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GRAPHICAL REPRESENTATION OF SOME DUALITY RELATIONS IN STOCHASTIC POPULATION MODELS

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Abstract
We derive a unified stochastic picture for the duality of a resampling-selection model with a branching-coalescing particle process (cf. [1]) and for the self-duality of Feller's branching diffusion with logistic growth (cf. [7]). The two dual processes are approximated by particle processes which are forward and backward processes in a graphical representation. We identify duality relations between the basic building blocks of the particle processes which lead to the two dualities mentioned above.

1 Introduction

Two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ with state spaces E_1 and E_2, respectively, are called dual with respect to the duality function H if $H\colon E_1 \times E_2 \to \mathbb{R}$ is a measurable and bounded function and if $\mathbf{E}^x[H(X_t, y)] = \mathbf{E}^y[H(x, Y_t)]$ holds for all $x \in E_1$, $y \in E_2$ and all $t \geq 0$ (see e.g. [9]). Here superscripts as in $\mathbf{P}^x$ or in $\mathbf{E}^x$ indicate the initial value of a process. In this paper, E_1 and E_2 will be subsets of $[0, \infty)$ or will be equal to $\{0, 1\}^N$. We speak of a *moment duality* if $H(x, y) = y^x$ or $H(x, y) = (1 - y)^x$, $x \in E_1 \subset \mathbb{N}_0$, $y \in [0, 1]$, and of a *Laplace duality* if $H(x, y) = \exp(-\lambda x{\cdot}y)$, $x, y \in E_1 = E_2 \subset [0, \infty)$, for some $\lambda > 0$.
We provide a unified stochastic picture for the following moment duality and the following Laplace duality of prominent processes from the field of stochastic population dynamics. For the moment duality, let $b, c, d \geq 0$. Denote by $X_t \in \mathbb{N}_0$ the number of particles at time $t \geq 0$ of the branching-coalescing particle process defined by the initial value $X_0 = n$ and

the following dynamics: Each particle splits into two particles at rate b, each particle dies at rate d and each ordered pair of particles coalesces into one particle at rate c. All these events occur independently of each other. In the notation of Athreya and Swart [1], this is the $(1, b, c, d)$-braco-process. Its dual process $(Y_t)_{t\geq 0}$ is the unique strong solution with values in $[0, 1]$ of the one-dimensional stochastic differential equation

$$\mathrm{d}Y_t = (b-d)Y_t\,\mathrm{d}t - bY_t^2\,\mathrm{d}t + \sqrt{2cY_t(1-Y_t)}\,\mathrm{d}B_t, \quad Y_0 = y, \tag{1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Athreya and Swart [1] call this process the resampling-selection process with selection rate b, resampling rate c and mutation rate d, or shortly the $(1, b, c, d)$-resem-process. They prove the moment duality

$$\mathbf{E}^n\left[(1-y)^{X_t}\right] = \mathbf{E}^y\left[(1-Y_t)^n\right] \qquad \forall\, n \in \mathbb{N}_0,\, y \in [0,1],\, t \geq 0. \tag{2}$$

For the Laplace duality, let $(X_t)_{t\geq 0}$ denote Feller's branching diffusion with logistic growth, i.e., the strong solution of

$$\mathrm{d}X_t = \alpha X_t\,\mathrm{d}t - \gamma X_t^2\,\mathrm{d}t + \sqrt{2\beta X_t}\,\mathrm{d}B_t, \tag{3}$$

where $\alpha, \gamma, \beta \geq 0$ and $(B_t)_{t\geq 0}$ is a standard Brownian motion. We call this process the logistic Feller diffusion with parameters (α, γ, β). Let $(Y_t)_{t\geq 0}$ be a logistic Feller diffusion with parameters $(\alpha, r\beta, \gamma/r)$ for some $r > 0$. Hutzenthaler and Wakolbinger [7] establish the Laplace duality

$$\mathbf{E}^x\left[e^{-rX_t\cdot y}\right] = \mathbf{E}^y\left[e^{-rx\cdot Y_t}\right], \qquad \forall\, x, y \in [0,\infty),\, t \geq 0. \tag{4}$$

The duality relations (2) and (4) include as special cases (see Remark 4.2 and Remark 4.4) the Laplace duality of Feller's branching diffusion with a deterministic process, the moment duality of the Fisher-Wright diffusion with Kingman's coalescent, and the moment duality of the (continuous time) Galton-Watson process with a deterministic process.

In the references [1] and [7], the duality relations (2) and (4) are proved analytically by means of a generator calculation. In this paper, we take a different approach by explaining the dynamics of the processes via *basic mechanisms* on the level of particles which lead to the above dualities. To this end, for every $N \in \mathbb{N}$, we construct approximating Markov processes $\left(X_t^N\right)_{t\geq 0}$ and $\left(Y_t^N\right)_{t\geq 0}$ with càdlàg sample paths and state space $\{0,1\}^N$ and with the following properties. The processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ are dual in the sense that

$$\mathbf{P}^{x^N}\left[X_t^N \wedge y^N = \underline{0}\right] = \mathbf{P}^{y^N}\left[x^N \wedge Y_t^N = \underline{0}\right], \qquad \forall\, x^N, y^N \in \{0,1\}^N \quad \forall\, t \geq 0. \tag{5}$$

The notation $x^N \wedge y^N$ denotes component-wise minimum and $\underline{0}$ denotes the zero configuration. If $|X_0^N| = n$, for some fixed $n \leq N$, then $\left(|X_t^N|\right)_{t\geq 0}$ converges weakly to a branching-coalescing particle process as $N \to \infty$. We use the notation $|x^N| := \sum_{i=1}^N x_i^N$ for $x^N \in \{0,1\}^N$. Assume that the set of càdlàg-paths is equipped with the Skorohod topology (see e.g. [4]). If $n = n(N)$ depends on N such that $n/N \to x \in [0,1]$ as $N \to \infty$, then $(|X_t^N|/N)_{t\geq 0}$ converges weakly to a resampling-selection model. If $n = n(N)$ satisfies $n/\sqrt{N} \to x \geq 0$, then $\left(|X_{t\sqrt{N}}^N|/\sqrt{N}\right)_{t\geq 0}$ converges weakly to Feller's branching diffusion with logistic growth. The process $(Y_t^N)_{t\geq 0}$ differs from $(X_t^N)_{t\geq 0}$ only by the set of parameters and by the initial condition.

We will derive the moment duality (2) and the Laplace duality (4) from (5) in the following way. Let the random variable X_0^N be uniformly distributed over all configurations $x^N \in \{0,1\}^N$ with total number of individuals of type 1 equal to $|x^N| = n = n(N)$ for a given $n(N) \leq N$.

