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ON THE SPECTRAL NORM OF A RANDOM TOEPLITZ MATRIX

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Abstract

Suppose that T_n is a Toeplitz matrix whose entries come from a sequence of independent but not necessarily identically distributed random variables with mean zero. Under some additional tail conditions, we show that the spectral norm of T_n is of the order $\sqrt{n \log n}$. The same result holds for random Hankel matrices as well as other variants of random Toeplitz matrices which have been studied in the literature

1 Introduction and results

Let $X_0, X_1, X_2, \ldots$ be a family of independent random variables. For $n \geq 2$, T_n denotes the $n \times n$ random symmetric Toeplitz matrix $T_n = \left[X_{|j-k|}\right]_{1 \leq j,k \leq n}$,

$$T_n = \begin{bmatrix} X_0 & X_1 & X_2 & \cdots & X_{n-2} & X_{n-1} \\ X_1 & X_0 & X_1 & & & X_{n-2} \\ X_2 & X_1 & X_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ X_{n-2} & & & & X_0 & X_1 \\ X_{n-1} & X_{n-2} & \cdots\cdots & & X_1 & X_0 \end{bmatrix}.$$

In [1], Bai asked whether the spectral measure of $n^{-1/2}T_n$ approaches a deterministic limit measure μ as $n \to \infty$. Bryc, Dembo, and Jiang [5] and Hammond and Miller [9] independently proved that this is so when the X_j are identically distributed with variance 1, and that with these assumptions μ does not depend on the distribution of the X_j. The measure μ does not appear to be a previously studied probability measure, and is described via rather complicated expressions for its moments.

This limiting spectral measure μ has unbounded support, which raises the question of the asymptotic behavior of the spectral norm $\|T_n\|$, i.e., the maximum absolute value of an eigen-

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value of T_n. (This problem is explicitly raised in [5, Remark 1.3].) This paper shows, under slightly different assumptions from [5, 9], that $\|T_n\|$ is of the order $\sqrt{n \log n}$. Here the X_j need not be identically distributed, but satisfy stronger moment or tail conditions than in [5, 9]. The spectral norm is also of the same order for other related random matrix ensembles, including random Hankel matrices. In the case of Hankel matrices, Theorems 1 and 3 below generalize in a different direction a special case of a result of Masri and Tonge [9] on multilinear Hankel forms with ± 1 Bernoulli entries.

A random variable X will be called *subgaussian* if

$$\mathbb{P}\big[|X| \geq t\big] \leq 2e^{-at^2} \quad \forall t > 0 \tag{1}$$

for some constant $a > 0$. A family of random variables is *uniformly subgaussian* if each satisfies (1) for the same constant a.

Theorem 1. *Suppose $X_0, X_1, X_2, \dots$ are independent, uniformly subgaussian random variables with $\mathbb{E}X_j = 0$ for all j. Then*

$$\mathbb{E}\|T_n\| \leq c_1 \sqrt{n \log n},$$

where $c_1 > 0$ depends only on the constant a in the subgaussian estimate (1) *for the X_j.*

Simple scaling considerations show that one can take $c_1 = Ca^{-1/2}$ for some absolute constant $C > 0$. In principle an explicit value for C can be extracted from the proof of Theorem 1. No attempt has been made to do so, since the techniques used in this paper are suited for determining rough orders of growth, and not precise constants. Similar remarks apply to the constants which appear in the statements of Theorems 2 and 3 below.
After this paper was written, the author learned from W. Bryc that the special case of Theorem 1 in which each X_j is standard normal was previously shown in unpublished work by W. Bednorz.

By strengthening the subgaussian assumption, the statement of Theorem 1 can be improved from a bound on expectations to an almost sure asymptotic bound. Recall that a real-valued random variable X (or more properly, its distribution) is said to satisfy a *logarithmic Sobolev inequality* with constant A if

$$\mathbb{E}\big[f^2(X) \log f^2(X)\big] \leq 2A \, \mathbb{E}\big[f'(X)^2\big]$$

for every smooth $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}f^2(X) = 1$. Standard normal random variables satisfy a logarithmic Sobolev inequality with constant 1. Furthermore, it is well known that independent random variables with bounded logarithmic Sobolev constants are uniformly subgaussian and possess the same concentration properties as independent normal random variables (see [13] or [14, Chapter 5]).

