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MOMENT ESTIMATES FOR SOLUTIONS OF LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY ANALYTIC FRACTIONAL BROWNIAN MOTION

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Abstract

As a general rule, differential equations driven by a multi-dimensional irregular path Γ are solved by constructing a rough path over Γ. The domain of definition – and also estimates – of the solutions depend on upper bounds for the rough path; these general, deterministic estimates are too crude to apply e.g. to the solutions of stochastic differential equations with linear coefficients driven by a Gaussian process with Hölder regularity $\alpha < 1/2$.

We prove here (by showing convergence of Chen's series) that linear stochastic differential equations driven by analytic fractional Brownian motion [6, 7] with arbitrary Hurst index $\alpha \in (0,1)$ may be solved on the closed upper half-plane, and that the solutions have finite variance.

1 Introduction

Assume $\Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d))$ is a smooth d-dimensional path, and $V_1, \dots, V_d : \mathbb{R}^r \to \mathbb{R}^r$ be smooth vector fields. Then (by the classical Cauchy-Lipschitz theorem for instance) the differential equation driven by Γ

$$dy(t) = \sum_{i=1}^{d} V_i(y(t)) d\Gamma_t(i) \tag{1.1}$$

admits a unique solution with initial condition $y(0) = y_0$. The usual way to prove this is by showing (by a functional fixed-point theorem) that iterated integrals

$$y_n(t) \to y_{n+1}(t) := y_0 + \int_0^t \sum_i V_i(y_n(s)) d\Gamma_s(i) \tag{1.2}$$

converge when $n \to \infty$.

Expanding this expression to all orders yields formally for an arbitrary analytic function f

$$f(y_t) = f(y_s) + \sum_{n=1}^{\infty} \sum_{1\le i_1,\ldots,i_n\le d} \left[V_{i_1}\ldots V_{i_n} f\right](y_s)\mathbf{\Gamma}^n_{ts}(i_1,\ldots,i_n), \tag{1.3}$$

where

$$\mathbf{\Gamma}^n_{ts}(i_1,\ldots,i_n) := \int_s^t d\Gamma_{t_1}(i_1)\int_s^{t_1} d\Gamma_{t_2}(i_2)\ldots\int_s^{t_{n-1}} d\Gamma_{t_n}(i_n), \tag{1.4}$$

provided, of course, the series converges. By specializing to the identity function $f = \mathrm{Id} : \mathbb{R}^r \to \mathbb{R}^r$, $x \to x$, one gets a series expansion for the solution (y_t).
Let

$$\mathscr{E}_V^{N,t,s}(y_s) = y_s + \sum_{n=1}^{N} \sum_{1\le i_1,\ldots,i_n\le d} \left[V_{i_1}\ldots V_{i_n}\mathrm{Id}\right](y_s)\mathbf{\Gamma}^n_{ts}(i_1,\ldots,i_n) \tag{1.5}$$

be the N-th order truncation of (1.3). It may be interpreted as *one* iteration of the numerical Euler scheme of order N, which is defined by

$$y^{Euler;D}_{t_k} := \mathscr{E}_V^{N,t_k,t_{k-1}} \circ \ldots \circ \mathscr{E}_V^{N,t_1,t_0}(y_0) \tag{1.6}$$

for an arbitrary partition $D = \{0 = t_0 < \ldots < t_n = T\}$ of the interval $[0,T]$. When Γ is only α-Hölder with $\frac{1}{N+1} < \alpha \le \frac{1}{N}$, the iterated integrals $\mathbf{\Gamma}^n(i_1,\ldots,i_n)$, $n = 2,\ldots,N$ do not make sense a priori and must be substituted with a *geometric rough path* over Γ. A *geometric rough path* over Γ is a family

$$\left((\mathbf{\Gamma}^1_{ts}(i_1))_{1\le i_1\le d}, (\mathbf{\Gamma}^2_{ts}(i_1,i_2))_{1\le i_1,i_2\le d},\ldots,(\mathbf{\Gamma}^N_{ts}(i_1,\ldots,i_N)_{1\le i_1,\ldots,i_N\le d})\right) \tag{1.7}$$

