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PERCOLATION DIMENSION OF BROWNIAN MOTION IN $\mathbb{R}^3$

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Abstract
Let $B(t)$ be a Brownian motion in $\mathbb{R}^3$. A subpath of the Brownian path $B[0,1]$ is a continuous curve $\gamma(t)$, where $\gamma[0,1] \subseteq B[0,1]$, $\gamma(0) = B(0)$, and $\gamma(1) = B(1)$. It is well-known that any subset S of a Browni an path must have Hausdorff dimension $\dim_h(S) \leq 2$. This paper proves that with probability one there exist subpaths of $B[0,1]$ with Hausdorff dimension strictly less than 2. Thus the percolation dimension of Brownian motion in $\mathbb{R}^3$ is strictly less than 2.

1. Introduction

Let $B(t) = B(\omega, t)$ denote a Brownian motion in $\mathbb{R}^d$, and denote the image of $[a,b]$ by $B[a,b]$. If $\gamma : \mathbb{R} \to \mathbb{R}^d$ is a continuous function, then we say $\gamma[0,1]$ is a *subpath* of $B[0,1]$ if $\gamma[0,1] \subset B[0,1]$ and $gamma(0) = B(0)$, $\gamma(1) = B(1)$. In this paper we prove that for $d = 3$, with probability one, there exists a subpath of $B[0,1]$ with Hausdorff dimension strictly less than 2.

A *double point* of $B[a,b]$ is any point $x \in \mathbb{R}^d$ such that there exist $a \leq t_1 < t_2 \leq b$ with

$$B(t_1) = B(t_2) = x.$$

For such t_1, t_2 it is natural to call $B[t_1, t_2]$ a "loop". For $d \geq 4$, Brownian motion has no double po ints [7] and hence has no loops. However, for $d = 3$ Brownian motion has double points (but no triple points [8]), and thus in order to find a lower dimensional subpath one might proceed by erasing the loops. This arises naturally as an e xtension of loop-erased random walk (e.g. see Lawler [11]).

Define the *cut times* of a Brownian motion B on $[a,b]$ by,

$$L[a,b] = \{t \in [a,b] : B[a,t) \cap B(t,b] = \emptyset\}.$$

The *cut points* are points x such that $x = B(t)$, where t is a cut time. Clearly, any subpath of $B[0,1]$ must contain the set of cut points $B(L[0,1])$. Lawler [12] has shown that for $d = 3$, the set $B(L[0,1])$ has Hausdorff dimensi on $\dim_h(B(L[0,1])) > 1$. The infimum of the dimensions of all subpaths of $B[0,1]$ is known as the percolation dimension of Brownian motion [3]. The result of Lawler shows the percolation dimension of three dimensional Brownian motion is stri ctly greater than 1. The main result of this paper is the following theorem,

Theorem 1.1. *If $B[0,1]$ is a Brownian path in $\mathbb{R}^3$, then there exists an $a > 0$ such that with probability one there exists a subpath $\gamma(t)$ of $B[0,1]$ with*

$$\dim_h(\gamma([0,1])) \leq 2(1-a).$$

As a corollary,

Corollary 1.2. *The percolation dimension of three dimensional Brownian motion is strictly less than 2.*

This result tells us that the percolation dimension of three dimensional Brownian motion is distinct from the Hausdorff dimension, 2, and the topological dimension, 1. In four or more dimensions, the percolation dimension is 2, and in two dimensions the conjectured value for the percolation dimension is 1.

Another useful dimensional quantity is boundary dimension, which takes on a non-trivial value in $d = 2$. The boundary dimension of $B[0,1]$ is the Hausdorff dimension of the "frontier" of Brownian motion, where the frontier of planar Brownian motion is t he boundary of the unbounded connected component of $\mathbb{R}^2 \backslash B[0,1]$. Lawler and Werner [13, 17, 18] and Bishop, Jones, Pemantle, and Peres [2], have shown independently that the dimension of the frontier of planar Bro wnian motion is strictly greater than 1. Burdzy and Lawler [4] have shown that the boundary dimension of planar Brownian motion is less than or equal to 3/2 - $1/4\pi^2$.

Two other papers also address the issue of the Hausdorff dimension of certain subsets of the random set $B[0,1]$. Burdzy [5] proves that a planar Brownian motion $B[0,1]$ (and hence the set $B[0,1]$ in dimensions 3 and higher) contains Jordan arcs with Hausdorff dimension arbitrarily close to 2. The supremum of the Hausdorff dimensions of all Jordan arcs contained in a fractal is known as its labyrinth dimension, and hence the labyrinth dimension of $B[0,1]$ is 2. In the other direction, Pe mantle [14] has shown that in $d \geq 2$, $B[0,1]$ almost surely contains no line segment. This is a start to deciding whether the percolation dimension of $B[0,1]$ in $d = 2$ is possibly 1, as conjectured.