Theorem 2. *Suppose $X_0, X_1, X_2, \dots$ are independent, $\mathbb{E}X_j = 0$ for all j, and for some constant A, either:*

1. *for all j, $|X_j| \leq A$ almost surely; or*
2. *for all j, X_j satisfies a logarithmic Sobolev inequality with constant A.*

Then

$$\limsup_{n\to\infty} \frac{\|T_n\|}{\sqrt{n \log n}} \le c_2$$

almost surely, where $c_2 > 0$ *depends only on* A.

We remark that according to the definition used here, T_n is a submatrix of T_{n+1}, but this is only a matter of convenience in notation. Theorem 2 remains true regardless of the dependence among the random matrices T_n for different values of n.
It seems unlikely that the stronger hypotheses of Theorem 2 are necessary. In fact a weaker version can be proved under the hypotheses of Theorem 1 alone; see the remarks following the proof of Theorem 2 in Section 2.

When the X_j have variance 1, the upper bound $\sqrt{n \log n}$ of Theorems 1 and 2 is of the correct order. In fact the matching lower bound holds under less restrictive tail assumptions, as the next result shows.

Theorem 3. *Suppose* $X_0, X_1, X_2, \ldots$ *are independent and for some constant* $B > 0$*, each* X_j *satisfies*

$$\mathbb{E}X_j = 0, \quad \mathbb{E}X_j^2 = 1, \quad \mathbb{E}|X_j| \ge B.$$

Then

$$\mathbb{E}\|T_n\| \ge c_3\sqrt{n \log n},$$

where $c_3 > 0$ *depends only on* B.

In the case that $\mathbb{E}X_j^2 = 1$ and $\mathbb{E}|X_j|^3 < \infty$, it is a consequence of Hölder's inequality that $\mathbb{E}|X_j| \ge (\mathbb{E}|X_j|^3)^{-1}$. Thus the lower bound on first absolute moments assumed in Theorem 3 is weaker than an upper bound on absolute third moments, and is in particular satisfied for uniformly subgaussian random variables.

Section 2 below contains the proofs of Theorems 1–3. As mentioned above, Theorems 1–3 also hold for other ensembles of random Toeplitz matrices, as well as for random Hankel matrices. Section 3 discusses these extensions of the theorems and makes some additional remarks.

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2 Proofs

The proof of Theorem 1 is based on Dudley's entropy bound [6] for the supremum of a subgaussian random process. Given a random process $\{Y_x : x \in M\}$, a pseudometric on M may be defined by

$$d(x, y) = \sqrt{\mathbb{E}|Y_x - Y_y|^2}.$$

The process $\{Y_x : x \in M\}$ is called *subgaussian* if

$$\forall x, y \in M,\ \forall t > 0, \quad \mathbb{P}\big[|Y_x - Y_y| \ge t\big] \le 2\exp\left[-\frac{b\, t^2}{d(x,y)^2}\right] \tag{2}$$

for some constant $b > 0$. For $\varepsilon > 0$, the ε-covering number of (M, d), $N(M, d, \varepsilon)$, is the smallest cardinality of a subset $\mathcal{N} \subset M$ such that

$$\forall x \in M \ \exists y \in \mathcal{N} : \ d(x, y) \leq \varepsilon.$$

Dudley's entropy bound is the following (see [22, Proposition 2.1] for the version given here).

Proposition 4. *Let $\{Y_x : x \in M\}$ be a subgaussian random process with $\mathbb{E} Y_x = 0$ for every $x \in M$. Then*

$$\mathbb{E} \sup_{x \in M} |Y_x| \leq K \int_0^\infty \sqrt{\log N(M, d, \varepsilon)} \ d\varepsilon,$$

where $K > 0$ depends only on the constant b in the subgaussian estimate (2) *for the process.*

We will also need the following version of the classical Azuma-Hoeffding inequality. This can be proved by a standard Laplace transform argument; see e.g. [16, Fact 2.1].

Proposition 5. *Let $X_1, \ldots, X_n$ be independent, symmetric, uniformly subgaussian random variables. Then for any $a_1, \ldots, a_n \in \mathbb{R}$ and $t > 0$,*

$$\mathbb{P}\left[\left| \sum_{j=1}^n a_j X_j \right| \geq t \right] \leq 2 \exp\left[- \frac{b \ t^2}{\sum_{j=1}^n a_j^2} \right],$$

where $b > 0$ depends only on the constant a in the subgaussian estimate (1) *for the X_j.*

Proof of Theorem 1. We first reduce to the case in which each X_j is symmetric. Let T_n' be an independent copy of T_n. Since $\mathbb{E} T_n = 0$, by Jensen's inequality,

$$\mathbb{E}\|T_n\| \leq \mathbb{E}\big[\mathbb{E}\big[\|T_n - T_n'\| \big| T_n\big]\big] = \mathbb{E}\|T_n - T_n'\|.$$

The random Toeplitz matrix $(T_n - T_n')$ has entries $(X_j - X_j')$ which are independent, symmetric, uniformly subgaussian random variables (with a possibly smaller constant a in the subgaussian estimate). Thus we may assume without loss of generality that the X_j are symmetric random variables.