of functions of two variables such that: $\mathbf{\Gamma}^1_{ts} = \Gamma^1_t - \Gamma^1_s$ and satisfying a natural Hölder regularity condition, namely, $\sup_{s,t\in\mathbb{R}}\left(\frac{|\mathbf{\Gamma}^k_{ts}(i_1,\ldots,i_k)|}{|t-s|^{k\alpha}}\right) < \infty$, $k = 1,\ldots,N$, along with two algebraic compatibility properties (Chen/multiplicativity and shuffle/geometricity properties) for which we refer e.g. to [2]. To such data one may associate a theory of integration along Γ, so that (1.1), rewritten in its integral form, makes sense, see e.g. [2] or [3] for *local* solutions of differential equations in this setting.

In this article, we prove convergence of the series (1.3) when the vector fields V_i are linear and Γ is *analytic fBm* (afBm for short). This process – first defined in [7] –, depending on an index $\alpha \in (0,1)$, is a complex-valued process, a.s. κ-Hölder for every $\kappa < \alpha$, which has an analytic continuation to the upper half-plane $\Pi^+ := \{z = x + \mathrm{i}y \mid x \in \mathbb{R}, y > 0\}$. Its real part $(2\mathrm{Re}\ \Gamma_t, t \in \mathbb{R})$ has the same law as fBm with Hurst index α. Trajectories of Γ on the closed upper half-plane $\bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$ have the same regularity as those of fBm, namely, they are κ-Hölder for every $\kappa < \alpha$. As shown in [6], the regularized rough path – constructed by moving inside the upper half-plane through an imaginary translation $t \to t + \mathrm{i}\varepsilon$ – converges in the limit $\varepsilon \to 0$ to a geometric rough path over Γ for *any* $\alpha \in (0,1)$, which makes it possible to produce strong, local pathwise solutions of stochastic differential equations driven by Γ with analytic coefficients.
We do not enquire about the convergence of the series (1.3) in the general case (as mentioned before, it diverges e.g. when V is quadratic), but only in the *linear* case. One obtains, see section 3:

Main Theorem.

Let $V_1, \ldots, V_d$ be linear vector fields on $\mathbb{C}^r$. Then the series (1.3), associated to afBm Γ with Hurst index $\alpha \in (0,1)$, converges in $L^2(\Omega)$ on the closed upper half-plane $\bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$. Furthermore, there exists a constant C such that the solution $(y_t)_{t \in \bar{\Pi}^+}$, defined as the limit of the series, satisfies

$$\mathbb{E}|y_t - y_s|^2 \leq C|t-s|^{2\alpha}, \quad s,t \in \bar{\Pi}^+. \tag{1.8}$$

The Main Theorem depends essentially on an explicit estimate of the variance of iterated integrals of Γ proved in Lemma 2.2 below, which states the following:
Lemma 2.2.
There exists a constant C' such that, for every $s,t \in \bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$,

$$\mathrm{Var}\mathbf{\Gamma}^n_{ts}(i_1, \ldots, i_n) \leq \frac{(C'|t-s|)^{2n\alpha}}{n!}. \tag{1.9}$$

Notation. Constants (possibly depending on α) are generally denoted by C, C', C_1, c_α and so on.

2 Definition of afBm and first estimates

We briefly recall to begin with the definition of the analytic fractional Brownian motion Γ, which is a complex-valued process defined on the closed upper half-plane $\bar{\Pi}^+$ [6]. Its introduction was initially motivated by the possibility to construct quite easily iterated integrals of Γ by a contour deformation. Alternatively, its Fourier transform is supported on $\mathbb{R}_+$, which makes the regularization procedure in [8, 9] void.