The other area of interest to which this paper relates is the self-avoiding walk. Erasing loops from a Brownian path may make it possible to create a self-avoiding Brownian motion. Although this would not be the limit of the usual self-avoiding walk, it may be the continuum limit of the loop-erased walk. It should be noted that loop-erased random walk is conjectured to be in a different universality class than self-avoiding walk. [11]

The main idea of the paper is as follows. We define a random subset of the Brownian motion $B(F)$, where $F \subset [0,1]$. The random set F is defined as the intersection of the nested sets F_n, i.e. $F_{n+1} \subseteq F_n$, so that standard techni ques may be used to give an upper bound for the Hausdorff dimension of $B(F)$. We define a particular subpath of the Brownian motion, $\gamma(t) = \gamma(\omega, t)$, as the subsequential limit of subpaths $\{\gamma_n(t)\}$ such that $\gamma([0,1]) \subset B(F)$. One should note that the method of proof (using subsequences, etc.) does not lend itself to proving that this subpath $\gamma([0,1])$ is a *random* curve. That is, there is no reason to believe that the mapping taking each ω in Wiene r space to $\gamma(\omega)$ is measurable. Nonetheless, we prove that there exists an $a > 0$ such that $\dim_h(B(F)) \leq 2(1-a)$ and that $\gamma([0,1]) \subset B(F)$ is a subpath of $B[0,1]$. This proves the main result.

The crucial definitions of the random set F and the subpath $\gamma(t)$ are given in the next section, and the main theorem is proven in section 3. The main technical tools needed to obtain this result are detailed estimates concerning the dimension of sets of cut points. These technical estimates are left to the end of the paper.

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2. Notation and Definitions

To define the particular choice of Brownian subset $B(F)$ studied below, it will be useful to define the cut points of $B[c,d]$ relative to $B[a,b]$ as

$$C_{[a,b]}[c,d] = B([c,d] \cap L[a,b]),$$

which gives the points in the set $B[c,d]$ that are cut points of $B[a,b]$. We also need to define *local cut times* for $[a,b]$, which are defined as

$$L^{loc} = \{t \in [a,b] : \text{there exists } \epsilon > 0 \text{ with } B[t-\epsilon, t) \cap B(t, t+\epsilon] = \emptyset\}.$$

The *local cut points* are points x such that $x = B(t)$, where t is a local cut time.

Let

$$I(k,n) = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right].$$

Definition 2.1. *For $1 \le k \le 2^n$ let $A(k,n)$ be the event*

$$A(k,n) = \left\{ C_{[0,\frac{k}{2^n}]}[0, \frac{k-1}{2^n}] \cap B[\frac{k}{2^n}, 1] = \emptyset \right\}.$$

Let F_n and F be the random sets

$$F_n = \bigcup_{\{k \,:\, A(k,n) \text{ holds}\}} I(k,n),$$

$$F = \bigcap_{n=1}^{\infty} F_n.$$

Lemma 2.2. *For every n, $F_{n+1} \subseteq F_n$.*

Proof. To prove $F_{n+1} \subseteq F_n$, it is sufficient to prove that if $I(k,n)$ is not in F_n, then $I(2k-1, n+1)$ and $I(2k, n+1)$ are both not in F_{n+1}. If $I(k,n) \not\subset F_n$, then

$$B[\frac{k}{2^n}, 1] \cap B([0, \frac{k-1}{2^n}] \cap L[0, \frac{k}{2^n}]) \neq \emptyset$$

$$\Longrightarrow B[\frac{k}{2^n}, 1] \cap B([0, \frac{2k-1}{2^{n+1}}] \cap L[0, \frac{k}{2^n}]) \neq \emptyset$$

$$\Longrightarrow I(2k, n+1) \not\subset F_{n+1}$$

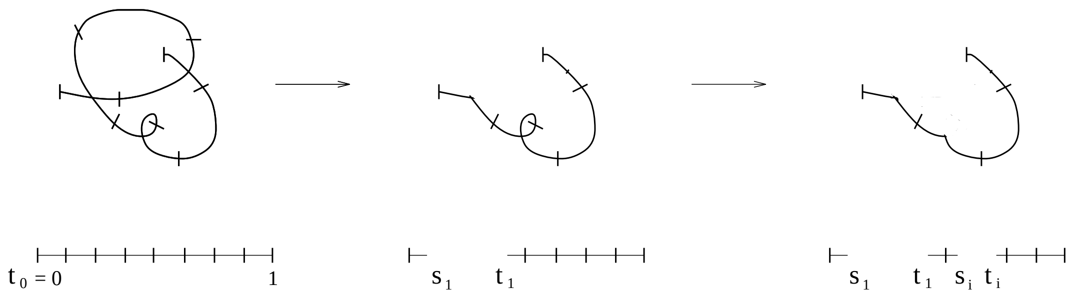

FIGURE 1. Construction of $\gamma(t)$

because there is now a larger set intersecting the same piece of the Brownian motion, so the intersection is non-empty. Likewise, $I(2k-1, n+1) \not\subset F_{n+1}$, because

$$L[0, \frac{2k-1}{2^{n+1}}] \cap [0, \frac{2k-2}{2^{n+1}}] \supseteq L[0, \frac{k}{2^n}] \cap [0, \frac{k-1}{2^n}]$$

$$\Longrightarrow B[\frac{2k-1}{2^{n+1}}, 1] \cap B\big([0, \frac{2k-2}{2^{n+1}}] \cap L[0, \frac{2k-1}{2^{n+1}}]\big) \neq \emptyset.$$

This says that the relative cut points of a section of Brownian motion only becomes larger as the set is made larg er and the relative set stays the same. Thus $F_{n+1} \subseteq F_n$. $\square$

Also, notice that since F_n is compact for all n, standard topological arguments imply that $\bigcap_{n=1}^{\infty} B(F_n) = B(\bigcap_{n=1}^{\infty} F_n) = B(F)$.