We next bound $\|T_n\|$ by the supremum of a subgaussian random process. A basic feature of the theory of Toeplitz matrices is their relationship to multiplication operators (cf. [4, Chapter 1]). Specifically, the finite Toeplitz matrix T_n is an $n \times n$ submatrix of the infinite Laurent matrix

$$L_n = \big[X_{|j-k|} \mathbf{1}_{|j-k| \leq n-1}\big]_{j,k \in \mathbb{Z}}.$$

Consider L_n as an operator on $\ell^2(\mathbb{Z})$ in the canonical way, and let $\psi : \ell^2(\mathbb{Z}) \to L^2[0, 1]$ denote the usual trigonometric isometry $\psi(e_j)(x) = e^{2\pi i j x}$. Then $\psi L_n \psi^{-1} : L^2 \to L^2$ is the multiplication operator corresponding to the L^∞ function

$$f(x) = \sum_{j=-(n-1)}^{n-1} X_{|j|} e^{2\pi i j x} = X_0 + 2 \sum_{j=1}^{n-1} \cos(2\pi j x) X_j.$$

Therefore

$$\|T_n\| \leq \|L_n\| = \|f\|_\infty = \sup_{0 \leq x \leq 1} |Y_x|, \tag{3}$$

where

$$Y_x = X_0 + 2\sum_{j=1}^{n-1} \cos(2\pi jx)X_j.$$

By Proposition 5, the random process $\{Y_x : x \in [0,1]\}$ becomes subgaussian if $M = [0,1]$ is equipped with the pseudometric

$$d(x,y) = \sqrt{\sum_{j=1}^{n-1} \big[\cos(2\pi jx) - \cos(2\pi jy)\big]^2}.$$

Finally, we bound $N([0,1], d, \varepsilon)$ in order to apply Proposition 4. Since $|\cos t| \le 1$ always, it follows that $d(x,y) < 2\sqrt{n}$ and therefore $N([0,1], d, \varepsilon) = 1$ if $\varepsilon > 2\sqrt{n}$. Next, since $|\cos s - \cos t| \le |s - t|$,

$$d(x,y) \le 2\pi|x-y|\sqrt{\sum_{j=1}^{n-1} j^2} < 4n^{3/2}|x-y|,$$

which implies that

$$N\big([0,1], d, \varepsilon\big) \le N\left([0,1], |\cdot|, \frac{\varepsilon}{4n^{3/2}}\right) \le \frac{4n^{3/2}}{\varepsilon}.$$

By (3), Proposition 4, and the substitution $\varepsilon = 4n^{3/2}e^{-t^2}$,

$$\mathbb{E}\|T_n\| \le K\int_0^{2\sqrt{n}} \sqrt{\log\left(\frac{4n^{3/2}}{\varepsilon}\right)}\, d\varepsilon = 2\sqrt{2}n^{3/2}K\int_{\sqrt{2\log 2n}}^{\infty} t^2 e^{-t^2/2}\, dt. \tag{4}$$

Integration by parts and the classical estimate $\frac{1}{\sqrt{2\pi}}\int_s^\infty e^{-t^2/2}\, dt \le e^{-s^2/2}$ for $s > 0$ yield

$$\int_s^\infty t^2 e^{-t^2/2}\, dt \le \big(s + \sqrt{2\pi}\big)e^{-s^2/2}.$$

Combining the case $s = \sqrt{2\log 2n}$ of this estimate with (4) completes the proof. □

The proof of Theorem 2 is based on rather classical measure concentration arguments commonly applied to probability in Banach spaces.