Similarly, choose Y_0^N uniformly in $\{0,1\}^N$ with $|Y_0^N| = k = k(N)$ for a given $k(N) \le N$. We will prove in Proposition 3.1 that property (5) implies a prototype duality relation, namely

$$\lim_{N\to\infty} \mathbf{E}\Big[1 - \frac{k}{N}\Big]^{\left|X^N_{tT_N}\right|} = \lim_{N\to\infty} \mathbf{E}\Big[1 - \frac{\left|Y^N_{tT_N}\right|}{N}\Big]^n, \quad t \ge 0, \tag{6}$$

under some assumptions – including the convergence of both sides – on the two processes and on the sequence $(T_N)_{N\ge 1} \subset \mathbb{R}_{\ge 0}$. Choosing n fixed, k such that $\frac{k}{N} \to y \ge 0$ and letting $T_N = 1$, we deduce from (6) (and from the convergence properties of $(X_t^N)_{t\ge 0}$ and of $(Y_t^N)_{t\ge 0}$) the moment duality of a branching-coalescing particle process with a resampling-selection model (cf. Theorem 4.1). In order to obtain a Laplace duality of logistic Feller diffusions, choose n, k such that $\frac{n}{\sqrt{N}} \to x \ge 0$, $\frac{k}{\sqrt{N}} \to y \ge 0$ and $T_N = \sqrt{N}$. Notice that $(1 - \frac{y}{\sqrt{N}})^{x\sqrt{N}}$ converges to e^{-xy} uniformly in $0 \le x, y \le \tilde{x}$ as $N \to \infty$ for every $\tilde{x} \ge 0$. This together with the weak convergence of the rescaled processes will imply

$$\lim_{N\to\infty} \mathbf{E}\Big[e^{-|X^N_{t\sqrt{N}}|\cdot y\big/\sqrt{N}}\Big] = \lim_{N\to\infty} \mathbf{E}\Big[e^{-x\cdot|Y^N_{t\sqrt{N}}|\big/\sqrt{N}}\Big]. \tag{7}$$

For the construction of the approximating processes, we interpret the elements of $\{1, \ldots, N\}$ as "individuals" and the elements of $\{0,1\}$ as the "type" of an individual. In the terminology of population genetics, individuals are denoted as "genes", whereas in population dynamics, the statement "individual i is of type 1 (resp. 0)" would be phrased as "site i is occupied (resp. not occupied) by a particle". Throughout the paper, we assume that whenever a change of the configuration happens at most two individuals are involved. We call every function $f\colon \{0,1\}^2 \to \{0,1\}^2$ a *basic mechanism*. A finite tuple $(f_1, ..., f_m)$, $m \in \mathbb{N}$, of basic mechanisms together with rates $\lambda_1, ..., \lambda_m \in [0,\infty)$ defines a process with state space $\{0,1\}^N$ by means of the following graphical representation, which is in the spirit of Harris [6]. With every $k \le m$ and every ordered pair $(i,j) \in \{1, ..., N\}^2$, $i \ne j$, of individuals, we associate a Poisson process with rate parameter λ_k. At every time point of this Poisson process, the configuration of (i,j) changes according to f_k. For example, if the pair of types was $(1,0)$ before, then it changes to $f_k(1,0) \in \{0,1\}^2$. All Poisson processes are supposed to be independent. This construction can be visualised by drawing arrows from i to j at the time points of the Poisson processes associated with the pair (i,j) (cf. Figure 1).
As an example, consider the following continuous time *Moran model* $(M_t^N)_{t\ge 0}$ with state space $\{0,1\}^N$. This is a population genetic model where ordered pairs of individuals resample at rate β/N, $\beta > 0$. When a resampling event occurs at (i,j), individual i bequeaths its type to individual j. Thus, the basic mechanism is f^R defined by

$$f^R(1,\cdot) := (1,1),\ f^R(0,\cdot) := (0,0). \tag{8}$$

Figure 1 shows a realisation with three resampling events. At time t_1, the pair $(2,1)$ resamples. The arrow in Figure 1 at time t_1 indicates that individual 2 bequeaths its type to individual 1. Furthermore, individual 5 inherits the type of individual 3 at time t_3. The dual process of the Moran model is a coalescent process. This process is defined by the coalescent mechanism f^C given by

$$f^C(1,\cdot) := (0,1),\ f^C(\underline{z}) := \underline{z}, \quad \underline{z} \in \{(0,0),(0,1)\}, \tag{9}$$

and by the rate β/N. To put it differently, the coalescent process is a coalescing random walk on the complete oriented graph of $\{1, \ldots, N\}$. In Section 2, we will specify in which sense

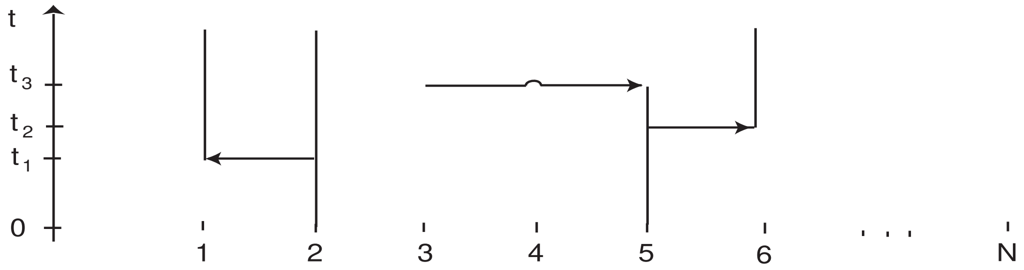

Figure 1: Three resampling events. Type 1 is indicated by black lines, absent lines correspond to type 0.

f^R and f^C are dual, and why this implies (5) (see Proposition 2.3). More generally, we will identify all dual pairs of basic mechanisms.

Our method elucidates the role of the square in (3) for the duality of the logistic Feller diffusion with another logistic Feller diffusion. We illustrate this by the Laplace duality of Feller's branching diffusion $(F_t)_{t\geq 0}$, which is the logistic Feller diffusion with parameters $(0,0,\beta)$, $\beta > 0$. Its dual process $(y_t)_{t\geq 0}$ is the logistic Feller diffusion with parameters $(0,\beta,0)$, i.e., the solution of the ordinary differential equation

$$\frac{d}{dt}y_t = -\beta\, y_t^2, \qquad y_0 = y \in [0,\infty). \tag{10}$$

The duality relation between these two processes is $\mathbf{E}^x[e^{-F_t y}] = e^{-x y_t}$, $t \geq 0$. In Theorem 4.3, we prove that the rescaled Moran model $\left(|M^N_{t\sqrt{N}}|/\sqrt{N}\right)_{t\geq 0}$ converges weakly to $(F_t)_{t\geq 0}$ as $N \to \infty$. To get an intuition for this convergence, notice that $(|M_t^N|)_{t\geq 0}$ is a pure birth-death process with size-dependent transition rates ("birth" corresponds to creation of an individual with type 1, whereas "death" corresponds to creation of an individual with type 0). It remains to prove that the birth and death events become asymptotically independent as $N \to \infty$. It is known, see e.g. Section 2 in [3], that the dual process of the Moran model $(M_t^N)_{t\geq 0}$, $N \geq 1$, is a coalescing random walk. Furthermore, the total number of particles of this coalescing random walk is a pure death process on $\{1,...,N\}$ which jumps from k to $k-1$ at exponential rate $\frac{\beta}{N}k(k-1)$, $2 \leq k \leq N$. This rate is essentially quadratic in k for large k. We will see that a suitably rescaled pure death process converges to a solution of (10); see Remark 4.5. The square in (10) originates in the quadratic rate of the involved pure death process; see the equations (42) and (29) for details.