Proposition 2.1. *There exists a unique analytic Gaussian process $(\Gamma'_z, z \in \Pi^+)$ with the following properties (see [6] or [7] for its definition):*

(1) *Γ' is a well-defined analytic process on Π^+, with Hermitian covariance kernel*

$$\mathbb{E}\Gamma'_z\Gamma'_w = 0, \quad \mathbb{E}\Gamma'_z\bar{\Gamma}'_w = \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}(-\mathrm{i}(z-\bar{w}))^{2\alpha-2}. \tag{2.1}$$

(2) *Let $\gamma : (0,1) \to \Pi^+$ be any continuous path with endpoints $\gamma(0) = 0$ and $\gamma(1) = z$, and set $\Gamma_z = \int_\gamma \Gamma'_u \, du$. Then Γ is an analytic process on Π^+. Furthermore, as z runs along any path in Π^+ going to $t \in \mathbb{R}$, the random variables Γ_z converge almost surely to a random variable called again Γ_t.*

(3) *The family $\{\Gamma_t;\, t \in \mathbb{R}\}$ defines a Gaussian centered complex-valued process, whose covariance function is given by:*

$$\mathbb{E}[\Gamma_s\Gamma_t] = 0, \quad \mathbb{E}[\Gamma_s\bar{\Gamma}_t] = \frac{e^{-\mathrm{i}\pi\alpha\,\mathrm{sgn}(s)}|s|^{2\alpha} + e^{\mathrm{i}\pi\alpha\,\mathrm{sgn}(t)}|t|^{2\alpha} - e^{\mathrm{i}\pi\alpha\,\mathrm{sgn}(t-s)}|s-t|^{2\alpha}}{4\cos(\pi\alpha)}.$$

The paths of this process are almost surely κ-Hölder for any $\kappa < \alpha$.

(4) *Both real and imaginary parts of $\{\Gamma_t;\, t \in \mathbb{R}\}$ are (non independent) fractional Brownian motions indexed by $\mathbb{R}$, with covariance given by*

$$\mathbb{E}[\mathrm{Re}\,\Gamma_s \mathrm{Im}\,\Gamma_t] = -\frac{\tan\pi\alpha}{8}\left[-\mathrm{sgn}(s)|s|^{2\alpha} + \mathrm{sgn}(t)|t|^{2\alpha} - \mathrm{sgn}(t-s)|t-s|^{2\alpha}\right]. \tag{2.2}$$

Definition 2.2. *Let $Y_t := \mathrm{Re}\,\Gamma_{it}$, $t \in \mathbb{R}_+$. More generally, $Y_t = (Y_t(1), \ldots, Y_t(d))$ is a vector-valued process with d independent, identically distributed components.*

The above results imply that Y_t is real-analytic on $\mathbb{R}_+^*$.

Lemma 2.3. *The infinitesimal covariance function of Y_t is:*

$$\mathbb{E}Y_s'Y_t' = \frac{\alpha(1-2\alpha)}{4\cos\pi\alpha}(s+t)^{2\alpha-2}. \tag{2.3}$$

Proof. Let $X_t := \mathrm{Im}\,\Gamma_{it}$. Since $\mathbb{E}\Gamma_s\Gamma_t = 0$, $(Y_s, s \geq 0)$ and $(X_s, s \geq 0)$ have same law, with covariance kernel $\mathbb{E}Y_sY_t = \mathbb{E}X_sX_t = \frac{1}{2}\mathrm{Re}\,\Gamma_{is}\bar{\Gamma}_{it}$. Hence

$$\mathbb{E}[Y_s'Y_t'] = \frac{1}{2}\mathrm{Re}\,\mathbb{E}\Gamma_{is}'\bar{\Gamma}_{it}' = \frac{\alpha(1-2\alpha)}{4\cos\pi\alpha}(s+t)^{2\alpha-2}. \tag{2.4}$$

□

Note that $\mathbb{E}Y_s'Y_t' > 0$. From this simple remark follows (see proof of a similar statement in [5] concerning usual fractional Brownian motion with Hurst index $\alpha > 1/2$):

Lemma 2.4. *Let $\mathbf{Y}_{ts}^n(i_1,\ldots,i_n)$, $n \geq 2$ be the iterated integrals of Y. Then there exists a constant $C > 0$ such that*

$$\mathrm{Var}\mathbf{Y}_{ts}^n(i_1,\ldots,i_n) \leq C\frac{(C|t-s|)^{2n\alpha}}{n!}. \tag{2.5}$$