Roughly, the construction of $\gamma(t)$ is as follows (see Figure 1 below).
Split the interval $[0,1]$ into 2^n pieces, labeled $I(k,n)$. Let t_1 be the last time that $B(t)$ intersects $B(I(1,n))$, the first piece of the Brownian path. Let $s_1 \in I(1,n)$ be the unique time such that $B(s_1) = B(t_1)$. Remove $[s_1, t_1]$ from $[0,1]$, thereby removing a loop (see Figure 1, first arrow). Let k_2 be such that $t_1 \in I(k_2, n)$, making $B(I(k_2,n))$ the second piece of the loop-erased Brownian path. Let t_2 be the last time that $B(t)$ intersects $B(I(k_2,n))$, and choose $s_2 \in I(k_2, n)$ such that $B(s_2) = B(t_2)$. Remove the loop $[s_2, t_2]$. Continue this process until $t_i = 1$. Parametrize the remaining pieces of $[0,1]$ to give a continuous curve, $\gamma_n(t)$, connecting $B(0)$ to $B(1)$. Define $\gamma(t)$ as a subsequential limit of $\{\gamma_n(t)\}$.

The precise definition of $\gamma_n(t)$ is as follows. Most of the elements in the definition of $\gamma_n(t)$ depend on n in an obvious way, but we often suppress the dependence on n. Let

$$k_1 = 1, \Gamma_1 = \Gamma_{1,n} = I(k_1, n), \text{ and } t_1 = \sup_{r \in [0,1]} \{B(r) \in \Gamma_1\}.$$

Then for $i > 1$ define

$$\begin{aligned} k_i &= \sup\left\{k : \frac{k-1}{2^n} < t_{i-1}\right\}, \\ \Gamma_i &= \Gamma_{i,n} = \Gamma_{i-1} \cup \left[t_{i-1}, \frac{k_i}{2^n}\right], \\ t_i &= \sup_{r\in[0,1]} \{B(r) \in \Gamma_i\}. \end{aligned} \tag{1}$$

Since, with probability one, $\frac{k_i}{2^n}$ is not a local cut point [12], then $t_i > \frac{k_i}{2^n}$. Hence for $k_i < 2^n$, we have $k_i > k_{i-1} > \ldots > k_1$. Let $i^* = \inf\{i : k_i = 2^n\}$ and notice that $t_{i^*} = 1$. Define s_i as the unique time such that

$$B(s_i) = B(t_i),\ s_i \neq t_i.$$

Since, with probability one, the Brownian path in $d = 3$ has only double po ints (no triple points, etc.), this is well-defined. Note that $s_i \in \Gamma_i$, $t_i \in \Gamma_{i+1}$, and $0 < s_1 < t_1 < s_2 < t_2 < \ldots < s_{i^*} \leq t_{i^*}$.

To parametrize the pieces $I(k_i, n)$, we define a time change $\phi_n(t)$. Let $t_0 = 0$ and notice that, almost surely, $s_{i^*} = t_{i^*} = 1$. Then define

$$\phi_n(t) = \begin{cases} t & \text{for } t_i \leq t \leq s_{i+1},\ 0 \leq i \leq i^* - 1 \\ s_i & \text{for } s_i < t < t_i,\ 1 \leq i \leq i^*. \end{cases}$$

Notice that with this definition, $B(\phi_n(t))$ is continuous, since $B(s_i) = B(t_i)$. We define $\gamma_n(t) = B(\phi_n(t))$.

Below we show that there exists a subsequence $\{n(i)\}$ such that the subsequential limit of $\{\gamma_{n(i)}(t)\}$, called $\gamma(t)$, exists, is continuous, and is a subpath of the $B[0,1]$. Also, we show that the sets $B(F_n)$ contain the subpath $\gamma(t)$. In Section 3 we prove the following theorem:

Theorem 1.1$'$. *If $B[0,1]$ is a Brownian path in $\mathbb{R}^3$ and F and $\gamma(t)$ are as described above, then $B(F)$ is a subset of the Brownian motion containing $\gamma([0,1])$, $\gamma(t)$ is a subpath of $B[0,1]$, and ther e exists $a > 0$ such that*

$$\mathbb{P}\{\dim_h(F) \leq 1 - a\} = 1.$$

Using this theorem and the well-known result that Brownian motion in $d \geq 2$ maps all Borel subsets of $[0,1]$ into sets of twice the dimension [10, 15] proves Theorem ?? and Corollary 1.2.

A convention that is used throughout this paper is that c, c_1, c_2 will be used to stand for constants that may change from line to line. The constants c_3, c_4, etc. will remain the same throughout the paper.

Start two independent Brownian motions B^1, B^2 at the origin. Let the cut points of the second Brownian motion be denoted by $C^2_{[a,b]}[c,d]$. Also denote stopping times

$$T^j_i = \inf\{t : |B^j(t)| \geq i\}.$$

Then the main technical estimate nee ded is that there exists a $c_3 < \infty$, $a > 0$ such that

$$\mathbb{P}\{B^1[0,T^1_n] \cap C^2_{[0,T^2_m]}[T^2_1, T^2_m] = \emptyset\} \leq c_3(\min(m,n))^{-2a} \tag{2}$$

$$\mathbb{P}\{B^1[0,n] \cap C^2_{[0,m]}[1,m] = \emptyset\} \leq c_3(\min(m,n))^{-a}. \tag{3}$$

To obtain these results, we need an estimate on the *intersection exponent* ζ_3 for Brownian motion in $\mathbb{R}^3$ [4]. The intersection exponent is defined as follows. If B^1, B^2 are independent Brownian motions in $\mathbb{R}^d$ $(d < 4)$ s tarting at distinct x, y respectively, with $|x| = |y| = 1$, then the intersection exponent is the number ζ_d such that

$$\mathbb{P}\{B^1[0,T^1_n] \cap B^2[0,T^2_n] \neq \emptyset\} \asymp n^{-2\zeta_d}.$$

The symbol $\asymp$ denotes that the two side s are bounded within a constant as $n \to \infty$. The existence and rigorous bounds for this exponent are given in [4, 11, 12],

$$\frac{1}{4} \leq \zeta_3 < \frac{1}{2}, \; d = 3. \tag{4}$$

In p articular, this paper uses the fact that $\zeta_3 < 1/2$. To be precise, also notice that $a \leq \zeta_3 < 1/2$. This fact is used in the next section.