In the literature, e.g. [9], the duality function $H(x^N, y^N) = \mathbb{1}_{x^N \leq y^N}$, $x^N, y^N \in \{0,1\}^N$, can be found frequently, where $x^N \leq y^N$ denotes component-wise comparison. Processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ with state space $\{0,1\}^N$ are dual with respect to this duality function if they satisfy

$$\mathbf{P}^{x^N}\left[X_t^N \leq y^N\right] = \mathbf{P}^{y^N}\left[x^N \leq Y_t^N\right] \qquad \forall\, x^N, y^N \in \{0,1\}^N, t \geq 0. \tag{11}$$

The biased voter model is dual to a coalescing branching random walk in this sense (see [8]). Property (11) could also be used to derive the dualities mentioned in this introduction. In fact, the two properties (5) and (11) are equivalent in the following sense: If $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ satisfy (5) then $(X_t^N)_{t\geq 0}$ and $(\underline{1} - Y_t^N)_{t\geq 0}$ satisfy (11) and vice versa. In the configuration $\underline{1}$ every individual has type 1 and $\underline{1} - y$ denotes component-wise subtraction. The dynamics of the process $(\underline{1} - Y_t^N)_{t\geq 0}$ is easily obtained from the dynamics of $(Y_t^N)_{t\geq 0}$ by interchanging the roles of the types 0 and 1.

2 Dual basic mechanisms

Fix $m \in \mathbb{N}$ and let $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ be two processes defined by basic mechanisms $(f_1, ..., f_m)$ and $(g_1, ..., g_m)$, respectively. Suppose that the Poisson processes associated with $k \leq m$ have the same rate parameter $\lambda_k \geq 0$, $k = 1, \ldots, m$. We introduce a property of basic mechanisms which will imply (5).

Definition 2.1 *Let $f, g : \{0,1\}^2 \to \{0,1\}^2$ and for $x = (x_1, x_2) \in \{0,1\}^2$ let $x^\dagger := (x_2, x_1)$. The basic mechanisms f and g are said to be **dual** iff the following two conditions hold:*

$$\forall\, x, y \in \{0,1\}^2 \colon \;\; y \wedge \big(f(x)\big)^\dagger = (0,0) \quad \Longrightarrow \quad g(y) \wedge x^\dagger = (0,0), \tag{12}$$

$$\forall\, x, y \in \{0,1\}^2 \colon \;\; x \wedge \big(g(y)\big)^\dagger = (0,0) \quad \Longrightarrow \quad f(x) \wedge y^\dagger = (0,0). \tag{13}$$

To see how this connects to the duality relation in (5), we illustrate this definition by an example.

Example 2.2 The resampling mechanism f^R defined in (8) and the coalescent mechanism f^C defined in (9) are dual. We check condition (12) with $f = f^R$ and $g = f^C$ by looking at Figure 2. The resampling mechanism acts in upward time (solid lines), the coalescent

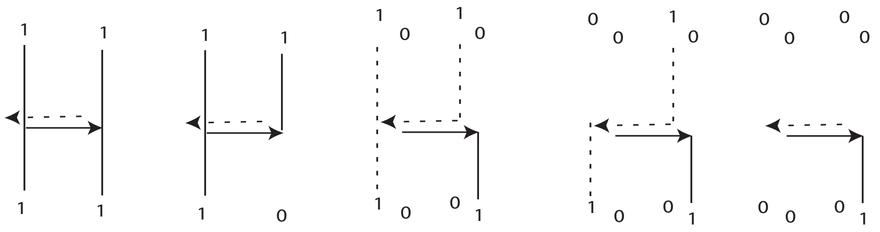

Figure 2: The resampling mechanism and the coalescent mechanism satisfy (12)

mechanism in downward time (dashed lines). There are three nontrivial configurations for x, i.e., $(1,1)$, $(1,0)$ and $(0,1)$. In the first two cases, we have $f^R(x) = (1,1)$. Then only $y = (0,0)$ satisfies $y \wedge (f^R(x))^\dagger = (0,0)$. In the third case, every y satisfies $y \wedge (f^R(0,1))^\dagger = (0,0)$ and has to be checked separately. We see that whenever the configuration y is disjoint from $(f(x))^\dagger$, i.e., $y \wedge (f(x))^\dagger = (0,0)$, then $g(y)$ is disjoint from $x^\dagger$. The coalescent mechanism is the natural dual mechanism of the resampling mechanism. Type 1 of the coalescent mechanism "traces back" the lines of descent of type 0 of the resampling mechanism. The "birth event" $(0,1) \mapsto (0,0)$ of an individual of type 0 results in a coalescent event of ancestral lines.
Figure 3 is useful to verify condition (13). Again, the coalescent mechanism is drawn with dashed lines. Here, the coalescent process is started in the nontrivial configurations $(1,1)$, $(1,0)$ and $(0,1)$. In any case we obtain $(f^C(y))^\dagger = (1,0)$. Hence, all admissible x are of the form $(0,\cdot)$. Condition (13) then follows from $f^R(0,\cdot) = (0,0)$.

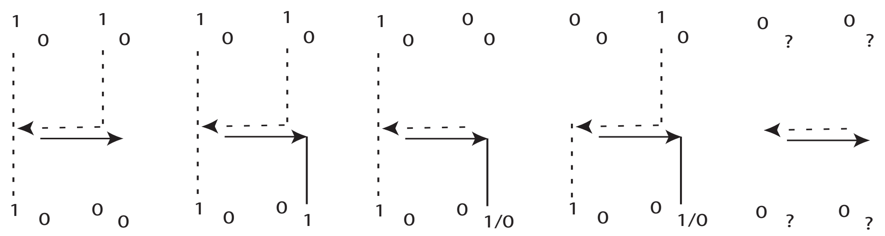

Figure 3: The resampling mechanism and the coalescent mechanism satisfy (13)

The following proposition shows that two processes are dual in the sense of (5) if their defining basic mechanisms are dual (cf. Definition 2.1). The proofs of both Proposition 2.3 and Proposition 3.1 follow similar ideas as in [5].

Proposition 2.3 *Let $m \in \mathbb{N}$ and let the processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ be defined by basic mechanisms $(f_1, ..., f_m)$ and $(g_1, ..., g_m)$, respectively. Suppose that the Poisson processes associated with $k \in \{1, \dots, m\}$ in $(X_t^N)_{t\geq 0}$ and in $(Y_t^N)_{t\geq 0}$ have the same rate parameter $\lambda_k \geq 0$. If f_k and g_k are dual for every $k = 1, \dots, m$, then $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ satisfy the duality relation* (5).

Proof: Fix $T > 0$ and initial values $X_0^N, Y_0^N \in \{0,1\}^N$. Assume for simplicity that $m = 1$ and let $f := f_1$, $g := g_1$. Define the process $\left(\hat{Y}_t^N\right)_{0\leq t\leq T}$ in backward time in the following way. Reverse all arrows in the graphical representation of $(X_t^N)_{t\geq 0}$. At (forward) time T, start with a type configuration given by $\hat{Y}_0^N := Y_0^N$. Now proceed until (forward) time 0: Whenever you encounter an arrow, change the configuration according to g. Recall that the direction of the arrow indicates the order of the involved individuals. We show that the processes $(X_t^N)_{t\geq 0}$ and $(\hat{Y}_t^N)_{0\leq t\leq T}$ satisfy

$$X_0^N \wedge \hat{Y}_T^N = \underline{0} \iff X_T^N \wedge \hat{Y}_0^N = \underline{0} \qquad \forall\, X_0^N, \hat{Y}_0^N \in \{0,1\}^N, \tag{14}$$

for every realisation. We prove the implication "$\Longrightarrow$" by contradiction. Hence, assume that for some initial configuration there is a (random) time $t \in [0, T]$ such that

$$X_0^N \wedge \hat{Y}_T^N = \underline{0} \text{ and } X_t^N \wedge \hat{Y}_{T-t}^N \neq \underline{0}. \tag{15}$$