Proof. Let Π be the set of all pairings π of the set $\{1,\ldots,2n\}$ such that $\big((k_1,k_2)\in\pi\big) \Rightarrow \left(i_{k_1'} = i_{k_2'}\right)$, where $k_1' = k_1$ if $k_1 \leq n$, $k_1 - n$ otherwise, and similarly for k_2'. By Wick's formula,

$$\begin{aligned} &\mathrm{Var}\mathbf{Y}_{ts}^n(i_1,\ldots,i_n) \\ &= \sum_{\pi\in\Pi}\left(\int_s^t dx_1\ldots\int_s^{x_{n-1}} dx_n\right)\left(\int_s^t dx_{n+1}\ldots\int_s^{x_{2n-1}} dx_{2n}\right) \\ &\qquad \prod_{(k_1,k_2)\in\pi}\mathbb{E}[Y_{x_{k_1}}'Y_{x_{k_2}}']. \end{aligned} \tag{2.6}$$

Since the process Y' is positively correlated, and Π is largest when all indices $i_1,\ldots,i_n$ are equal, one gets $\mathrm{Var}\mathbf{Y}_{ts}^n(i_1,\ldots,i_n) \leq \mathrm{Var}\mathbf{Y}_{ts}^n(1,\ldots,1)$. On the other hand, $\mathbf{Y}_{ts}^n(1,\ldots,1) = \frac{1}{n!}(Y_t - Y_s)^n$, hence

$$\mathrm{Var}\mathbf{Y}_{ts}^n(1,\ldots,1) = \frac{[\mathrm{Var}(Y_t-Y_s)]^n}{(n!)^2}\cdot\frac{(2n)!}{2^n\,.\,n!} \leq \frac{[2\mathrm{Var}(Y_t-Y_s)]^n}{n!}. \tag{2.7}$$

Now (assuming for instance $0 < s < t$)

$$\mathrm{Var}(Y_t - Y_s) = c_\alpha\int_s^t\int_s^t (u+v)^{2\alpha-2}dudv \leq c_\alpha s^{2\alpha-2}(t-s)^2 \leq c_\alpha(t-s)^{2\alpha} \tag{2.8}$$

if $\frac{t}{2} \leq s \leq t$, and

$$\mathrm{Var}(Y_t - Y_s) = \frac{c_\alpha}{2\alpha(2\alpha-1)}\left[(2t)^{2\alpha} + (2s)^{2\alpha} - 2(t+s)^{2\alpha}\right] \leq Ct^{2\alpha} \leq C'(t-s)^{2\alpha} \tag{2.9}$$

if $s < t/2$. Hence the result. □

3 Estimates for iterated integrals of Γ

The main tool for the study of Γ is the use of contour deformation. Iterated integrals of Γ are particular cases of analytic iterated integrals, see [7] or [6]. In particular, the following holds:

Lemma 3.1. *Let $\gamma : (0,1) \to \Pi^+$ be the piecewise linear contour with affine parametrization defined by :*

(i) $\gamma([0,1/3]) = [s, s + i|\mathrm{Re}\,(t-s)|]$;

(ii) $\gamma([1/3,2/3]) = [s + \mathrm{i}|\mathrm{Re}\,(t-s)|, t + \mathrm{i}|\mathrm{Re}\,(t-s)|]$;

(iii) $\gamma([2/3,1]) = [t + \mathrm{i}|\mathrm{Re}\,(t-s)|, t]$.

If $z = \gamma(x) \in \gamma([0,1])$, we let γ_z be the same path stopped at z, i.e. $\gamma_z = \gamma([0,x])$, with the same parametrization. Then (letting $c_\alpha = \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}$)