3. Proof of the Main Theorem

The proof of Theorem 1.1′ proceeds as follows. First we show that the functions $\gamma_n(t)$ are uniformly equicontinuous. This implies that a subsequence of $\{\gamma_n(t)\}$ converges to a continuous function $\gamma(t)$, which is a subpath of the Brownian path. Lemma 3.3 shows that $\gamma([0,1])$ is actually contained in $B(F)$. The final argument proves an upper bound on the Hausdorff dimension of F and gives the main result.

The delta oscillation of a function $f(t)$ over an interval $[a,b]$ can be defined as

$$os(f,\delta) = os(f,\delta,[a,b]) = \sup_{\{s,t\in[a,b]\,:\,|t-s|<\delta\}} |f(t)-f(s)|.$$

The following lemma uses a path-by-path comparison of $\gamma_n(t)$ and $B(t)$ to show that the path of $\gamma_n(t)$ has at most the same oscillation as the path of the Brownian motion from which it is generated. In particular, Lemma 3.1 proves that the functions $\gamma_n(t)$ are uniformly equiconti nuous.

Lemma 3.1. *With probability 1, over the interval* $[0,1]$*,*

$$os(\gamma_n,\delta) \leq os(B,\delta).$$

Proof. Fix $\delta > 0$. For any $s < t$ such that $|t-s| < \delta$, let

$$r_1 = \begin{cases} s & \text{if } t_i \leq s \leq s_{i+1},\ 0 \leq i \leq i^* \\ t_i & \text{if } s_i < s < t_i,\ 1 \leq ileqi^* \end{cases} \quad r_2 = \begin{cases} t & \text{if } t_i \leq t \leq s_{i+1},\ 0 \leq i \leq i^* \\ s_i & \text{if } s_i < s < t_i,\ 1 \leq i \leq i^*. \end{cases} \tag{5}$$

Then $|r_1 - r_2| \leq |s-t|$, and $\gamma_n(r_i) = B(r_i)$, which implies that

$$\begin{aligned} |\gamma_n(t) - \gamma_n(s)| &= |B(r_2) - B(r_1)| \\ &\leq os(B, |r_2 - r_1|) \\ &\leq os(B,\delta). \end{aligned} \tag{6}$$

This proves the lemma. □

Corollary 3.2. *There exists a subsequence* $\{n(i)\}$ *such that* $\gamma_{n(i)}(t)$ *converges to a continuous function* $\gamma(t)$*.*

Proof. Using the previous lemma and the Arzela-Ascoli theorem, we can construct a function $\gamma(\omega,t)$ which is a subsequential limit of the $\{\gamma_n(\omega,t)\}$. □

Now, it is clear that the set $\gamma([0,1])$ is a subset of the Brownian path (since the Brownian path is compact). Also, $B(0) \in \gamma([0,1])$, because $\gamma_n(0) = B(0)$ for all n, by construction. Likewise, $B(1) \in \gamma([0,1])$. Thus, $gamma([0,1])$ is a subpath of $B[0,1]$. The only other fact we need about $\gamma([0,1])$ is:

Lemma 3.3. *With probability one, there exists a curve* γ *such that* $\gamma([0,1]) \subseteq B(F)$*.*

Proof. This proof uses the notation in the definition of $\gamma_n(t)$. Specifically,

$$\Gamma_{i^*,n} = \bigcup_{i \le i^*} \left[t_{i-1}, \frac{k_i}{2^n} \right].$$

Since $B(F_n)$ is compact, it is sufficient to prove that $\Gamma_{i^*,n}$ is contained in F_n for all n. Given $r \in \Gamma_{i^*,n}$, there is a j such that $r \in [t_{j-1}, \frac{k_j}{2^n}]$. Since $r \in \Gamma_{i^*,n}$, then $B(r,1] \cap B\left([0, \frac{k_j - 1}{2^n}] \cap \Gamma_{i^*,n}\right) = \emptyset$. S o, to prove $r \in F_n$, it is sufficient to show that

$$L\left[0, \frac{k_j}{2^n}\right] \cap \left[0, \frac{k_j - 1}{2^n}\right] \subset \Gamma_{i^*,n}. \tag{7}$$

This fact follows directly from the construction of $\Gamma_{i^*,n}$. Specifically, if $0 \le t \le \frac{k_j-1}{2^n}$ and $t \in L[0, \frac{k_j}{2^n}]$, then either $t_{i-1} \le t \le s_i$, or $s_i < t < t_i$, for some $i \le j-1$. The second possibility violates the fact that $t \in L[0, \frac{k_j}{2^n}]$, since $s_i < t_i \le \frac{k_j}{2^n}$, for all $i \le j-1$. Thus, $t_{i-1} \le t \le s_i \le \frac{k_i}{2^n}$ for some $i \le j-1$, which implies that $t \in \Gamma_{i^*,n}$. Thus $\Gamma_{i^*,n} \subseteq F_n$ for all n, and so $\gamma([0,1]) \subseteq B(F_n)$ for every n. Therefore, $\gamma([0,1]) \subseteq \bigcap_{n=1}^{\infty} B(F_n) = B(F)$. □

Therefore, proving the set F has $\dim_h(F) \le 1 - a$, with probability 1, proves Theorem 1.1′.