There are only finitely many arrows until time T and no two arrows occur at the same time almost surely. Hence, there is a first time τ such that the processes are disjoint before this time but not after this time. The arrow at time τ points from i to j, say. Denote by $(x_i^-, x_j^-) \in \{0,1\}^2$ and (x_i^+, x_j^+) the types of the pair $(i,j) \in \{1, ..., N\}^2$ according to the process $(X_t^N)_{t\geq 0}$ immediately before and after forward time τ, respectively. By the definition of the process, we then have $f(x_i^-, x_j^-) = (x_i^+, x_j^+)$. Furthermore, denote by (y_j^-, y_i^-) the types of the pair (j,i) according to $(Y_t^N)_{t\geq 0}$ immediately before backward time $T - \tau$. We have chosen τ, i, j such that

$$(x_i^-, x_j^-) \wedge \left(g(y_j^-, y_i^-)\right)^\dagger = (0,0) \quad \text{and} \quad (x_i^+, x_j^+) \wedge (y_i^-, y_j^-) \neq (0,0). \tag{16}$$

However, this contradicts the duality of f and g. The proof of the other implication is analogous.

It remains to prove that Y_T^N and $\hat{Y}_T^N$ are equal in distribution. The assertion then follows from

$$\mathbf{P}\big[X_0^N \wedge Y_T^N = \underline{0}\big] = \mathbf{P}\big[X_0^N \wedge \hat{Y}_T^N = \underline{0}\big] \overset{(14)}{=} \mathbf{P}\big[X_T^N \wedge \hat{Y}_0^N = \underline{0}\big] = \mathbf{P}\big[X_T^N \wedge Y_0^N = \underline{0}\big]. \quad (17)$$

If a Poisson process is conditioned on its value at some fixed time $T > 0$, then the time points are uniformly distributed over the interval $[0, T]$. The uniform distribution is invariant under time reversal. In addition, the Poisson processes of $(Y_t^N)_{t\geq 0}$ nd $(X_t^N)_{t\geq 0}$ have the same rate parameter. Thus, $(Y_t^N)_{0\leq t\leq T}$ and $(\hat{Y}_t^N)_{0\leq t\leq T}$ have the same one-dimensional distributions. $\Box$

We will now give a list of those maps $f : \{0,1\}^2 \to \{0,1\}^2$ for which there exists a dual basic mechanism (see Definition 2.1). The maps f and g in every row of the following table are dual to each other. As in Example 2.2, it is elementary to check this.

N^o	$f(0,0)$	$f(0,1)$	$f(1,0)$	$f(1,1)$	$g(0,0)$	$g(0,1)$	$g(1,0)$	$g(1,1)$
i)	(0,0)	(0,0)	(1,1)	(1,1)	(0,0)	(0,1)	(0,1)	(0,1)
ii)	(0,0)	(0,1)	(1,1)	(1,1)	(0,0)	(0,1)	(1,1)	(1,1)
iii)	(0,0)	(0,0)	(0,1)	(0,1)	(0,0)	(0,0)	(0,1)	(0,1)
iv)	(0,0)	(0,1)	(1,0)	(1,1)	(0,0)	(0,1)	(1,0)	(1,1)
v)	(0,0)	(1,1)	(1,1)	(1,1)	(0,0)	(1,1)	(1,1)	(1,1)
vi)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Check that the pair (f, g) is dual if and only if the pair $(f^\dagger, g^\dagger)$ is dual where $f^\dagger(x) := (f(x^\dagger))^\dagger$. Furthermore, the pair (f, g) is dual if and only if $(\hat{f}, \hat{g}^\dagger)$ is dual where $\hat{f}(x) := f(x^\dagger)$ and $\hat{g}^\dagger(x) = (g(x))^\dagger$ for $x \in \{0,1\}^2$. Thus, for each of the listed dual pairs (f, g), the pairs $(f^\dagger, g^\dagger)$, $(\hat{f}, \hat{g}^\dagger)$ and $(\hat{f}^\dagger, \hat{g})$ are also dual. Modulo this relation, the listing of dual basic mechanisms is complete. The proof of this assertion is elementary but somewhat tedious and is thus omitted. Readers interested in the proof are invited to contact the authors in order to get the detailed classification of dual basic mechanisms.

Of particular interest are the dualities in i)-iii). The first of these is the duality between the resampling mechanism and the coalescent mechanism, which we already encountered in Example 2.2. The duality in ii) is the self-duality of the **pure birth mechanism**

$$f^B : \{0,1\}^2 \to \{0,1\}^2,\ (1,0) \mapsto (1,1) \text{ and } x \mapsto x\ \forall\, x \in \{(0,0),(0,1),(1,1)\} \quad (18)$$

and iii) is the self-duality of the **death/coalescent mechanism**

$$f^{DC} : \{0,1\}^2 \to \{0,1\}^2,\ (1,\cdot) \mapsto (0,1) \text{ and } (0,\cdot) \mapsto (0,0). \quad (19)$$

We are only interested in the effect of a basic mechanism on the total number of individuals of type 1. The identity map in iv) does not change the number of individuals of type 1 in the configuration. The effect of v) and vi) on the number of individuals of type 1 is similar to the effect of ii) and iii), respectively. Furthermore, both $f^\dagger$ and $\hat{f}$ have the same effect on the number of individuals of type 1 as f.

Closing this section, we define processes which satisfy the duality relation (5). These processes will play a major role in deriving the dualities (2) and (4) in Section 4. For $u, e, \gamma, \beta \geq 0$, let $(X_t^N)_{t\geq 0} = (X_t^{N,(u,e,\gamma,\beta)})_{t\geq 0}$ be the process on $\{0,1\}^N$ with the following transition rates (of independent Poisson processes):

- With rate $\frac{u}{N}$, the pure birth mechanism f^B occurs (cf.(18)).
- With rate $\frac{e}{N}$, the death/coalescent mechanism f^{DC} occurs (cf. (19)).
- With rate $\frac{\gamma}{N}$, the coalescent mechanism f^C occurs (cf. (9)).
- With rate $\frac{\beta}{N}$, the resampling mechanism f^R occurs (cf. (9)).

Together with an initial configuration, this defines the process. The process $(X_t^{N,(u,e,\gamma,\beta)})_{t\geq 0}$ is defined by the basic mechanisms (f^B, f^{DC}, f^C, f^R), and the process $(X_t^{N,(u,e,\beta,\gamma)})_{t\geq 0}$ is defined by the basic mechanisms (f^B, f^{DC}, f^R, f^C). Proposition 2.3 then yields the following corollary.

Corollary 2.4 *Let $u, e, \gamma, \beta \geq 0$. The two processes $(X_t^{N,(u,e,\gamma,\beta)})_{t\geq 0}$ and $(X_t^{N,(u,e,\beta,\gamma)})_{t\geq 0}$ satisfy the duality relation* (5).

3 Prototype duality

In this section, we derive the prototype duality (6) from (5). The main idea for this is to integrate equation (5) in the variables x^N and y^N with respect to a suitable measure. Furthermore, we will exploit the fact that drawing from an urn with replacement and without replacement, respectively, is almost surely the same if the urn contains infinitely many balls.