$$\begin{aligned}\mathrm{Var}\boldsymbol{\Gamma}^n_{ts}(i_1,\dots,i_n) = \\ c_\alpha^n \sum_{\sigma\in\Sigma_I} \int_\gamma dz_1 \int_{\bar\gamma} d\bar w_1 (-\mathrm{i}(z_1 - \bar w_{\sigma(1)}))^{2\alpha-2} \cdot \int_{\gamma_{z_1}} dz_2 \int_{\bar\gamma_{\bar w_1}} d\bar w_2 (-\mathrm{i}(z_2 - \bar w_{\sigma(2)}))^{2\alpha-2} \dots \\ \int_{\gamma_{z_{n-1}}} dz_n \int_{\bar\gamma_{\bar w_{n-1}}} d\bar w_n (-\mathrm{i}(z_n - \bar w_{\sigma(n)}))^{2\alpha-2} \end{aligned} \tag{3.1}$$

where Σ_I is the subset of permutations of $\{1,\dots,n\}$ such that $(i_j = i_k) \Rightarrow (\sigma(j) = \sigma(k))$.

Proof. Note first that, similarly to eq. (2.6),

$$\begin{aligned}\mathrm{Var}\boldsymbol{\Gamma}^n_{ts}(i_1,\dots,i_n) = \sum_{\sigma\in\Sigma_I} \left(\int_1^t dz_1 \dots \int_s^{z_{n-1}} dz_n \right) \left(\int_{\bar s}^{\bar t} d\bar w_1 \dots \int_{\bar s}^{\bar w_{n-1}} d\bar w_n \right) \\ \prod_{j=1}^n \mathbb{E}\left[\Gamma'_{z_j} \bar\Gamma'_{\bar w_{\sigma(j)}} \right] \end{aligned} \tag{3.2}$$

(the difference with respect to eq. (2.6) comes from the fact that contractions only operate between Γ's and $\bar\Gamma$'s, since $\mathbb{E}[\Gamma_{z_j}\Gamma_{z_k}] = \mathbb{E}[\bar\Gamma_{\bar w_j}\bar\Gamma_{\bar w_k}] = 0$ by Proposition 2.1). Now the result comes from a deformation of contour, see [7]. □

Lemma 3.2. *There exists a constant C' such that, for every $s,t \in \bar\Pi^+ = \Pi^+ \cup \mathbb{R}$,*

$$\mathrm{Var}\boldsymbol{\Gamma}^n_{ts}(i_1,\dots,i_n) \le \frac{(C'|t-s|)^{2n\alpha}}{n!}. \tag{3.3}$$

Proof. We assume (without loss of generality) that $\mathrm{Im}\, s \le \mathrm{Im}\, t$. If $|\mathrm{Im}\,(t-s)| \ge c\mathrm{Re}\,|t-s|$ for some positive constant c (or equivalently $|\mathrm{Re}\,(t-s)| \le c'|t-s|$ for some $0 \le c' < 1$) then it is preferable to integrate along the straight line $[s,t] = \{z \in \mathbb{C} \mid z = (1-u)s + ut,\ 0 \le u \le 1\}$ instead of γ, and use the parametrization $y = \mathrm{Im}\, z$. If $z_1, z_2 \in [s,t]$, $y_1 = \mathrm{Im}\, z_1, y_2 = \mathrm{Im}\, z_2$, then $|(-\mathrm{i}(z_1 - \bar z_2))^{2\alpha-2}| \le C(y_1+y_2)^{2\alpha-2}$, hence $\mathrm{Var}\boldsymbol{\Gamma}^n_{ts}(i_1,\dots,i_n) \le C'^n \mathrm{Var}\mathbf{Y}^n_{y_2,y_1}(i_1,\dots,i_n)$, which yields the result by Lemma 2.4. So we shall assume that $|\mathrm{Re}\,(t-s)| > c|t-s|$ for some constant $c > 0$.

Let us use as new variable the parametrization coordinate x along γ. Then formula (3.1) reads

$$\mathrm{Var}\boldsymbol{\Gamma}^n_{ts}(i_1,\ldots,i_n) = c_\alpha^n \sum_{\sigma\in\Sigma_I} \int_0^1 dx_1 \int_0^1 dy_1 K'(x_1,y_{\sigma(1)}) \, . \int_0^{x_1} dx_2 \int_0^{y_1} dy_2 K'(x_2,y_{\sigma(2)}) \ldots \int_0^{x_{n-1}} dx_n \int_0^{x_n} dy_n K'(x_n,y_{\sigma(n)}), \tag{3.4}$$

