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# HSU-ROBBINS AND SPITZER'S THEOREMS FOR THE VARIATIONS OF FRACTIONAL BROWNIAN MOTION

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*Abstract*

Using recent results on the behavior of multiple Wiener-Itô integrals based on Stein's method, we prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables related to the increments of the fractional Brownian motion.

## 1 Introduction

A famous result by Hsu and Robbins [7] says that if $X_1, X_2, \ldots$ is a sequence of independent identically distributed random variables with zero mean and finite variance and $S_n := X_1 + \ldots + X_n$, then

$$\sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right) < \infty$$

for every $\varepsilon > 0$. Later, Erdös ([3], [4]) showed that the converse implication also holds, namely if the above series is finite for every $\varepsilon > 0$ and $X_1, X_2, \ldots$ are independent and identically distributed, then $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. Since then, many authors extended this result in several directions. Spitzer's showed in [13] that

$$\sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right) < \infty$$

for every $\varepsilon > 0$ if and only if $\mathbf{E}X_1 = 0$ and $\mathbf{E}|X_1| < \infty$. Also, Spitzer's theorem has been the object of various generalizations and variants. One of the problems related to the Hsu-Robbins' and Spitzer's theorems is to find the precise asymptotic as $\varepsilon \to 0$ of the quantities $\sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right)$ and $\sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right)$. Heyde [5] showed that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right) = \mathbf{E}X_1^2 \tag{1}$$

whenever $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. In the case when $X$ is attracted to a stable distribution of exponent $\alpha > 1$, Spataru [12] proved that

$$\lim_{\varepsilon\to 0} \frac{1}{-\log\varepsilon} \sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right) = \frac{\alpha}{\alpha - 1}. \tag{2}$$

The purpose of this paper is to prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables, related to the increments of fractional Brownian motion, in the spirit of [5] or [12]. Recall that the fractional Brownian motion $(B_t^H)_{t\in[0,1]}$ is a centered Gaussian process with covariance function $R^H(t,s) = \mathbf{E}(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. It can be also defined as the unique self-similar Gaussian process with stationary increments. Concretely, in this paper we will study the behavior of the tail probabilities of the sequence

$$V_n = \sum_{k=0}^{n-1} H_q\left(n^H \left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}\right)\right) \tag{3}$$

where $B$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ (in the sequel we will omit the superscript $H$ for $B$) and $H_q$ is the Hermite polynomial of degree $q \geq 1$ given by $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}(e^{-\frac{x^2}{2}})$. The sequence $V_n$ behaves as follows (see e.g. [9], Theorem 1; the result is also recalled in Section 3 of our paper): if $0 < H < 1-\frac{1}{2q}$, a central limit theorem holds for the renormalized sequence $Z_n^{(1)} = \frac{V_n}{c_{1,q,H}\sqrt{n}}$ while if $1-\frac{1}{2q} < H < 1$, the sequence $Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}}$ converges in $L^2(\Omega)$ to a Hermite random variable of order $q$ (see Section 2 for the definition of the Hermite random variable and Section 3 for a rigorous statement concerning the convergence of $V_n$). Here $c_{1,q,H}, c_{2,q,H}$ are explicit positive constants depending on $q$ and $H$.
We note that the techniques generally used in the literature to prove the Hsu-Robbins and Spitzer's results are strongly related to the independence of the random variables $X_1, X_2, \ldots$. In our case the variables are correlated. Indeed, for any $k, l \geq 1$ we have

$$\mathbf{E}\left(H_q(B_{k+1} - B_k) H_q(B_{l+1} - B_l)\right) = \frac{1}{(q!)^2} \rho_H(k-l)$$

where the correlation function is $\rho_H(k) = \frac{1}{2}\left((k+1)^{2H} + (k-1)^{2H} - 2k^{2H}\right)$ which is not equal to zero unless $H = \frac{1}{2}$ (which is the case of the standard Brownian motion). We use new techniques based on the estimates for the multiple Wiener-Itô integrals obtained in [2], [10] via Stein's method and Malliavin calculus. Concretely, we study in this paper the behavior as $\varepsilon \to 0$ of the quantities