Define the random variable K_n as

$$K_n = \#\{k : 1 \le k \le 2^n \text{ and } A(k,n) \text{ holds}\}.$$

By (3), there exists a constant $c > 0$ such that

$$\mathbb{E}(K_n) \le \frac{c}{1-a}(2^n)^{1-a}.$$

Then for any $\epsilon > 0$, Markov's Inequality implies

$$\mathbb{P}\{K_n \ge (2^n)^{1-a+\epsilon}\} \le c(2^n)^{-\epsilon},$$

and then using the Borel-Cantelli Lemma,

$$\mathbb{P}\{K_n \ge (2^n)^{1-a+\epsilon} \text{ i.o.}\} = 0.$$

Thus, for all sufficiently large n, $K_n \le (2^n)^{1-a+\epsilon}$, which says that the set F can be covered by $(2^n)^{1-a+\epsilon}$ intervals of length $(2^n)^{-1}$. Now, using standard arguments, $\dim_h(F) \le 1 - a + \epsilon$. With probability one, this holds for every $\epsilon > 0$, so

$$\mathbb{P}\{\dim_h(F) \le 1 - a\} = 1.$$

This completes the proof of the upper bound on the dimension of the set F.

4. Cut Point Estimates

In this section we prove two technical results about the cut point properties of the Brownian path. These results, Propositions 4.1 and 4.2, are needed in the next section to prove (3). First, some definitions and notation are pr esented:

In the following, $B(t)$ is a Brownian motion starting at the origin. The following notation is used throughout the rest of the paper:

$\mathcal{S}_i$ = sphere of radius e^i centered at the origin.
$\mathcal{B}(a,r)$ = closed ball of radius r centered at a.
$\sigma_i = \sup\{t < \infty : B_t \in \mathcal{S}_{i-1}\}$

$\tau_i = T_{e^i} = \inf\{t : B_t \in \mathcal{S}_i\}$ (Recall that $T_i = \inf\{t : |B(t)| \geq i\}$.)
$G_i = I\{(\sigma_i < \tau_i) \cap (\sigma_{i+2} < \tau_{i+2})\}$
$\rho_0 = 0$
$\rho_{l+1} = \inf\{j \in \mathbb{Z} : B[0, \sigma_{\rho_l+2}] \cap \mathcal{S}_j = \emptyset\}$

With these definitions, notice that $(B(t),\, t \geq \tau_{\rho_l})$ conditioned on $(B(t), t \leq \tau_{\rho_l})$ is a Brownian motion conditioned never to hit $\mathcal{S}_{\rho_{l-1}+1}$.

We will think of G_i as the indicator of a "good" number: an i such that after traveling distance e^i from the origin, the Brownian motion never returns to the ball of radius e^{i-1} and after hitting the ball of radius e^{i+2} never returns t o the ball of radius e^{i+1}. Proposition 4.1 proves that there exists a fraction $\epsilon > 0$ such that, with high probability, at least this fraction ϵ of the G_i equal 1. Specifically,

Proposition 4.1. *There exists $\epsilon > 0$, $b > 0$, and $c_4 < \infty$, such that*

$$\mathbb{P}\{\sum_{i=1}^{n} G_i < n\epsilon\} < c_4 e^{-bn}.$$

This proposition says that a positive fraction of the sets $B[0, \tau_n] \cap (\mathcal{B}(0, e^{i+1}) \backslash \mathcal{B}(0, e^i))$ are in effect isolated from the rest of the Brownian path. This includes information from both the past *and the f uture.* This "isolation" of annuli leads us to suspect that the probability of having cut points in each annulus is bounded below by a positive constant. This is true and is proven as:

Proposition 4.2. *There exists $c_5 > 0$ such that, for $n \geq 2$, the following is true:*

$$\mathbb{P}\{\dim_h(L[0, \tau_n] \cap [\tau_1, \tau_2]) = 1 - \zeta; B[\tau_3, \infty] \subset \mathcal{B}(0, e^2)^c; B[\tau_1, \infty] \subset \mathcal{B}(0, 1)^c\} \geq c_5,$$

where $\zeta = \zeta_3$, the intersection exponent, and $\mathcal{B}(0, r)^c$ is the complement of the ball of radius r.

This proposition could also be written as: There exists $c_5 > 0$ such that for all $i \geq 0$, for all $n \geq i + 1$,

$$\mathbb{P}\{\dim_h(L[0, \tau_n] \cap [\tau_i, \tau_{i+1}]) = 1 - \zeta; B[\tau_{i+2}, \infty] \subset \mathcal{B}(0, e^{i+1})^c; B[\tau_i, \infty] \subset \mathcal{B}(0, e^{i-1})^c\} \geq c_5.$$

We start by proving Proposition 4.1. It is well-known (see for example [6, p. 29]) that in $d = 3$, for $r < |x| < R$,

$$\mathbb{P}^x\{T_r < T_R\} = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}, \tag{8}$$

where $\mathbb{P}^x$ is the probability given that $B(0) = x$. Using this fact,

$$\mathbb{P}\{G_j = 1\} = \frac{e - 1}{e^3 - 1}$$

for any j. Therefore,

$$\mathbb{P}\{G_{\rho_l} = 1 \mid \rho_0, \ldots, \rho_{l-1}\} \geq \frac{e - 1}{e^3 - 1}, \tag{9}$$

since knowing that the Brownian motion never returns to $\mathcal{S}_{\rho_{l-1}+1}$ only increases this probability.