Proof of Theorem 2. Denote by M_0 the $n \times n$ identity matrix, and for $m = 1, \dots, n-1$ let $M_m = \big[\mathbf{1}_{|j-k|=m}\big]_{1\le j,k\le n}$. Then T_n can be written as the sum

$$T_n = \sum_{j=0}^{n-1} X_j M_j$$

of independent random vectors in the finite-dimensional Banach space $\mathcal{M}_n$ equipped with the spectral norm. Observe that $\|M_j\| \le 2$ for every j.
Under the assumption (1), up to the precise values of constants the estimate

$$\mathbb{P}\big[\|T_n\| \ge \mathbb{E}\|T_n\| + t\big] \le e^{-t^2/32A^2n} \quad \forall t > 0$$

follows from any of several standard approaches to concentration of measure (cf. Corollary 1.17, Corollary 4.5, or Theorem 7.3 of [14]; the precise statement can be proved from Corollary 1.17). Combining this with Theorem 1 yields

$$\mathbb{P}\Big[\|T_n\| \geq (c_1 + 8A)\sqrt{n \log n}\Big] \leq \frac{1}{n^2},$$

which completes the proof via the Borel-Cantelli lemma.
The proof under the assumption (2) is similar. By the triangle inequality and the Cauchy-Schwarz inequality,

$$\|T_n\| \leq 2\sqrt{n \sum_{j=0}^{n-1} X_j^2},$$

so that the map $(X_0, \ldots, X_{n-1}) \mapsto \|T_n\|$ has Lipschitz constant bounded by $2\sqrt{n}$. By the well-known tensorization and measure concentration properties of logarithmic Sobolev inequalities (cf. [13, Sections 2.1–2.3] or [14, Sections 5.1–5.2]),

$$\mathbb{P}\big[\|T_n\| \geq \mathbb{E}\|T_n\| + t\big] \leq e^{-t^2/4An} \quad \forall t > 0.$$

The proof is completed in the same way as before (with a different dependence of c_2 on A). $\square$

As remarked above, a weaker version of Theorem 2 may be proved under the assumptions of Theorem 1 alone. From the proof of Proposition 4 in [22] one can extract the following tail inequality under the assumptions of Proposition 4:

$$\mathbb{P}\left[\sup_{x \in M} |Y_x| \geq t\right] \leq 2e^{-ct^2/\alpha^2} \quad \forall t > 0, \quad \text{where} \quad \alpha = \int_0^\infty \sqrt{\log N(M, d, \varepsilon)}\, d\varepsilon. \tag{5}$$

The explicit statement here is adapted from lecture notes of Rudelson [19]. Using the estimates derived in the proof of Theorem 1 and applying the Borel-Cantelli lemma as above, one directly obtains

$$\limsup_{n \to \infty} \frac{\|T_n\|}{\sqrt{n} \log n} \leq c_4 \quad \text{almost surely} \tag{6}$$

under the assumptions that the X_j are symmetric and uniformly subgaussian. The general (nonsymmetric but mean 0) case can be deduced from the argument for the symmetric case. Let T_n' be an independent copy of T_n. By independence, the triangle inequality, and the tail estimate which follows from (5),

$$\mathbb{P}\big[\|T_n'\| \leq s\big]\mathbb{P}\big[\|T_n\| \geq s + t\big] \leq \mathbb{P}\big[\|T_n - T_n'\| \geq t\big] \leq 2e^{-ct^2/n \log n}$$

for some constant c which depends on the subgaussian estimate for the X_j. By Theorem 1 and Chebyshev's inequality,

$$\mathbb{P}\big[\|T_n'\| \leq s\big] \geq 1 - \frac{1}{s} c_1 \sqrt{n \log n}.$$

Picking $s = 2c_1\sqrt{n \log n}$ and $t = \sqrt{\frac{2n}{c}} \log n$ yields

$$\mathbb{P}[\big[\|T_n\| \geq c_4 \sqrt{n} \log n\big] \leq \frac{4}{n^2}$$

for some constant c_4, and (6) then follows from the Borel-Cantelli lemma.

The proof of Theorem 3 amounts to an adaptation of the proof of the lower bound in [17], with much of the proof abstracted into a general lower bound for the suprema of certain random processes due to Kashin and Tzafriri [10, 11]. The following is a special case of the result of [11].