$$\sum_{n\geq 1} \frac{1}{n} P\left(V_n > \varepsilon n\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right), \tag{4}$$

and

$$\sum_{n\geq 1} P\left(V_n > \varepsilon n\right) = \sum_{n\geq 1} P\left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right), \tag{5}$$

if $0 < H < 1 - \frac{1}{2q}$ and of

$$\sum_{n\geq 1} \frac{1}{n} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right) \tag{6}$$

and

$$\sum_{n\geq 1} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n\geq 1} P\left(Z_n^{(2)} > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right) \tag{7}$$

if $1-\frac{1}{2q} < H < 1$. The basic idea in our proofs is that, if we replace $Z_n^{(1)}$ and $Z_n^{(2)}$ by their limits (standard normal random variable or Hermite random variable) in the above expressions, the behavior as $\varepsilon \to 0$ can be obtained by standard calculations. Then we need to estimate the difference between the tail probabilities of $Z_n^{(1)}, Z_n^{(2)}$ and the tail probabilities of their limits. To this end, we will use the estimates obtained in [2], [10] via Malliavin calculus and we are able to prove that this difference converges to zero in all cases. We obtain that, as $\varepsilon \to 0$, the quantities (4) and (6) are of order of $\log \varepsilon$ while the functions (5) and (7) are of order of $\varepsilon^2$ and $\varepsilon^{1-q(1-H)}$ respectively.

The paper is organized as follows. Section 2 contains some preliminaries on the stochastic analysis on Wiener chaos. In Section 3 we prove the Spitzer's theorem for the variations of the fractional Brownian motion while Section 4 is devoted to the Hsu-Robbins theorem for this sequence.

Throughout the paper we will denote by $c$ a generic strictly positive constant which may vary from line to line (and even on the same line).

## 2 Preliminaries

Let $(W_t)_{t\in[0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathscr{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of $f$ with respect to $W$. The basic reference is [11].

Let $f \in \mathscr{S}_m$ be an elementary function with $m$ variables that can be written as

$$f = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} 1_{A_{i_1}\times\ldots\times A_{i_m}}$$

where the coefficients satisfy $c_{i_1,\ldots i_m} = 0$ if two indices $i_k$ and $i_l$ are equal and the sets $A_i \in \mathscr{B}([0,1])$ are disjoint. For such a step function $f$ we define

$$I_m(f) = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} W(A_{i_1})\ldots W(A_{i_m})$$

where we put $W(A) = \int_0^1 1_A(s)dW_s$ if $A \in \mathscr{B}([0,1])$. It can be seen that the mapping $I_n$ constructed above from $\mathscr{S}_m$ to $L^2(\Omega)$ is an isometry on $\mathscr{S}_m$ , i.e.

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = n!\langle f,g\rangle_{L^2([0,1]^n)} \text{ if } m = n \tag{8}$$

and

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = 0 \text{ if } m \neq n.$$

Since the set $\mathscr{S}_n$ is dense in $L^2([0,1]^n)$ for every $n \geq 1$ the mapping $I_n$ can be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension.

We will need the following bound for the tail probabilities of multiple Wiener-Itô integrals (see [8], Theorem 4.1)

$$P\left(|I_n(f)| > u\right) \leq c \exp\left(\left(\frac{-cu}{\sigma}\right)^{\frac{2}{n}}\right) \tag{9}$$

for all $u > 0$, $n \geq 1$, with $\sigma = \|f\|_{L^2([0,1]^n)}$.
The Hermite random variable of order $q \geq 1$ that appears as limit in Theorem 1, point ii. is defined as (see [9])

$$Z = d(q,H) I_q(L) \tag{10}$$

where the kernel $L \in L^2([0,1]^q)$ is given by

$$L(y_1, \dots, y_q) = \int_{y_1 \vee \dots \vee y_q}^{1} \partial_1 K^H(u, y_1) \dots \partial_1 K^H(u, y_q) du.$$

The constant $d(q,H)$ is a positive normalizing constant that guarantees that $\mathbf{E}Z^2 = 1$ and $K^H$ is the standard kernel of the fractional Brownian motion (see [11], Section 5). We will not need the explicit expression of this kernel. Note that the case $q = 1$ corresponds to the fractional Brownian motion and the case $q = 2$ corresponds to the Rosenblatt process.

