote a centered ball such that $\mu(B) \leq b$. Then there exists a constant $R(b) = \frac{3}{b\sqrt{1-b}}$, depending only on b, such that for all $0 \leq t \leq 1$

$$\mu(tB) \leq R(b)t^{\alpha/2}\mu(B). \tag{1}$$

Of course, for small values of t, the quantity $t^{\alpha/2}$ is much larger than t. The authors of [1] stated in their Conjecture 7.4 that (1) is true for all symmetric α-stable measures with t instead of $t^{\alpha/2}$ and some $R(b)$ depending only on b.

In our earlier paper [3], we also gave an estimate of a stable measure of a small ball. Namely, we proved the following.
Let μ be a symmetric α-stable measure, $0 < \alpha \leq 2$, on a separable Banach space, put $B = \{x : \|x\| \leq 1\}$, let $0 < r < \alpha$ and suppose that μ is so normalized that $\int \|x\|^r \mu(dx) = 1$. Then there exists a constant $K = K(\alpha, r)$ such that for all $0 \leq t \leq 1$

$$\mu(tB) \leq K(\alpha, r)\, t. \tag{2}$$

Some estimates of $K(\alpha, r)$ were also given in [3], we recall one of them in the final Remark. Some normalization of μ is needed, as we will show in the sequel (see Example), in the paper [3] we chose the normalizing condition $\int \|x\|^r \mu\,(dx) = 1$. But proving the inequality (2), we also obtained the inequality

$$\mu(tB) \le K(\alpha, r)[1 - \mu(B)]^{-1/r}\, t. \tag{3}$$

In this note we will show that using (3) we can prove an estimate that is very close to the above-mentioned conjecture, however, the constant $R(b)$ must depend also on α.

The following is a generalization of (1).

Theorem 1 *Let μ be a symmetric α-stable measure, $0 < \alpha \le 2$, on a separable Banach space F. Then for every closed, symmetric, convex set $B \subset F$ and for each $b < 1$ there exists $R(\alpha, b)$ such that for all $0 \le t \le 1$*

$$\mu(tB) \le R(\alpha, b)\, t\, \mu(B), \quad \textit{if} \;\; \mu(B) \le b. \tag{4}$$

First we show that the constant R must depend on α.

Example. Suppose that there exists positive function $R(b)$ that fulfills (4), does not depend on α and is bounded on every closed subinterval of $(0, 1)$ Let X_α be an α-stable random variable with the characteristic function $e^{-|t|^\alpha}$. It is known (see e.g. [4]) that

$$|X_\alpha|^\alpha \xrightarrow{d} \frac{1}{W}, \qquad \text{as} \;\; \alpha \to 0+, \tag{5}$$

where W is a random variable having the exponential distribution with mean 1. Consider one-dimensional ball $B = [-1, 1]$. From (5) we infer that

$$b_\alpha = P(X_\alpha \in B) = P(-1 \le X_\alpha \le 1) = P(|X_\alpha|^\alpha \le 1) \to_{\alpha\to 0} P(\frac{1}{W} \le 1) = \frac{1}{e}.$$

Denote by μ the distribution of X_α. It is easy to compute the value of the density of μ at zero:

$$p_\alpha(0) = \frac{1}{\pi} \int_0^\infty e^{-t^\alpha}\, dt = \frac{1}{\pi}\Gamma(\frac{1}{\alpha}).$$

Now

$$\lim_{\alpha\to 0} \lim_{t\to 0+} \frac{1}{t}\mu(tB) = \lim_{\alpha\to 0} \lim_{t\to 0+} \frac{1}{t} \int_0^t p(x)\, dx = \lim_{\alpha\to 0} p_\alpha(0) = \lim_{\alpha\to 0} \frac{1}{\pi}\Gamma(\frac{1}{\alpha}) = \infty,$$

and

$$\lim_{\alpha\to 0} R(b_\alpha)\, b_\alpha = R\left(\frac{1}{e}\right) \frac{1}{e},$$

contradicting the inequality (4).

This implies that $R(b)$ must also depend on α.

The proof of the theorem is almost the same as the proof of (1) in the paper [1], the difference is that instead of Kanter inequality we use our estimate (3). For the sake of completeness we repeat this proof.
We start with two lemmas.