Proposition 3.1 *Let $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ be processes with state space $\{0,1\}^N$, $N \geq 1$. Assume that $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ satisfy the duality relation* (5)*. Choose $n, k \in \{0, ..., N\}$ which may depend on N. Define $\mu_n^N(x^N) := \binom{N}{n}^{-1} \mathbb{1}_{|x^N|=n}$ for every $x^N \in \{0,1\}^N$ where $|x^N| = \sum_{i=1}^N x_i^N$ is the total number of individuals of type* 1*. Assume $\mathcal{L}\left(X_0^N\right) = \mu_n^N$ and $\mathcal{L}\left(Y_0^N\right) = \mu_k^N$. Suppose that the process $(X_t^N)_{t\geq 0}$ satisfies*

$$\frac{n}{N} \to 0 \quad \textit{and} \quad \frac{\mathbf{E}\big[\big|X_{t_N}^N\big|\big]}{N} \longrightarrow 0 \qquad \textit{as } N \to \infty, \tag{20}$$

where $t_N \geq 0$. Then

$$\lim_{N\to\infty} \mathbf{E}\left[\left(1 - \tfrac{k}{N}\right)^{\left|X_{t_N}^N\right|}\right] = \lim_{N\to\infty} \mathbf{E}\left[\left(1 - \tfrac{\left|Y_{t_N}^N\right|}{N}\right)^n\right] \tag{21}$$

under the assumption that the limits exist.

Proof: A central idea of the proof is to make use of the well known fact that the hypergeometric distribution $\mathrm{Hyp}(N, R, l)$, $R, l \in \{0, ..., N\}$, can be approximated by the binomial distribution $\mathrm{B}(l, \frac{R}{N})$ as $N \to \infty$ provided that l is sufficiently small compared to N. In fact, by Theorem 4 of [2],

$$\left|\mathrm{B}(l, \tfrac{R}{N})\big[\{0\}\big] - \mathrm{Hyp}(N, R, l)\big[\{0\}\big]\right| \leq d_{TV}\Big(\mathrm{B}\big(l, \tfrac{R}{N}\big), \mathrm{Hyp}(N, R, l)\Big) \leq \frac{4 \cdot l}{N} \qquad \forall\, R, l \leq N, \tag{22}$$

where d_{TV} is the total variation distance. By assumption (20), we have (with $R := k, l := |X^N_{t_N}|$)

$$\mathbf{E}\Big[\Big(1-\tfrac{k}{N}\Big)^{|X^N_{t_N}|}\Big] = \mathbf{E}\Big[\mathrm{B}\Big(|X^N_{t_N}|, \tfrac{k}{N}\Big)[\{0\}]\Big] = \mathbf{E}\Big[\mathrm{Hyp}\big(N,k,|X^N_{t_N}|\big)[\{0\}]\Big] + o(1) \tag{23}$$

as $N \to \infty$. Similarly, we have (with $R := |Y^N_{t_N}|$, $l := n$)

$$\mathbf{E}\Big[\Big(1-\tfrac{|Y^N_{t_N}|}{N}\Big)^{n}\Big] = \mathbf{E}\Big[\mathrm{B}\Big(n, \tfrac{|Y^N_{t_N}|}{N}\Big)[\{0\}]\Big] = \mathbf{E}\Big[\mathrm{Hyp}\big(N,|Y^N_{t_N}|,n\big)[\{0\}]\Big] + o(1) \tag{24}$$

as $N \to \infty$. By definition of the hypergeometric distribution, we get

$$\mathrm{Hyp}\big(N,|Y^N_t|,n\big)[\{0\}] = \binom{N}{n}^{-1} \sum_{x^N:\,|x^N|=n} \mathbb{1}_{\{x^N \wedge Y^N_t = \underline{0}\}} = \mu^N_n\big[x^N : x^N \wedge Y^N_t = \underline{0}\big]. \tag{25}$$

By the same argument, we also obtain

$$\mathrm{Hyp}\big(N,k,|X^N_t|\big)[\{0\}] = \mathrm{Hyp}\big(N,|X^N_t|,k\big)[\{0\}] = \mu^N_k\big[y^N : X^N_t \wedge y^N = \underline{0}\big]. \tag{26}$$

We denote by $\mathbf{P}^{x^N}$ the law of the process $(X^N_t)_{t\ge 0}$ started in the fixed initial configuration $x^N \in \{0,1\}^N$. Starting from the left-hand side of (21), the above considerations yield

$$\begin{aligned}
&\mathbf{E}\Big[\big(1-\frac{k}{N}\big)^{|X^N_{t_N}|}\Big] + o(1) \overset{(23)}{=} \mathbf{E}\Big[\mathrm{Hyp}(N,k,|X^N_{t_N}|)[\{0\}]\Big] \\
&\overset{(26)}{=} \int \mathbf{E}^{x^N}\Big[\mu^N_k\big[X^N_{t_N} \wedge y^N = \underline{0}\big]\Big]\mu^N_n(\mathrm{d}x^N) \\
&\overset{(5)}{=} \int\int \mathbf{P}^{y^N}\Big[x^N \wedge Y^N_{t_N} = \underline{0}\Big]\mu^N_k(\mathrm{d}y^N)\,\mu^N_n(\mathrm{d}x^N) = \mathbf{E}\Big[\mu^N_n\big[x^N \wedge Y^N_{t_N} = \underline{0}\big]\Big] \\
&\overset{(25)}{=} \mathbf{E}\Big[\mathrm{Hyp}(N,|Y^N_{t_N}|,n)[\{0\}]\Big] \overset{(24)}{=} \mathbf{E}\Big[\Big(1-\frac{|Y^N_{t_N}|}{N}\Big)^n\Big] + o(1),
\end{aligned} \tag{27}$$

which proves the assertion. □

4 Various scalings

Recall the definition of the process $(X^{N,(u,e,\gamma,\beta)}_t)_{t\ge 0}$ from the end of Section 2. Define $X^N_t := X^{N,(u,e,\gamma,\beta)}_t$ and $Y^N_t := X^{N,(u,e,\beta,\gamma)}_t$ for $t \ge 0$ and $N \in \mathbb{N}$. Notice that the Poisson process attached to the resampling mechanism in the process $(Y^N_t)_{t\ge 0}$ has rate γ. By Corollary 2.4, the two processes $(X^N_t)_{t\ge 0}$ and $(Y^N_t)_{t\ge 0}$ satisfy the duality relation (5). Let $\mathcal{L}\left(X^N_0\right) = \mu^N_n$ and $\mathcal{L}\left(Y^N_0\right) = \mu^N_k$ for some $n, k \in \mathbb{N}$ to be chosen later, where μ^N_n is defined in Proposition 3.1. In order to apply Proposition 3.1, we essentially have to prove existence of the limits in (21). Depending on the scaling, this will result in the moment duality (2) of a resampling-selection model with a branching-coalescing particle process and in the Laplace duality (4) of the logistic Feller diffusion with another logistic Feller diffusion, respectively. Both dualities could be derived simultaneously. However, in order to keep things simple, we consider the two cases separately.

Theorem 4.1 *Assume that $b, c, d \geq 0$. Denote by $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ the $(1, b, c, d)$-braco-process and the $(1, b, c, d)$-resem-process, respectively. The initial values are $X_0 = n \in \mathbb{N}_0$ and $Y_0 = y \in [0, 1]$. Then*

$$\mathbf{E}^n\big[(1-y)^{X_t}\big] = \mathbf{E}^y\big[(1-Y_t)^n\big], \qquad t \geq 0. \tag{28}$$

Remark 4.2 In the special case $b = 0 = d$ and $c > 0$, this is the moment duality of the Fisher-Wright diffusion with Kingman's coalescent. Furthermore, choosing $c = 0$ and $b, d > 0$ results in the moment duality of the Galton-Watson process with a deterministic process.