# 3 Spitzer's theorem

Let us start by recalling the following result on the convergence of the sequence $V_n$ (3) (see [9], Theorem 1).

**Theorem 1.** *Let $q \geq 2$ an integer and let $(B_t)_{t \geq 0}$ a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then, with some explicit positive constants $c_{1,q,H}, c_{2,q,H}$ depending only on q and H we have*

*i. If $0 < H < 1 - \frac{1}{2q}$ then*

$$\frac{V_n}{c_{1,q,H}\sqrt{n}} \xrightarrow{\text{Law}}_{n \to \infty} N(0,1) \tag{11}$$

*ii. If $1 - \frac{1}{2q} < H < 1$ then*

$$\frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \xrightarrow{\text{L}^2}_{n \to \infty} Z \tag{12}$$

*where Z is a Hermite random variable given by (10).*

In the case $H = 1 - \frac{1}{2q}$ the limit is still Gaussian but the normalization is different. However we will not treat this case in the present work.
We set

$$Z_n^{(1)} = \frac{V_n}{c_{1,q,H}\sqrt{n}}, \quad Z_n^{(2)} = \frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \tag{13}$$

with the constants $c_{1,q,H}, c_{2,q,H}$ from Theorem 1.
Let us denote, for every $\varepsilon > 0$,

$$f_1(\varepsilon) = \sum_{n \geq 1} \frac{1}{n} P\left(V_n > \varepsilon n\right) = \sum_{n \geq 1} \frac{1}{n} P\left(Z_n^{(1)} > c_{1,q,H}^{-1} \varepsilon \sqrt{n}\right) \tag{14}$$

and

$$f_2(\varepsilon) = \sum_{n \geq 1} \frac{1}{n} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n \geq 1} \frac{1}{n} P\left(Z_n^{(2)} > c_{2,q,H}^{-1} \varepsilon n^{1-q(1-H)}\right) \tag{15}$$

**Remark 1.** *It is natural to consider the tail probability of order $n^{2-2q(1-H)}$ in (15) because the $L^2$ norm of the sequence $V_n$ is in this case of order $n^{1-q(1-H)}$.*

We are interested to study the behavior of $f_i(\varepsilon)$ $(i=1,2)$ as $\varepsilon \to 0$. For a given random variable $X$, we set $\Phi_X(z) = 1 - P(X < z) + P(X < -z)$.
The first lemma gives the asymptotics of the functions $f_i(\epsilon)$ as $\varepsilon \to 0$ when $Z_n^{(i)}$ are replaced by their limits.

**Lemma 1.** *Consider $c > 0$.*

*i. Let $Z^{(1)}$ be a standard normal random variable. Then as*

$$\frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) \to_{\varepsilon \to 0} 2.$$

*ii. Let $Z^{(2)}$ be a Hermite random variable or order $q$ given by (10). Then, for any integer $q \geq 1$*

$$\frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \to_{\varepsilon \to 0} \frac{1}{1-q(1-H)}.$$