where $K'(x,y) = (3|\mathrm{Re}\,(t-s)|)^2(3(x+y)|\mathrm{Re}\,(t-s)| + 2\mathrm{Im}\,s)^{2\alpha-2}$ if $0<x,y<1/3$, $(3|\mathrm{Re}\,(t-s)|)^2(3((1-x)+(1-y))|\mathrm{Re}\,(t-s)| + 2\mathrm{Im}\,t)^{2\alpha-2}$ if $2/3<x,y<1$, and is bounded by a constant times $|t-s|^{2\alpha}$ otherwise thanks to the condition $|\mathrm{Re}\,(t-s)| > c|t-s|$. Note that $(x+y)^{2\alpha-2} > 2^{2\alpha-2}$ if $0<x,y<1$. Hence (if $0<x,y<1$) $|K'(x,y)| \le (C_1|t-s|)^{2\alpha}\left[(x+y)^{2\alpha-2} + ((1-x)+(1-y))^{2\alpha-2}\right]$, which is (up to a coefficient) the infinitesimal covariance of $|t-s|^\alpha(Y_x+\tilde{Y}_{1-x}, 0<x<1)$ if $\tilde{Y} \overset{(law)}{=} Y$ is independent of Y. A slight modification of the argument of Lemma 2.4 yields

$$\begin{aligned}\mathrm{Var}\boldsymbol{\Gamma}^n_{ts}(i_1,\ldots,i_n) &\le (C_1|t-s|)^{2n\alpha}\frac{[\mathrm{Var}(Y_1-Y_0)+\mathrm{Var}(\tilde{Y}_1-\tilde{Y}_0)]^n}{(n!)^2}\cdot\frac{(2n)!}{2^n\,.\,n!} \\ &\le C_1^{2n\alpha}\,.\,\frac{(2C|t-s|)^{2n\alpha}}{n!}.\end{aligned} \tag{3.5}$$

□

4 Proof of main theorem

We now prove the theorem stated in the introduction, which is really a simple corollary of Lemma 3.2.
Let C be the maximum of the matrix norms $|||V_i||| = \sup_{||x||_\infty=1}||V_i x||_\infty$ for the supremum norm $||x||_\infty = \sup(|x_1|,\ldots,|x_r|)$. Rewrite eq. (1.5) as

$$\mathscr{E}_V^{N,t,s}(y_s) = y_s + \sum_{n=1}^N \sum_{1\le i_1,\ldots,i_n\le d} a_{i_1,\ldots,i_n}\boldsymbol{\Gamma}^n_{ts}(i_1,\ldots,i_n). \tag{4.1}$$

Then $||a_{i_1,\ldots,i_n}||_\infty \le C^n$ and $\mathbb{E}|\boldsymbol{\Gamma}^n_{ts}(i_1,\ldots,i_n)|^2 \le \frac{(C|t-s|)^{2n\alpha}}{n!}$. Hence (by the Cauchy-Schwarz inequality)

$$\begin{aligned}\mathbb{E}\left(\mathscr{E}_V^{N,t,s}(y_s)-y_s\right)^2 &\le \sum_{m,n=1}^N \frac{(C''|t-s|)^{(m+n)\alpha}}{\sqrt{m!n!}} \\ &= \left(\sum_{m=1}^N \frac{(C''|t-s|)^{m\alpha}}{\sqrt{m!}}\right)^2 \le C'''.|t-s|^{2\alpha}\end{aligned} \tag{4.2}$$

independently of N. The series obviously converges and yields eq. (1.8) for $p=1$. □
It should be easy to prove along the same lines that the series defining $\mathbb{E}|y_t-y_s|^{2p}$ converges for every $p\ge 1$, and that there exists a constant C_p such that $\mathbb{E}|y_t-y_s|^{2p} \le C_p|t-s|^{2\alpha p}$ for every $s,t\in\bar{\Pi}^+$. The most obvious consequence – using Kolmogorov's lemma – would be that y_t has Hölder regularity of any order less than α. But this follows from standard rough path theory, so we skip the proof.

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