Proof: Choose $u, e, \beta \geq 0$ and $\gamma = \gamma(N)$ such that $b = u + \beta$, $d = e + \beta$ and $\gamma/N \to c$ as $N \to \infty$. In the first step, we prove that the process $(|X_t^N|)_{t\geq 0}$ of the total number of individuals of type 1 converges weakly to $(X_t)_{t\geq 0}$. The total number of individuals of type 1 increases by one if a "birth event" occurs (f^B or f^R) and if the type configuration of the respective ordered pair of individuals is $(1, 0)$. If the total number of individuals of type 1 is equal to k, then the probability of the type configuration of a randomly chosen ordered pair to be $(1, 0)$ is $\frac{k}{N}\frac{N-k}{N-1}$. The number of Poisson processes associated with a fixed basic mechanism is $N(N-1)$. Thus, the process of the total number of individuals of type 1 has the following transition rates:

$$\begin{aligned} k \to k+1: &\quad \tfrac{u+\beta}{N} \cdot N(N-1) \cdot \tfrac{k}{N}\tfrac{N-k}{N-1}, \\ k \to k-1: &\quad \tfrac{e+\beta}{N} \cdot N(N-1) \cdot \tfrac{N-k}{N}\tfrac{k}{N-1} + \tfrac{e+\gamma}{N} \cdot N(N-1) \cdot \tfrac{k}{N}\tfrac{k-1}{N-1}, \end{aligned} \tag{29}$$

where $k \in \mathbb{N}_0$. Notice that the coalescent mechanism produces the quadratic term $k(k-1)$ because the probability of the type configuration of a randomly chosen ordered pair to be $(1, 1)$ is $\frac{k}{N}\frac{k-1}{N-1}$ if there are k individuals of type 1. The transition rates determine the generator $\mathcal{G}^N = \mathcal{G}^{N,(u,e,\gamma,\beta)}$ of $(|X_t^N|)_{t\geq 0}$, namely

$$\begin{aligned} \mathcal{G}^N f(k) =& \frac{u+\beta}{N} \cdot k(N-k) \cdot \big(f(k+1) - f(k)\big) \\ &+ \frac{e+\beta}{N} \cdot k(N-k) \cdot \big(f(k-1) - f(k)\big) \\ &+ \frac{e+\gamma}{N} \cdot k(k-1) \cdot \big(f(k-1) - f(k)\big), \quad k \in \{0, \ldots, N\}, \end{aligned} \tag{30}$$

for $f\colon \{0, \ldots, N\} \to \mathbb{R}$. The $(1, u+\beta, c, e+\beta)$-braco-process $(X_t)_{t\geq 0}$ is the unique solution of the martingale problem for $\mathcal{G}$ (see [1]) where

$$\mathcal{G}f(k) := (u+\beta)k\,\big(f(k+1) - f(k)\big) + \big((e+\beta) + c(k-1)\big)k\,\big(f(k-1) - f(k)\big), \quad k \in \mathbb{N}_0, \tag{31}$$

for $f\colon \mathbb{N}_0 \to \mathbb{R}$ with finite support. Letting $N \to \infty$, we see that

$$\mathcal{G}^N f(k) \longrightarrow \mathcal{G}f(k) \qquad \text{as } N \to \infty, \quad k \in \mathbb{N}_0, \tag{32}$$

for $f\colon \mathbb{N}_0 \to \mathbb{R}$ with finite support. We aim at using Lemma 5.1 which is given below (with $E_N = \{0, \ldots, N\}$ and $E = \mathbb{N}_0$), to infer from (30) the weak convergence of the corresponding Markov processes. A coupling argument shows that $(|X_t^N|)_{t\geq 0}$ is dominated by $(Z_t^N)_{t\geq 0} := (|X_t^{N,(u,0,0,\beta)}|)_{t\geq 0}$. The process $(Z_t^N)_{t\geq 0}$ solves the martingale problem for $\mathcal{G}^{N,(u,0,0,\beta)}$. Thus, we obtain

$$Z_t^N - Z_0^N = \int_0^t \mathcal{G}^{N,(u,0,0,\beta)} Z_s^N\, ds + C_t^N = \int_0^t u Z_s^N \tfrac{N-Z_s^N}{N}\, ds + C_t^N \tag{33}$$

where $(C_t^N)_{t\geq 0}$ is a martingale. Hence, $(Z_t^N)_{t\geq 0}$ is a submartingale. Taking expectations, Gronwall's inequality implies

$$\mathbf{E}[Z_t^N] \leq \mathbf{E}[Z_0^N]e^{ut}, \qquad \forall\, t \geq 0. \tag{34}$$

Let $S_N = T_N = 1$, $s_N = u$ and recall $|X_0^N| = n$. With this, the assumptions of Lemma 5.1 are satisfied. Thus, Lemma 5.1 implies that $(|X_t^N|)_{t\geq 0}$ converges weakly to $(X_t)_{t\geq 0}$ as $N \to \infty$. Let $k = k_N \in \{0, ..., N\}$ be such that $k/N \to y$ as $N \to \infty$. For every $\bar{n} \in \mathbb{N}$, $(1-\frac{k}{N})^n$ converges uniformly in $n \leq \bar{n}$ to $(1-y)^n$ as $N \to \infty$. In general, if the sequence $(\tilde{X}_n)_{n\in\mathbb{N}}$ of random variables with complete and separable state space converges weakly to $\tilde{X}$ and if the sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in C_b$, converges uniformly on compact sets to $f \in C_b$, then $\mathbf{E}[f_n(\tilde{X}_n)] \to \mathbf{E}[f(\tilde{X})]$ as $n \to \infty$. Hence,

$$\mathbf{E}^n\Big[(1-y)^{X_t}\Big] = \lim_{N\to\infty} \mathbf{E}\Big[\Big(1 - \frac{k}{N}\Big)^{|X_t^N|}\Big]. \tag{35}$$

The next step is to prove that the rescaled processes $(|Y_t^N|/N)_{t\geq 0}$ converge weakly to $(Y_t)_{t\geq 0}$ as $N \to \infty$. The generator of $(|Y_t^N|/N)_{t\geq 0}$ is given by

$$\begin{aligned} \mathcal{G}^{N,(u,e,\beta,\gamma)} f\big(\tfrac{k}{N}\big) =& \gamma k \frac{N-k}{N}\Big(f\big(\tfrac{k+1}{N}\big) + f\big(\tfrac{k-1}{N}\big) - 2f\big(\tfrac{k}{N}\big)\Big) \\ &+ uk\frac{N-k}{N}\Big(f\big(\tfrac{k+1}{N}\big) - f\big(\tfrac{k}{N}\big)\Big) + ek\frac{N-k}{N}\Big(f\big(\tfrac{k-1}{N}\big) - f\big(\tfrac{k}{N}\big)\Big) \\ &+ \tfrac{e+\beta}{N}k(k-1)\Big(f\big(\tfrac{k-1}{N}\big) - f\big(\tfrac{k}{N}\big)\Big), \qquad k \in \{0, ..., N\}, \end{aligned} \tag{36}$$

for $f \in C_c^2([0,1])$. Choose $k = k_N \leq N$ such that $\frac{k}{N} \to y \in [0,1]$ as $N \to \infty$. Notice that

$$N^2 \cdot \Big(f\big(\tfrac{k+1}{N}\big) + f\big(\tfrac{k-1}{N}\big) - 2f\big(\tfrac{k}{N}\big)\Big) \to f^{''}(y) \quad \text{as } N \to \infty. \tag{37}$$