Also, the ρ_l cannot be very far apart. In particular, we can prove similarly that

$$\mathbb{P}\{\rho_{l+1} - \rho_l \geq k \mid \rho_1, \ldots, \rho_l\} \leq \frac{e-1}{e^{k-1}-1}. \tag{10}$$

Now, using standard techniques, there exists an $s > 0$ such that (9) implies that $\mathbb{P}\{\sum_{l=1}^m G_{\rho_l} \leq sm\}$ is exponentially decreasing in m. Similarly, there exists an $r > 0$ such that (10) implies that $p\{\sum_{l=1}^m (\rho_l - \rho_{l-1}) \geq rm\}$ is exponentially decreasing in m. Therefore, since $G_{\rho_1} + \ldots + G_{\rho_m} \leq \rho_m = \sum_{l=1}^m (\rho_l - \rho_{l-1})$ there exist $c_4 < \infty$, $b_1 > 0$ such that

$$\mathbb{P}\{sm \leq G_{\rho_1} + \ldots + G_{\rho_m} \leq \rho_m \leq rm\} \geq 1 - c_4 e^{-b_1 m}. \tag{11}$$

Choose $\frac{n}{2r} \leq m \leq \frac{n}{r}$ and then equation (11) becomes

$$\mathbb{P}\{\frac{s}{2r} n \leq G_1 + \ldots + G_n\} \geq 1 - c_4 e^{-b_1 m}. \tag{12}$$

Letting $b = \frac{b_1}{2r}$ finishes the proof of Proposition 4.1.

The proof of Proposition 4.2 uses two main ideas. First, if $G_i = 1$ then the Brownian motion never returns to $\mathcal{S}_{i-1}$ after hitting $\mathcal{S}_i$ and never returns to $\mathcal{S}_{i+1}$ after hitting $\mathcal{S}_{i+2}$. Thus, assuming $G_i = 1$ allows the possibility of local cut times $t \in [\tau_i, \tau_{i+1}]$, because the Brownian motion is "pulled away" from the annulus $\mathcal{B}(0, e^{i+1}) \backslash \mathcal{B}(0, e^i)$. Second, we use the fact [12] t hat, with probability one, the dimension of the local cut points of a Brownian motion $B[0,1]$ is $2 - 2\zeta$. Using these two ideas, we prove that there is a positive probability of having local cut times $t \in [\tau_i, \tau_{i+1}]$. Moreover, the di mension of the cut points corresponding to these cut times is $2 - 2\zeta$. These heuristics are now made precise.

Specifically, in order to prove Proposition 4.2, we need to show: There exists a $c_5 > 0$ such that

$$\mathbb{P}\left\{\dim_h(L[0,\tau_n] \cap [\tau_1,\tau_2]) = 1 - \zeta; B[\tau_3,\infty] \subset \mathcal{B}(0,e^2)^c; B[\tau_1,\infty] \subset \mathcal{B}(0,1)^c\right\} \geq c_5.$$

It suffices to prove that there exists $c_5 > 0$ such that the following is true:

$$\mathbb{P}\left\{\dim_h(L[0,\tau_n] \cap [\frac{1}{4},\frac{3}{4}]) = 1 - \zeta; B[\tau_3,\infty] \subset \mathcal{B}(0,e^2)^c; B[\tau_1,\infty] \subset \mathcal{B}(0,1)^c; \tau_1 < \frac{1}{4}; \frac{3}{4} < \tau_2\right\} \geq c_5.$$

To prove this, let

$$\begin{aligned} A(k,n) = \{C_{[0,\frac{k}{2^n}]}[0, \frac{k-1}{2^n}] \cap B[\frac{k}{2^n}, 1] = \emptyset; B[\tau_3,\infty] \subset \mathcal{B}(0,e^2)^c; \\ B[\tau_1,\infty] \subset \mathcal{B}(0,1)^c; \tau_1 < \frac{1}{4}; \frac{3}{4} < \tau_2\} \end{aligned} \tag{13}$$

and let

$$J_n = \#\{k : \frac{1}{4} 2^n < k \leq \frac{3}{4} 2^n \text{ and } A(k,n) \text{ holds }\}.$$

The next lemma provides the technical estimate to prove that $\mathbb{E}(J_n) \geq c_1 (2^n)^{1-\zeta}$, for some $c_1 > 0$, where $\zeta = \zeta_3$, the in tersection exponent. A result of Lawler[12, Lemma 2.3] says that there exists $c_2 < \infty$ such that $\mathbb{E}(J_n^2) \leq c_2 (2^n)^{2(1-\zeta)}$. Using these two bounds on the first and second moments, standard second moment arguments yield the result (see for example [12, Proposition 2.2]).