**Proof:** The case when $Z^{(1)}$ follows the standard normal law is hidden in [12]. We will give the ideas of the proof. We can write (see [12])

$$\sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) = \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx - \frac{1}{2} \Phi_{Z^{(1)}}(c\varepsilon) - \int_1^\infty P_1(x) d\left[\frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right].$$

with $P_1(x) = [x] - x + \frac{1}{2}$. Clearly as $\varepsilon \to 0$, $\frac{1}{\log \varepsilon} \Phi_{Z^{(1)}}(c\varepsilon) \to 0$ because $\Phi_{Z^{(1)}}$ is a bounded function and concerning the last term it is also trivial to observe that

$$\begin{aligned}
&\frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) d\left[\frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right] \\
&= \frac{1}{-\log c\varepsilon} \left( -\int_1^\infty P_1(x) \left( \frac{1}{x^2} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx + c\varepsilon \frac{1}{2} x^{-\frac{1}{2}} \frac{1}{x} \Phi'_{Z^{(1)}}(\varepsilon\sqrt{x}) \right) dx \right) \to_{\varepsilon \to 0} 0
\end{aligned}$$

since $\Phi_{Z^{(1)}}$ and $\Phi'_{Z^{(1)}}$ are bounded. Therefore the asymptotics of the function $f_1(\varepsilon)$ as $\varepsilon \to 0$ will be given by $\int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx$. By making the change of variables $c\varepsilon\sqrt{x} = y$, we get

$$\lim_{\varepsilon \to 0} \frac{1}{-\log c\varepsilon} \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x}) dx = \lim_{\varepsilon \to 0} \frac{1}{-\log c\varepsilon} 2 \int_{c\varepsilon}^\infty \frac{1}{y} \Phi_{Z^{(1)}}(y) dy = \lim_{\varepsilon \to 0} 2\Phi_{Z^{(1)}}(c\varepsilon) = 2.$$

Let us consider now the case of the Hermite random variable. We will have as above

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{-\log c\varepsilon} \sum_{n \geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \\
= \quad &\lim_{\varepsilon \to 0} \frac{1}{-\log c\varepsilon} \left( \int_1^\infty \frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx - \int_1^\infty P_1(x) d\left[\frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]\right)
\end{aligned}$$

By making the change of variables $c\varepsilon x^{1-q(1-H)} = y$ we will obtain

$$\begin{aligned}
& \lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} \int_1^\infty \frac{1}{x} \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx \\
= & \lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} \frac{1}{1-q(1-H)} \int_{c\varepsilon}^\infty \frac{1}{y} \Phi_{Z^{(2)}}(y) dy = \lim_{\varepsilon\to 0} \frac{1}{1-q(1-H)} \Phi_{Z^{(2)}}(c\varepsilon) = \frac{1}{1-q(1-H)}
\end{aligned}$$

where we used the fact that $\Phi_{Z^{(2)}}(y) \leq y^{-2}\mathbf{E}|Z^{(2)}|^2$ and so $\lim_{y\to\infty} \log y \Phi_{Z^{(2)}}(y) = 0$.
It remains to show that $\frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) d\left[\frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]$ converges to zero as $\varepsilon$ tends to 0 (note that actually it follows from a result by [1] that a Hermite random variable has a density, but we don't need it explicitly, we only use the fact that $\Phi_{Z^{(2)}}$ is differentiable almost everywhere). This is equal to

$$\begin{aligned}
& \lim_{\varepsilon} \frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) c\varepsilon (1-q(1-H)) x^{-q(1-H)-1} \Phi'_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) dx \\
= & \; c \frac{\varepsilon}{-\log \varepsilon} (c\varepsilon)^{\frac{q(1-H)}{1-q(1-H)}} \int_{c\varepsilon}^\infty P_1\left(\left(\frac{y}{c\varepsilon}\right)^{\frac{1}{1-q(1-H)}}\right) \Phi'_{Z^{(2)}}(y) y^{-\frac{1}{1-q(1-H)}} dy \\
\leq & \; c \frac{1}{-\log \varepsilon} \int_{c\varepsilon}^\infty P_1\left(\left(\frac{1}{c\varepsilon}\right)^{\frac{1}{1-q(1-H)}}\right) \Phi'_{Z^{(2)}}(y) dy
\end{aligned}$$

which clearly goes to zero since $P_1$ is bounded and $\int_0^\infty \Phi'_{Z^{(2)}}(y) dy = 1$. ■

The next result estimates the limit of the difference between the functions $f_i(\varepsilon)$ given by (14), (15) and the sequence in Lemma 1.