As $N \to \infty$, the right-hand side of (36) converges to

$$\begin{aligned} &cy(1-y)\cdot f^{''}(y) + (u-e)y(1-y)\cdot f^{'}(y) - (e+\beta)y^2 \cdot f^{'}(y) \\ &= (u-e)y \cdot f^{'}(y) - (u+\beta)y^2 \cdot f^{'}(y) + cy(1-y)\cdot f^{''}(y) =: \mathcal{G}f(y) \end{aligned} \tag{38}$$

for every $f \in C_c^2([0,1])$. Athreya and Swart [1] show that the $(1,b,c,d)$-resem-process $(Y_t)_{t\geq 0}$ solves the martingale problem for $\mathcal{G}$ and that this solution is unique. Let $E_N = \{0, 1, \ldots, N\}$, $E = [0,1]$, $Z_t^N := |X_t^{N,(u,0,0,\gamma)}|$, $S_N = N$ and $T_N = 1$. With this, the assumptions of Lemma 5.1 are satisfied and we conclude that $(|Y_t^N|/N)_{t\geq 0}$ converges weakly to $(Y_t)_{t\geq 0}$. It follows that, for $k = k_N \in \{0, ..., N\}$ with $k/N \to y$,

$$\lim_{N\to\infty} \mathbf{E}\left[\Big(1 - \frac{|Y_t^N|}{N}\Big)^n\right] = \mathbf{E}^y\Big[(1-Y_t)^n\Big]. \tag{39}$$

This proves existence of the limits in (21) with $t_N := t$. Inequality (34) and $|X_0^N| = n << N$ imply condition (20). Thus, Proposition 3.1 establishes equation (21). The assertion follows from equations (35), (21) and (39). □

Next, we derive the Laplace duality of a logistic Feller diffusion with another logistic Feller diffusion. Recall that the logistic Feller diffusion with parameters (α, γ, β) solves equation (3).

Theorem 4.3 *Suppose that $\alpha, \gamma, \beta \geq 0$, $r > 0$ and $X_0 = x \geq 0, Y_0 = y \geq 0$. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be logistic Feller diffusions with parameters (α, γ, β) and $(\alpha, r\beta, \gamma/r)$, respectively. Then*

$$\mathbf{E}^x\left[e^{-rX_t\cdot y}\right] = \mathbf{E}^y\left[e^{-rx\cdot Y_t}\right] \tag{40}$$

for all $t \geq 0$.

Remark 4.4
[(a)]
For $\beta, \gamma > 0$ and $r = \gamma/\beta$, Theorem 4.3 yields the self-duality of the logistic Feller diffusion.

2. For $\alpha = 0$, $\gamma = 0$, $r = 1$ and $\beta > 0$, Theorem 4.3 specialises to the Laplace duality of Feller's branching diffusion.

Proof: Choose $u = u_N \geq 0$ and $e = e_N \geq 0$ such that $(u-e)\sqrt{N} \to \alpha$ as $N \to \infty$. We prove that the rescaled process $(|Y^N_{t\sqrt{N}}|/(r\sqrt{N}))_{t\geq 0}$ converges weakly to $(Y_t)_{t\geq 0}$ as $N \to \infty$. The generator of the rescaled process is given by (cf. (36))

$$\begin{aligned}
\sqrt{N}\mathcal{G}^N f\left(\tfrac{k}{r\sqrt{N}}\right) &= \sqrt{N}\cdot\gamma\cdot k\frac{(N-k)}{N}\cdot\left(f\left(\tfrac{k+1}{r\sqrt{N}}\right) + f\left(\tfrac{k-1}{r\sqrt{N}}\right) - 2f\left(\tfrac{k}{r\sqrt{N}}\right)\right)\\
&\quad + \sqrt{N}u_N\cdot k\frac{(N-k)}{N}\cdot\left(f\left(\tfrac{k+1}{r\sqrt{N}}\right) - f\left(\tfrac{k}{r\sqrt{N}}\right)\right)\\
&\quad + \sqrt{N}e_N\cdot k\frac{(N-k)}{N}\cdot\left(f\left(\tfrac{k-1}{r\sqrt{N}}\right) - f\left(\tfrac{k}{r\sqrt{N}}\right)\right)\\
&\quad + \sqrt{N}\cdot(e_N+\beta)\cdot\frac{k(k-1)}{r^2N}r^2\cdot\frac{r\sqrt{N}}{r\sqrt{N}}\left(f\left(\tfrac{k-1}{r\sqrt{N}}\right) - f\left(\tfrac{k}{r\sqrt{N}}\right)\right),
\end{aligned} \tag{41}$$

for $k \in \{0,\ldots,N\}$ and for $f \in C_c^2([0,\infty))$. Let $k = k(N) \in \{0,\ldots,N\}$ be such that $k/(r\sqrt{N}) \to y$. Letting $N \to \infty$, the right-hand side converges to

$$\tfrac{\gamma}{r}\, y\cdot f^{''}(y) + \alpha\, y\cdot f^{'}(y) - \beta r\, y^2\cdot f^{'}(y) =: \mathcal{G}f(y) \tag{42}$$

for every $f \in C_c^2([0,\infty))$. Notice that the quadratic term y^2 originates in the quadratic term $k(k-1)$. Hutzenthaler and Wakolbinger [7] prove that $(Y_t)_{t\geq 0}$ is the unique solution of the martingale problem for $\mathcal{G}$. Let $|Y_0^N| = k = k(N)$ be such that $k/(r\sqrt{N}) \to y \in [0,1]$ as $N \to \infty$ and define $Z_0^N := k$. As before, $(Z_t^N)_{t\geq 0} := (|X_t^{N,(u,0,0,\gamma)}|)_{t\geq 0}$ is a submartingale which dominates $(Y_t^N)_{t\geq 0}$ and which satisfies

$$\sup_N \frac{1}{r\sqrt{N}}\mathbf{E}[Z^N_{t\sqrt{N}}] \leq \sup_N \frac{1}{r\sqrt{N}}\mathbf{E}[Z_0^N]e^{u_N t\sqrt{N}} < \infty, \qquad \forall\, t \geq 0. \tag{43}$$