So, in order to finish the proof of Proposition 4.2 it suffices to prove the following:

Lemma 4.3. *Define the following events on a Brownian Motion: (suppress the k and n dependence)*

$$\begin{aligned}
\eta_1 &= \inf\{j : |B(\tfrac{j}{2^n}) - B(\tfrac{k+1}{2^n})| < \tfrac{1}{16} \text{ and } |\tfrac{j}{2^n} - \tfrac{k+1}{2^n}| < \tfrac{1}{16}\}\\
\eta_2 &= \sup\{j : |B(\tfrac{j}{2^n}) - B(\tfrac{k}{2^n})| < \tfrac{1}{16} \text{ and } |\tfrac{j}{2^n} - \tfrac{k}{2^n}| < \tfrac{1}{16}\}\\
Q^1 &= \{B[\eta_1, \tfrac{k}{2^n}] \cap B[\tfrac{k+1}{2^n}, \eta_2] = \emptyset\}\\
Q^2 &= \{\tau_1 < \tfrac{1}{4}\}\\
Q^3 &= \{B[\tfrac{1}{4}, \eta_1] \cup B[\eta_2, \tfrac{3}{4}] \subset \mathcal{B}(0, e^2 - \tfrac{1}{8})\}\\
Q^4 &= \{B[\tau_1, \tfrac{1}{4}] \subset \mathcal{B}(0,1)^c, B[\tfrac{1}{4}, \eta_1] \subset \mathcal{B}(0, \tfrac{9}{8})^c, B[\eta_2, \tau_3] \subset \mathcal{B}(0, \tfrac{9}{8})^c\}\\
Q^5 &= \{B[\tau_3, \infty] \subset \mathcal{B}(0, e^2)^c\}\\
Q^6 &= \{B[0, \eta_1] \cap B[\tfrac{k+1}{2^n}, \infty] = \emptyset\}\\
Q^7 &= \{B[0, \tfrac{k}{2^n}] \cap B[\eta_2, \infty] = \emptyset\}
\end{aligned}$$

Then there exists a $c_6 > 0$ such that $\mathbb{P}\{Q^1 \cap Q^2 \cap Q^3 \cap Q^4 \cap Q^5 \cap Q^6 \cap Q^7\} \geq c_6 (2^n)^{-\zeta}$, for $\frac{5}{16} 2^n \leq k \leq \frac{11}{16} 2^n$.

Proof. The events are chosen so that Q^1 is independent of the rest, so

$$\mathbb{P}\{Q^1 \cap Q^2 \cap Q^3 \cap Q^4 \cap Q^5 \cap Q^6 \cap Q^7\}$$

$$= \mathbb{P}\{Q^1\}\mathbb{P}\{Q^2 \cap Q^3 \cap Q^4 \cap Q^5 \cap Q^6 \cap Q^7\},$$

and by a result of Lawler [12, Proposi tion 3.16], there exists a $c_7 > 0$ such that

$$\geq c_7 (2^n)^{-\zeta} \mathbb{P}\{Q^2 \cap Q^3 \cap Q^4 \cap Q^5 \cap Q^6 \cap Q^7\}.$$

This last probability is clearly a positive probability event, because it doesn't depend on n or k in any sub stantial way. □

These events are defined such that

$$\mathbb{P}\{A(k,n) \text{ holds }\} \geq \mathbb{P}\{Q^1 \cap Q^2 \cap Q^3 \cap Q^4 \cap Q^5 \cap Q^6 \cap Q^7\} \geq c_6 (2^n)^{-\zeta},$$

for $\frac{5}{16} 2^n \leq k \leq \frac{11}{16} 2^n$. Therefore, $\mathbb{E}(J_n) \geq \frac{3}{8} c_6 (2^n)^{1-\zeta}$. This completes the proof of Proposition 4.2.

5. Proving Power Law Decay

The inequality (3) is the key estimate for this paper. This section is devoted to proving this estimate using Propositions 4.1 and 4.2. We use two standard facts and assume throughout this section that $d = 3$. The first is that in $d = 3$ any set of Hausdorff dimension strictly greater than 1 has positive capacity [9, Theorem 4.13]. The second is that Brownian motion hits sets of positive capacity with positive probability, i.e. there exists a $c > 0$ such that if $A \subset \mathcal{B}(0, r)$ with $Cap(A) > 0$, for every $x \in \mathcal{B}(0, r)$,

$$\mathbb{P}^x\{T_A < T_{2r}\} > c r^{-1} Cap(A), \tag{14}$$

where $T_A = \inf\{t : B(t) \in A\}$. Bass [1, Chap. II., sect. 5] discusse s this fact at length.

Using the Strong Markov Property and (14), we can prove the following lemma,

Lemma 5.1. *For every $\epsilon > 0$, there exist $c_8 > 0$, $c_9 < \infty$ such that if $A \subset \mathcal{B}(0, e^n)$ and $\#\{k : Cap(A \cap \mathcal{B}(0, e^k)) \geq \epsilon e^k\} \geq \epsilon n$, then for all $|x| = 1$, $\mathbb{P}^x\{\tau_n < T_A\} \leq c_9 e^{-n c_8}$.*

Combining the fact [10, 15] that Brownian motion $B[0,\infty)$ doubles the dimension of subsets of $[0,\infty)$ with Proposition 4.2 proves that there exists a $c>0$ (independent of n) such that

(15)
$$\mathbb{P}\left\{\dim_h\left(B(L[0,\tau_n]\cap[\tau_i,\tau_{i+1}])\right)=2-2\zeta \;\middle|\; i=\rho_l \text{ for some } l,\ G_i=1,\ \rho_0,\ldots,\rho_{l-1}\right\}\geq c.$$

Let $A=C_{[0,\tau_n]}[\tau_1,\tau_n]$. Since $\zeta<1/2$, then A has Hausdorff dimension strictly greater than 1, and hence $Cap(A)$ is positive. Proposition 4.1, Brownian scaling, and the scaling law for capacity combine to prove the following lemma,