**Proposition 1.** *Let $q \geq 2$ and $c > 0$.*

*i. If $H < 1 - \frac{1}{2q}$, let $Z_n^{(1)}$ be given by (13) and let $Z^{(1)}$ be standard normal random variable. Then it holds*

$$\frac{1}{-\log c\varepsilon} \left[ \sum_{n\geq 1} \frac{1}{n} P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1} \frac{1}{n} P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) \right] \to_{\varepsilon\to 0} 0.$$

*ii. Let $Z^{(2)}$ be a Hermite random variable of order $q \geq 2$ and $H > 1 - \frac{1}{2q}$. Then*

$$\frac{1}{-\log c\varepsilon} \left[ \sum_{n\geq 1} \frac{1}{n} P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - \sum_{n\geq 1} \frac{1}{n} P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) \right] \to_{\varepsilon\to 0} 0.$$

**Proof:** Let us start with the point i. Assume $H < 1 - \frac{1}{2q}$. We can write

$$\begin{aligned}
& \sum_{n\geq 1} \frac{1}{n} P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1} \frac{1}{n} P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) \\
= & \sum_{n\geq 1} \frac{1}{n} \left[ P\left(Z_n^{(1)} > c\varepsilon\sqrt{n}\right) - P\left(Z^{(1)} > c\varepsilon\sqrt{n}\right) \right] + \sum_{n\geq 1} \left[ \frac{1}{n} P\left(Z_n^{(1)} < -c\varepsilon\sqrt{n}\right) - P\left(Z^{(1)} < -c\varepsilon\sqrt{n}\right) \right] \\
\leq & \; 2 \sum_{n\geq 1} \frac{1}{n} \sup_{x\in\mathbb{R}} \left| P\left(Z_n^{(1)} > x\right) - P\left(Z^{(1)} > x\right) \right|.
\end{aligned}$$

It follows from [10], Theorem 4.1 that

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)}>x\right)-P\left(Z^{(1)}>x\right)\right|\le c\begin{cases}\frac{1}{\sqrt{n}}, & H\in(0,\frac{1}{2}]\\ n^{H-1}, & H\in[\frac{1}{2},\frac{2q-3}{2q-2})\\ n^{qH-q+\frac{1}{2}}, & H\in[\frac{2q-3}{2q-2},1-\frac{1}{2q}).\end{cases} \tag{16}$$

and this implies that

$$\sum_{n\ge1}\frac{1}{n}\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(i)}>x\right)-P\left(Z^{(i)}>x\right)\right|\le c\begin{cases}\sum_{n\ge1}\frac{1}{n\sqrt{n}}, & H\in(0,\frac{1}{2}]\\ \sum_{n\ge1}n^{H-2}, & H\in[\frac{1}{2},\frac{2q-3}{2q-2})\\ \sum_{n\ge1}n^{qH-q-\frac{1}{2}}, & H\in[\frac{2q-3}{2q-2},1-\frac{1}{2q}).\end{cases} \tag{17}$$

and the last sums are finite (for the last one we use $H<1-\frac{1}{2q}$). The conclusion follows.

Concerning the point ii. (the case $H>1-\frac{1}{2q}$), by using a result in Proposition 3.1 of [2] we have

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(i)}>x\right)-P\left(Z^{(i)}>x\right)\right|\le c\left(\mathbf{E}\left|Z_n^{(2)}-Z^{(2)}\right|^2\right)^{\frac{1}{2q}}\le cn^{1-\frac{1}{2q}-H} \tag{18}$$

and as a consequence

$$\sum_{n\ge1}\frac{1}{n}P\left(|Z_n^{(2)}|>c\varepsilon n^{1-q(1-H)}\right)-\sum_{n\ge1}\frac{1}{n}P\left(|Z^{(2)}|>c\varepsilon n^{1-q(1-H)}\right)\le c\sum_{n\ge1}n^{-\frac{1}{2q}-H}$$

and the above series is convergent because $H>1-\frac{1}{2q}$. ∎

We state now the Spitzer's theorem for the variations of the fractional Brownian motion.