Proposition 6. *Let $\varphi_j : [0,1] \to \mathbb{R}$, $j = 0, \ldots, n-1$ be a family of functions which are orthonormal in $L^2[0,1]$ and satisfy $\|\varphi_j\|_{L^3[0,1]} \leq A$ for every j, and let $X_0, \ldots, X_{n-1}$ be independent random variables such that for every j,*

$$\mathbb{E}X_j = 0, \quad \mathbb{E}X_j^2 = 1, \quad \mathbb{E}|X_j| \geq B.$$

Then for any $a_0, \ldots, a_{n-1} \in \mathbb{R}$,

$$\mathbb{E}\left[\sup_{0\leq x\leq 1}\left|\sum_{j=0}^{n-1} a_j X_j \varphi_j(x)\right|\right] \geq K\ \|a\|_2 \sqrt{\log \frac{\|a\|_2}{\|a\|_4}},$$

where $\|a\|_p = \left(\sum_{j=0}^{n-1} |a_j|^p\right)^{1/p}$ and $K > 0$ depends only on A and B.

Proof of Theorem 3. First make the estimate

$$\|T_n\| = \sup_{v\in\mathbb{C}^n\setminus\{0\}} \frac{|\langle T_n v, v\rangle|}{\langle v, v\rangle} \geq \sup_{0\leq x\leq 1} \frac{1}{n}\left|\langle T_n v_x, v_x\rangle\right|,$$

where $v_x \in \mathbb{C}^n$ is defined by $(v_x)_j = e^{2\pi i j x}$ for $j = 1, \ldots, n$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{C}^n$. Therefore

$$\begin{aligned}
\|T_n\| &\geq \frac{1}{n}\sup_{0\leq x\leq 1}\left|\sum_{j,k=1}^{n} X_{|j-k|} e^{2\pi i (j-k)x}\right| \\
&= \frac{1}{n}\sup_{0\leq x\leq 1}\left|\sum_{j=-(n-1)}^{n-1} (n-|j|) X_{|j|} e^{2\pi i j x}\right| \\
&= \sup_{0\leq x\leq 1}\left|X_0 + 2\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right) X_j \cos(2\pi j x)\right| \\
&= \sup_{0\leq x\leq 1}\left|\sum_{j=0}^{n-1} a_j X_j \varphi_j(x)\right|,
\end{aligned}$$

where we have defined $a_0 = 1$, $a_j = \sqrt{2}(1-j/n)$ for $j \geq 1$, $\varphi_0 \equiv 1$, and $\varphi_j(x) = \sqrt{2}\cos(2\pi j x)$ for $j \geq 1$. It is easy to verify that $\|a\|_2 > \sqrt{n}/2$ and $\|a\|_4 < 2n^{1/4}$. The theorem now follows from Proposition 6. ☐

We remark that by combining Theorem 3 with the proof of Theorem 2, one obtains a non-trivial bound on the left tail of $\|T_n\|$ under the assumptions of Theorem 2 and the additional assumption that $\mathbb{E}X_j^2 = 1$ for every j. Unfortunately, one cannot deduce an almost sure lower bound of the form

$$\liminf_{n\to\infty} \frac{\|T_n\|}{\sqrt{n \log n}} \geq c \quad \text{almost surely}$$

without more precise control over the constants in Proposition 6 and the concentration inequalities used in the proof of Theorem 2.

3 Extensions and additional remarks

3.1 Other random matrix ensembles

For simplicity Theorems 1–3 were stated and proved only for the case of real symmetric Toeplitz matrices. However, straightforward adaptations of the proofs show that the theorems hold for other related ensembles of random matrices. These include nonsymmetric real Toeplitz matrices $\left[X_{j-k}\right]_{1\leq j,k\leq n}$ for independent random variables X_j, $j \in \mathbb{Z}$, as well as complex Hermitian or general complex Toeplitz variants. In the complex cases one should consider matrix entries of the form $X_j = Y_j + iZ_j$, where Y_j and Z_j are independent and each satisfy the tail or moment conditions imposed on X_j in the theorems as stated.

Closely related to the case of nonsymmetric random Toeplitz matrices are random Hankel matrices $H_n = \left[X_{j+k-1}\right]_{1\leq j,k\leq n}$, which are constant along skew diagonals. This ensemble was also mentioned by Bai [1], and was shown to have a universal limiting spectral distribution in [5]. Independently, Masri and Tonge [17] considered a random r-linear Hankel form

$$(v_1, \dots, v_r) \mapsto \sum_{j_1,\dots,j_r=0}^{n} X_{j_1+\cdots+j_r} (v_1)_{j_1} \cdots (v_r)_{j_r}$$

in the case $\mathbb{P}[X_j = 1] = \mathbb{P}[X_j = -1] = 1/2$, and showed that the expected norm of this form is of the order $\sqrt{n^{r-1}\log n}$. As observed in [5, Remark 1.2], H_n has the same singular values, and so in particular the same spectral norm, as the (nonsymmetric) Toeplitz matrix obtained by reversing the order of the rows of H_n. Therefore Theorems 1–3 apply to H_n as well. As mentioned in the introduction, the versions of Theorems 1 and 3 for H_n generalize the $r = 2$ case of the result of [17] to subgaussian matrix entries X_j.