Lemma 1. Let μ be a symmetric α-stable measure, $0 < \alpha \leq 2$, on a separable Banach space F. Fix $0 < r < \alpha$. Then there exists a constant $K(\alpha, r) \geq 2$ such that for every convex, symmetric, closed set $B \subset F$, every $y \in F$ and all $t \in [0,1]$ there holds

$$\mu(tB + y) \leq K(\alpha, r)\, Rt\ \mu(2B + y),$$

where $R = (\mu(B))^{-1}(1-\mu(B))^{-1/r}$.
Proof. It is well-known that symmetric stable measures are conditionally Gaussian [2], hence they satisfy the Anderson property.
Case 1. If $y \in B$ then $B \subset 2B + y$ so that $\mu(B) \leq \mu(2B+y)$, hence by the Anderson property and (3)

$$\mu(tB+y) \leq \frac{K(\alpha,r)}{(1-\mu(B))^{1/r}}\, t \leq \frac{K(\alpha,r)\mu(B)}{\mu(B)(1-\mu(B))^{1/r}}\, t \leq \frac{K(\alpha,r)}{\mu(B)(1-\mu(B))^{1/r}}\, t\, \mu(2B+y).$$

Case 2. If $y \notin B$ then take $r = [t^{-1} - 2^{-1}]$. Then for $k = 0, 1, ..., r$ the balls $\{y_k + tB\}$ are disjoint and contained in $y + 2B$, where $y_k = (1 - 2t\|y\|^{-1}k)y$. By the Anderson property $\mu(y_k + tB) \geq \mu(y + tB)$ for $k = 0, 1, ..., r$. Therefore

$$\mu(tB+y) \leq (r+1)^{-1}\mu(2B+y) \leq \frac{2t}{2-t}\mu(2B+y)$$

$$\leq \frac{K(\alpha,r)}{(1-\mu(B))^{1/r}}\, \mu(2B+y)\, t,$$

because we assumed that $K(\alpha, r) > 2$ and $2 - t \geq 1 > (1-\mu(B))^{1/r}$.

Lemma 2. With the same assumptions as in Lemma 1, we have for all $0 \leq \kappa,\ t \leq 1$

$$\mu(\kappa t B) \leq R' t \mu(\kappa B),$$

where $R' = \frac{2K(\alpha,r)}{\mu(B/2)(1-\mu(B/2))^{1/r}}$.

Proof. For $0 \leq t \leq 1$ define a measure μ_t by the formula $\mu_t(C) = \mu(tC) = P(X/t \in C)$, where X is a symmetric α-stable random variable with the distribution μ. Then μ_t is also α-stable and we have the following equality:

$$\mu * \mu_s(C) = P(X + X'/s \in C) = P((1+s^{-\alpha})^{1/\alpha} X \in C) = \mu_t(C),$$

where $t = (1+s^{-\alpha})^{-1/\alpha}$ and X' is an independent copy of X. Now by Lemma 1

$$\mu(\kappa(tB)) = \mu(t(\kappa B)) = P(X/t \in \kappa B) = \mu * \mu_s(\kappa B) = \int_F \mu(\frac{2\kappa B}{2} + y)\mu_s(dy)$$

$$\leq \frac{K(\alpha,r)2\kappa}{\mu(B/2)(1-\mu(B/2))^{1/r}}\mu_t(B) = \frac{2K(\alpha,r)}{\mu(B/2)(1-\mu(B/2))^{1/r}}\kappa\mu(tB).$$

Proof of the Theorem. Fix B with $\mu(B) \leq b$ and take $s \geq 1$ such that $\mu(sB) = b$. Now, in Lemma 2, put $\kappa = t$ and $t = \frac{1}{2s}$. Then

$$\mu(tB) = \mu(t \cdot \frac{1}{2s} \cdot (2sB)) \leq t\frac{K(\alpha,r)2}{\mu(sB)(1-\mu(sB))^{1/r}}\mu(\frac{1}{2s} \cdot 2sB) \leq$$

$$\frac{K(\alpha,r)2}{\mu(sB)(1-\mu(sB))^{1/r}}t\,\mu(B) = R(b)K(\alpha,r)t\,\mu(B),$$

where $R(b) = 2b^{-1}(1-b)^{-1/r}$. Taking different values of $r \in (0,\alpha)$ we get different values of $K(\alpha,r)$. If, for simplicity, we take $r = \alpha/2$ we get $R(\alpha,b) = K(\alpha,\alpha/2)\frac{2}{b(1-b)^{1-\alpha/2}}$. This ends the proof of the theorem.

Remark. Let us recall some estimates of $K(\alpha,r)$ which were given in the paper [3]. If we take $r = \alpha/2$ then

$$K(\alpha,\frac{\alpha}{2}) = \frac{1}{2^{1/\alpha}\sqrt{\pi}}\Gamma^{\frac{2}{\alpha}}(\frac{\alpha}{4}+\frac{1}{2})\Gamma(1+\frac{2}{\alpha})\inf_{x>0}\frac{1}{x^{2/\alpha}(1-\Phi(x))},$$

where Φ is the distribution function of a standard normal variable. For different values of r other estimates are possible, it could be interesting to find the least value of $K(\alpha,r)$. Of course, if we consider $\alpha \geq \varepsilon > 0$ then we can find

$$R(b) = \sup_{\varepsilon \leq \alpha \leq 2} R(\alpha,b) < \infty$$

and then for all $0 \leq t \leq 1$ and $\alpha \geq \varepsilon$

$$\mu(tB) \leq R(b)\,t\,\mu(B), \quad \text{if } \mu(B) \leq b.$$

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