**Theorem 2.** *Let $f_1,f_2$ be given by (14), (15) and the constants $c_{1,q,H},c_{2,q,H}$ be those from Theorem 1.*

*i. If $0<H<1-\frac{1}{2q}$ then*

$$\lim_{\varepsilon\to0}\frac{1}{\log(c_{1,H,q}^{-1}\varepsilon)}f_1(\varepsilon)=2.$$

*ii. If $1>H>1-\frac{1}{2q}$ then*

$$\lim_{\varepsilon\to0}\frac{1}{\log(c_{2,H,q}^{-1}\varepsilon)}f_2(\varepsilon)=\frac{1}{1-q(1-H)}.$$

**Proof:** It is a consequence of Lemma 1 and Proposition 1. ∎

**Remark 2.** *Concerning the case $H=1-\frac{1}{2q}$, note that the correct normalization of $V_n$ (3) is $\frac{1}{(\log n)\sqrt{n}}$. Because of the appearance of the term $\log n$ our approach is not directly applicable to this case.*

# 4 Hsu-Robbins theorem for the variations of fractional Brownian motion

In this section we prove a version of the Hsu-Robbins theorem for the variations of the fractional Brownian motion. Concretely, we denote here by, for every $\varepsilon > 0$

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n\right) \tag{19}$$

if $H < 1 - \frac{1}{2q}$ and by

$$g_2(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2-2q(1-H)}\right) \tag{20}$$

if $H > 1 - \frac{1}{2q}$. and we estimate the behavior of the functions $g_i(\varepsilon)$ as $\varepsilon \to 0$. Note that we can write

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right), \quad g_2(\varepsilon) = \sum_{n\geq 1} P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right)$$

with $Z_n^{(1)}, Z_n^{(2)}$ given by (13).
We decompose it as: for $H < 1 - \frac{1}{2q}$

$$\begin{aligned} g_1(\varepsilon) &= \sum_{n\geq 1} P\left(|Z^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) \\ &+ \sum_{n\geq 1}\left[P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right)\right]. \end{aligned}$$

and for $H > 1 - \frac{1}{2q}$

$$\begin{aligned} g_2(\varepsilon) &= \sum_{n\geq 1} P\left(|Z^{(2)}| > \varepsilon c_{2,q,H}^{-1} n^{1-q(1-H)}\right) \\ &+ \sum_{n\geq 1}\left[P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right)\right]. \end{aligned}$$

We start again by consider the situation when $Z_n^{(i)}$ are replaced by their limits.

**Lemma 2.** *i. Let $Z^{(1)}$ be a standard normal random variable. Then*

$$\lim_{\varepsilon\to 0}(c\varepsilon)^2 \sum_{n\geq 1} P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) = 1.$$

*ii. Let $Z^{(2)}$ be a Hermite random variable with $H > 1 - \frac{1}{2q}$. Then*

$$\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n\geq 1} P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) = \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}.$$

**Proof:** The part i. is a consequence of the result of Heyde [5]. Indeed take $X_i \sim N(0,1)$ in (1). Concerning part ii. we can write

$$\begin{aligned}
& \lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}}\sum_{n\geq 1}\Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \\
= \ & \lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}}\left[\int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx - \int_1^\infty P_1(x)d\left[\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]\right] \\
& := \lim_{\varepsilon\to 0}(A(\varepsilon)+B(\varepsilon))
\end{aligned}$$

with $P_1(x) = [x] - x + \frac{1}{2}$. Moreover

$$\begin{aligned}
A(\varepsilon) &= (c\varepsilon)^{\frac{1}{1-q(1-H)}}\int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx \\
&= \frac{1}{1-q(1-H)}\int_{c\varepsilon}^\infty \Phi_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}-1}dy.
\end{aligned}$$