The methods of this paper can also be used to treat random Toeplitz matrices with additional restrictions. For example, the theorems apply to the ensemble of symmetric circulant matrices considered in [2, Remark 2] which is defined as T_n here except for the restriction that $X_{n-j} = X_j$ for $j = 1, \dots, n-1$, and the closely related symmetric palindromic Toeplitz matrices considered in [18], in which $X_{n-j-1} = X_j$ for $j = 0, \dots, n-1$. We remark that [2, 18] show that each of these ensembles, properly scaled and with some additional assumptions, have a limiting spectral distribution which is normal.

3.2 Weaker hypotheses

It is unclear how necessary the tail or moment conditions on the X_j are to the conclusions of the theorems. It appears likely (cf. [24, 3]) that versions of Theorems 1 and 2 remain true assuming only the existence of fourth moments, at least when the X_j are identically distributed. In particular it is very likely that the assumptions of Theorem 2 can be relaxed considerably.

Even within the present proof, the assumption of a logarithmic Sobolev inequality can be weakened slightly to that of a quadratic transportation cost inequality; cf. [14, Chapter 6].
If the X_j have nonzero means then the behavior of $\|T_n\|$ may change. Suppose first that the X_j are uniformly subgaussian and $\mathbb{E}X_j = m \neq 0$ for every j. If J_n denotes the $n \times n$ matrix whose entries are all 1, then (6) implies that

$$\limsup_{n\to\infty} \frac{\|T_n - mJ_n\|}{\sqrt{n}\log n} \leq c \quad \text{almost surely,} \tag{7}$$

where c depends on m and the subgaussian estimate for the X_j. Since $\|J_n\| = n$, (7) and the triangle inequality imply a strong law of large numbers:

$$\lim_{n\to\infty} \frac{\|T_n\|}{n} = |m| \quad \text{almost surely.} \tag{8}$$

In [3], (8) was proved using estimates from [5] under the assumption that the X_j are identically distributed and have finite variance. We emphasize again that while the methods of this paper require stronger tail conditions, we never assume the X_j to be identically distributed.
More generally, the behavior of $\|T_n\|$ depends on the rate of growth of the spectral norms of the deterministic Toeplitz matrices $\mathbb{E}T_n$. The same argument as above shows that

$$\lim_{n\to\infty} \frac{\|T_n\|}{\|\mathbb{E}T_n\|} = 1 \quad \text{almost surely}$$

if the random variables $(X_j - \mathbb{E}X_j)$ are uniformly subgaussian and $\lim_{n\to\infty} \frac{\sqrt{n}\log n}{\|\mathbb{E}T_n\|} = 0$. On the other hand, if $\|\mathbb{E}T_n\| = o(\sqrt{n \log n})$ then the conclusion of Theorem 1 holds.

3.3 Random trigonometric polynomials

The supremum of the random trigonometric polynomial

$$Z_x = \sum_{j=1}^{n} X_j \cos(2\pi j x),$$

has been well-studied in the special case $\mathbb{P}[X_j = 1] = \mathbb{P}[X_j = -1] = 1/2$, in work dating back to Salem and Zygmund [20]. Observe that Z_x is essentially equivalent to the process Y_x defined in the proof of Theorem 1, and is also closely related to the random process considered in the proof of Theorem 3. Halász [8] proved in particular that

$$\lim_{n\to\infty} \frac{\sup_{0\leq x\leq 1} |Z_x|}{\sqrt{n \log n}} = 1 \quad \text{almost surely.}$$

From this it follows that when $\mathbb{P}[X_j = 1] = \mathbb{P}[X_j = -1] = 1/2$ for every j, the conclusion of Theorem 2 holds with $c_2 = 2$. Numerical experiments suggest, however, that the optimal value of c_2 is 1 in this case, and more generally when the X_j are i.i.d. with mean 0 and variance 1. Conversely, adaptations of the proofs in this paper yield less numerically precise bounds for the supremum of Z_x under the same weaker assumptions on the X_j in the statements of the theorems. We remark that the techniques used to prove the results of [10, 11, 17] cited above (and hence indirectly also Theorem 3) were adapted from the work of Salem and Zygmund in [20].

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