Lemma 5.2. *There exists $\epsilon>0$, $b>0$, and $c_{10}<\infty$ such that*

$$\mathbb{P}\{\#\{k: Cap(A\cap\mathcal{B}(0,e^{k+1}))\geq\epsilon e^k\}\geq\epsilon n\}\geq 1-c_{10}e^{-bn}.$$

Proof. If $G_1=1$, then (15) implies that there exists a $c_1>0$ such that

$$\mathbb{P}\{\dim_h(C_{[0,\tau_n]}[\tau_1,\tau_n])>1\}\geq c_1.$$

Therefore, since sets of Hausdorff dimension greater than 1 have positive capacity in $\mathbb{R}^3$, there exists $\epsilon_1>0$ such that

$$\mathbb{P}\{Cap(C_{[0,\tau_n]}[\tau_1,\tau_2])\geq\epsilon_1\}\geq\frac{c_1}{2}.$$

Using a similar argument, if $G_k=1$ for $k\geq 1$,

$$\mathbb{P}\{\dim_h(C_{[0,\tau_{n+k-1}]}[\tau_k,\tau_{k+1}])>1\}\geq c_1.$$

Now, using the fact that capacity scales inversely with the diameter of the set in $\mathbb{R}^3$ [11], we see that

$$\mathbb{P}\{Cap(C_{[0,\tau_{n+k-1}]}[\tau_k,\tau_{k+1}])>\epsilon_1 e^{k-1}\}\geq\frac{c_1}{2}.$$

Since $C_{[0,\tau_{n+k-1}]}[\tau_k,\tau_{k+1}]\subset C_{[0,\tau_n]}[\tau_k,\tau_{k+1}]$,

$$\mathbb{P}\{Cap(C_{[0,\tau_n]}[\tau_k,\tau_{k+1}])>\epsilon_1 e^{k-1}\}\geq\frac{c_1}{2}.$$

Proposition 4.1 implies that there exists $\epsilon_2>0$, $b>0$, and $c<\infty$ such that

$$\mathbb{P}\{\sum_{i=1}^n G_i<n\epsilon_2\}<ce^{-bn}.$$

Choose $\epsilon=\min(\frac{\epsilon_1}{e},(\frac{c_1}{2})\epsilon_2)$. Then,

$$\sum_{k=1}^n G_k=\#\{k:G_k=1\}\leq\#\{k:\mathbb{P}\{Cap(A\cap\mathcal{B}(0,e^{k+1}))\geq\epsilon e^k\}\geq\frac{c_1}{2}\}.$$

Then, again using Proposition 4.1

$$1-ce^{-bn}\leq\mathbb{P}\{\sum_{k=1}^n G_k<n\epsilon_2\}\leq\mathbb{P}\{\#\{k:Cap(A\cap\mathcal{B}(0,e^{k+1}))\geq\epsilon e^k\}\geq\frac{c_1}{2}\epsilon_2 n\}.$$

□

Therefore, using Lemmas 5.1 and 5.2, if B^1, B^2 are independent Brownian motions starting at the origin, there exists an $a'>0$ and $c<\infty$ such that,

(16)
$$\mathbb{P}\{B^1[0,\tau_n^1]\cap C^2_{[0,\tau^2_m]}[\tau^2_1,\tau^2_m]=\emptyset\}\leq ce^{-a'n}+ce^{-a'm},$$

where $\tau_j^i = \inf\{t : B^i(t) \in \mathcal{S}_j\}$. Then rewriting the inequality for non-exponential stopping times, (16) implies

$$\mathbb{P}\{B^1[0, T_n^1] \cap C^2_{[0,T_m^2]}[T_1^2, T_m^2] = \emptyset\} \le c(\min(n,m))^{-a'}. \tag{17}$$

A few extra results are needed here about how this probability works for arbitrary times (not just distance stopping times). To this end, standard techniques yield the following:

Claim: For every $\beta > 0$ there exists a $c = c(\beta)$ such that $\mathbb{P}\{T_n \le n^2(\ln n)^{-2}\} \le cn^{-\beta}$.

Proof. This need only be proven for 1-dimensional Brownian motion.

$$\begin{aligned} \mathbb{P}\{T_n \le n^2(\ln n)^{-2}\} &= \mathbb{P}^0\{\sup_{t \le n^2(\ln n)^{-2}} |B(t)| \ge n\} \\ &= 2\mathbb{P}^0\{B(n^2(\ln n)^{-2}) \ge n\} \\ &= 2\mathbb{P}^0\{B(1) \ge \ln n\} \\ &\le ce^{-\frac{(\ln n)^2}{2}} \\ &\le cn^{-\frac{\ln n}{2}}. \end{aligned} \tag{18}$$

□

A similar proof shows that $\mathbb{P}\{T_n \ge n^2(\ln n)^2\} \le cn^{-\frac{\ln n}{2}}$. Therefore, we get that for every $\beta > 0$ there exists a $c < \infty$ such that

$$\mathbb{P}\{n^2(\ln n)^{-2} \le T_n \le n^2(\ln n)^2\} \ge q1 - cn^{-\beta} \tag{19}$$

Then, using (19) we see that there exists a constant $c_3 < \infty$ and $a > 0$ such that

$$\mathbb{P}\{B^1[0,n] \cap C^2_{[0,m]}[1,m] = \emptyset\} \le c_3(\min(m,n))^{-a}.$$

This completes the proof of (3).

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