Since $\Phi_{Z^{(2)}}(y) \leq y^{-2}$ we have $\Phi_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}} \to_{y\to\infty} 0$ and therefore

$$A(\varepsilon) = -\Phi_{Z^{(2)}}(c\varepsilon)(c\varepsilon)^{\frac{1}{1-q(1-H)}} - \int_{c\varepsilon}^\infty \Phi'_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}}dy$$

where the first terms goes to zero and the second to $\mathbf{E}\left|Z^{(2)}\right|^{\frac{1}{1-q(1-H)}}$. The proof that the term $B(\varepsilon)$ converges to zero is similar to the proof of Lemma 2, point ii. ∎

**Remark 3.** *The Hermite random variable has moments of all orders (in particular the moment of order $\frac{1}{1-q(1-H)}$ exists) since it is the value at time 1 of a selfsimilar process with stationary increments.*

**Proposition 2.** *i. Let $H < 1 - \frac{1}{2q}$ and let $Z_n^{(1)}$ be given by (13). Let also $Z^{(1)}$ be a standard normal random variable. Then*

$$(c\varepsilon)^2\sum_{n\geq 1}\left[P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \to_{\varepsilon\to 0} 0$$

*ii. Let $H > 1 - \frac{1}{2q}$ and let $Z_n^{(2)}$ be given by (13). Let $Z^{(2)}$ be a Hermite random variable. Then*

$$(c\varepsilon)^{\frac{1}{1-q(1-H)}}\sum_{n\geq 1}\left[P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \to_{\varepsilon\to 0} 0.$$

**Remark 4.** *Note that the bounds (16), (18) does not help here because the series that appear after their use are not convergent.*

**Proof of Proposition 2:** *Case* $H < 1 - \frac{1}{2q}$. We have, for some $\beta > 0$ to be chosen later,

$$\begin{aligned}
&\varepsilon^2 \sum_{n\geq 1} \left[ P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\
= \;& \varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} \left[ P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\
& + \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} \left[ P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\
:= \;& I_1(\varepsilon) + J_1(\varepsilon).
\end{aligned}$$

Consider first the situation when $H \in (0, \frac{1}{2}]$. Let us choose a real number $\beta$ such that $2 < \beta < 4$. By using (16),

$$I_1(\varepsilon) \leq c\varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} n^{-\frac{1}{2}} \leq c\varepsilon^2 \varepsilon^{-\frac{\beta}{2}} \to_{\varepsilon\to 0} 0$$

since $\beta < 4$. Next, by using the bound for the tail probabilities of multiple integrals and since $\mathbf{E}\left|Z_n^{(1)}\right|^2$ converges to 1 as $n \to \infty$

$$J_1(\varepsilon) = \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} P\left(Z_n^{(1)} > c\varepsilon\sqrt{n}\right) \leq c\varepsilon^{-2} \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\frac{-c\varepsilon\sqrt{n}}{\left(\mathbf{E}\left|Z_n^{(1)}\right|^2\right)^{\frac{1}{2}}}\right)^{\frac{2}{q}}$$

$$\leq \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\left(-cn^{-\frac{1}{\beta}}\sqrt{n}\right)^{\frac{2}{q}}\right)$$

and since converges to zero for $\beta > 2$. The same argument shows that $\varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} P\left(Z^{(1)} > c\varepsilon\sqrt{n}\right)$ converges to zero.
The case when $H \in (\frac{1}{2}, \frac{2q-3}{2q-2})$ can be obtained by taking $2 < \beta < \frac{2}{H}$ (it is possible since $H < 1$) while in the case $H \in (\frac{2q-3}{2q-2}, 1 - \frac{1}{2q})$ we have to choose $2 < \beta < \frac{2}{qH-q+\frac{3}{2}}$ (which is possible because $H < 1 - \frac{1}{2q}$!).

*Case* $H > 1 - \frac{1}{2q}$. We have, with some suitable $\beta > 0$

$$\begin{aligned}
&\varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n\geq 1} \left[ P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \\
= \;& \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n=1}^{[\varepsilon^{-\beta}]} \left[ P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \\
& + \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n\geq[\varepsilon^{-\beta}]} \left[ P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \\
:= \;& I_2(\varepsilon) + J_2(\varepsilon).
\end{aligned}$$

Choose $\frac{1}{1-q(1-H)} < \beta < \frac{1}{(1-q(1-H))(2-H-\frac{1}{2q})}$ (again, this is always possible when $H > 1 - \frac{1}{2q}$!). Then

$$I_2(\varepsilon) \leq c e^{\frac{1}{1-q(1-H)}} \varepsilon^{(-\beta)(2-H-\frac{1}{2q})} \to_{\varepsilon\to 0} 0$$

and by (9)

$$J_2(\varepsilon) \leq c \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\left(\frac{-c\varepsilon n^{1-q(1-H)}}{\left(\mathbf{E}\left|Z_n^{(2)}\right|^2\right)^{\frac{1}{2}}}\right)^{\frac{2}{q}}\right) \leq c \sum_{n>[\varepsilon^{-\beta}]} \exp\left(cn^{-\frac{1}{\beta}} n^{1-q(1-H)}\right)^{\frac{2}{q}} \to_{\varepsilon\to 0} 0$$

■

We state the main result of this section which is a consequence of Lemma 2 and Proposition 2.

**Theorem 3.** *Let $q \geq 2$ and let $c_{1,q,H}, c_{2,q,H}$ be the constants from Theorem 1. Let $Z^{(1)}$ be a standard normal random variable, $Z^{(2)}$ a Hermite random variable of order $q \geq 2$ and let $g_1, g_2$ be given by (19) and (20). Then*

*i. If $0 < H < 1 - \frac{1}{2q}$, we have $(c_{1,q,H}^{-1}\varepsilon)^2 g_1(\varepsilon) \to_{\varepsilon\to 0} 1 = \mathbf{E}Z^{(1)}$.*

*ii. If $1 - \frac{1}{2q} < H < 1$ we have $(c_{2,q,H}^{-1}\varepsilon)^{\frac{1}{1-q(1-H)}} g_2(\varepsilon) \to_{\varepsilon\to 0} \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}$.*

**Remark 5.** *In the case $H = \frac{1}{2}$ we retrieve the result (1) of [5]. The case $q = 1$ is trivial, because in this case, since $V_n = B_n$ and $\mathbf{E}V_n^2 = n^{2H}$, we obtain the following (by applying Lemma 1 and 2 with $q = 1$)*

$$\frac{1}{\log \varepsilon} \sum_{n\geq 1} \frac{1}{n} P\left(|V_n| > \varepsilon n^{2H}\right) \to_{\varepsilon\to 0} \frac{1}{H}$$

*and*

$$\varepsilon^2 \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2H}\right) \to_{\varepsilon\to 0} \mathbf{E}\left|Z^{(1)}\right|^{\frac{1}{H}}.$$

**Remark 6.** *Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of i.i.d. centered random variable with finite variance and let $(a_i)_{i\geq 1}$ a square summable real sequence. Define $X_n = \sum_{i\geq 1} a_i \varepsilon_{n-i}$. Then the sequence $S_N = \sum_{n=1}^{N} \left[K(X_n) - \mathbf{E}K(X_n)\right]$ satisfies a central limit theorem or a non-central limit theorem according to the properties of the measurable function $K$ (see [6] or [14]). We think that our tools can be applied to investigate the tail probabilities of the sequence $S_N$ in the spirit of [5] or [12] at least the in particular cases (for example, when $\varepsilon_i$ represents the increment $W_{i+1} - W_i$ of a Wiener process because in this case $\varepsilon_i$ can be written as a multiple integral of order one and $X_n$ can be decomposed into a sum of multiple integrals. We thank the referee for mentioning the references [6] and [14].*

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