Let $E_N := \{0,\ldots,N\}$, $E := [0,\infty)$, $s_N := u_N$, $S_N := r\sqrt{N}$ and $T_N := \sqrt{N}$. The assumptions of Lemma 5.1 are satisfied and we conclude that $(|Y^N_{t\sqrt{N}}|/(r\sqrt{N}))_{t\geq 0}$ converges weakly to $(Y_t)_{t\geq 0}$. This also proves that $(|X^N_{t\sqrt{N}}|/\sqrt{N})_{t\geq 0}$ converges weakly to $(X_t)_{t\geq 0}$ if $|X_0^N| = n = n(N)$ is such that $n/\sqrt{N} \to x$ as $N \to \infty$. It is not hard to see that, for every $\tilde{z} \geq 0$,

$$\left(1 - r\tfrac{k/(r\sqrt{N})}{\sqrt{N}}\right)^{\sqrt{N}z} \longrightarrow e^{-rzy} \quad \text{and} \quad \left(1 - r\tfrac{z}{\sqrt{N}}\right)^{\sqrt{N}\frac{n}{\sqrt{N}}} \longrightarrow e^{-rxz} \qquad \text{as } N \to \infty \tag{44}$$

uniformly in $0 \le z \le \tilde{z}$. Together with the weak convergence of the rescaled processes, this implies

$$\mathbf{E}^x\big[e^{-rX_t\cdot y}\big] = \lim_{N\to\infty} \mathbf{E}^n\left[\Big(1 - r\frac{k/(r\sqrt{N})}{\sqrt{N}}\Big)^{\sqrt{N}\cdot X^N_{t\sqrt{N}}/\sqrt{N}}\right] \tag{45}$$

and

$$\lim_{N\to\infty} \mathbf{E}^k\left[\Big(1 - r\frac{Y^N_{t\sqrt{N}}/(r\sqrt{N})}{\sqrt{N}}\Big)^n\right] = \mathbf{E}^y\big[e^{-rx\cdot Y_t}\big] \tag{46}$$

for $t \ge 0$. This proves existence of the limits in (21) with $t_N := t\sqrt{N}$. Inequality (43) and $|X_0^N| = n << N$ imply condition (20). Thus, Proposition 3.1 establishes equation (21). The assertion follows from equations (45), (21) and (46). $\square$

Remark 4.5 Assume $u = e = \gamma = \alpha = 0$ and $r = 1$ in the proof of Theorem 4.3. Then $(|Y_t^N|)_{t\ge0}$ is a pure death process on $\{1, ..., N\}$ which jumps from k to $k-1$ at exponential rate $\frac{\beta}{N}k(k-1)$, $2 \le k \le N$. Furthermore, $(Y_t)_{t\ge0}$ is a solution of (10). We have just shown that the rescaled pure death process $\big(|Y^N_{t\sqrt{N}}|/\sqrt{N}\big)_{t\ge0}$ converges weakly to $(Y_t)_{t\ge0}$ as $N \to \infty$.

5 Weak convergence of processes

In the proofs of Theorem 4.1 and Theorem 4.3, we have established convergence of generators plus a domination principle. In this section, we prove that this implies weak convergence of the corresponding processes. For the weak convergence of processes with càdlàg paths, let the topology on the set of càdlàg paths be given by the Skorohod topology (see [4], Section 3.5).

Lemma 5.1 *Let $E \subset \mathbb{R}_{\ge0}$ be closed. Assume that the martingale problem for $(\mathcal{G}, \nu)$ has at most one solution where $\mathcal{G}\colon C_c^2(E) \to C_b(E)$ is a linear operator and ν is a probability measure on E. Furthermore, for $N \in \mathbb{N}$, let $E_N \subset \mathbb{R}_{\ge0}$ and let $(Y_t^N)_{t\ge0}$ be an E_N-valued Markov process with càdlàg paths and generator $\mathcal{G}^N$. Let $(S_N)_{N\in\mathbb{N}}$ and $(T_N)_{N\in\mathbb{N}}$ be sequences in $\mathbb{R}_{>0}$ with $y^N/S_N \in E$ for all $y^N \in E_N$ and $N \in \mathbb{N}$. Suppose that*

$$y^N \in E_N,\ \lim_{N\to\infty}\frac{y^N}{S_N} = y \in E \ \ \textit{implies}\ \ T_N\mathcal{G}^N f\big(\tfrac{y^N}{S_N}\big) \to \mathcal{G}f(y) \qquad \textit{as } N\to\infty, \tag{47}$$

for every $f \in C_c^2(E)$. Assume that, for $N \in \mathbb{N}$, $(Y_t^N)_{t\ge0}$ is dominated by a process $(Z_t^N)_{t\ge0}$, i.e., $Y_t^N \le Z_t^N$ for all $t \ge 0$ almost surely, which is a submartingale satisfying $\mathbf{E}[Z_t^N] \le \mathbf{E}[Z_0^N]e^{ts_N}$ for all $t \ge 0$ and some constant s_N. In addition, suppose that $\limsup_{N\to\infty} s_N T_N < \infty$ and $\limsup_{N\to\infty}\frac{\mathbf{E}[Z_0^N]}{S_N} < \infty$. If Y_0^N/S_N converges in distribution to ν as $N\to\infty$, then

$$\mathcal{L}\Big(\big(Y^N_{tT_N}/S_N\big)_{t\ge0}\Big) \Longrightarrow \mathcal{L}^\nu\Big(\big(Y_t\big)_{t\ge0}\Big) \qquad \textit{as } N\to\infty \tag{48}$$

where $(Y_t)_{t\ge0}$ is a solution of the martingale problem $(\mathcal{G}, \nu)$ with initial distribution ν.

Proof: We aim at applying Corollary 4.8.16 of Ethier and Kurtz [4]. For this, define

$$\tilde{E}_N := \{\tfrac{y^N}{S_N} : y^N \in E_N\}, \quad \tilde{\mathcal{G}}^N f(\tilde{y}^N) := T_N\mathcal{G}^N f\big(\tfrac{y^N}{S_N}\big)\Big|_{y^N=\tilde{y}^N S_N}, \tilde{y}^N \in \tilde{E}^N, \tag{49}$$

for $f \in C_c^2(E)$ and let $\eta_N \colon \tilde{E}_N \to E$ be the embedding function. The process $\big(Y_{tT_N}^N / S_N\big)_{t\geq 0}$ has state space $\tilde{E}_N$ and generator $\tilde{\mathcal{G}}^N$. Now we prove the compact containment condition, i.e., for fixed $\varepsilon, t > 0$ we show

$$\big(\exists K > 0\big)\ \big(\forall N \in \mathbb{N}\big)\ \mathbf{P}\Big[\sup_{s\leq t} \frac{Y_{sT_N}^N}{S_N} \leq K\Big] \geq 1 - \varepsilon. \tag{50}$$

Using $Y_t^N \leq Z_t^N$, $t \geq 0$, and Doob's Submartingale Inequality, we conclude for all $N \in \mathbb{N}$

$$\begin{aligned}\mathbf{P}\Big[\sup_{s\leq t} Y_{sT_N}^N \geq KS_N\Big] &\leq \mathbf{P}\Big[\sup_{s\leq t} Z_{sT_N}^N \geq KS_N\Big] \leq \frac{1}{KS_N}\mathbf{E}\big[Z_{tT_N}^N\big] \\ &\leq \frac{1}{K}\sup_{N\in\mathbb{N}} \frac{\mathbf{E}\big[Z_0^N\big]}{S_N} \cdot \exp\big(t \cdot \sup_{N\in\mathbb{N}}(s_N T_N)\big) =: \frac{C}{K}.\end{aligned} \tag{51}$$

Thus, choosing $K := \frac{C}{\varepsilon}$ completes the proof of the compact containment condition.
It remains to verify condition (f) of Corollary 4.8.7 of [4]. Condition (47) implies that for every $f \in C_c^2$ and every compact set $K \subset E$

$$\sup_{y\in K\cap\tilde{E}_N} |\tilde{\mathcal{G}}^N f(y) - \mathcal{G}f(y)| \to 0 \qquad \text{as } N \to \infty. \tag{52}$$

Choose a sequence K_N such that (52) still holds with K replaced by K_N. This together with the compact containment condition implies condition (f) of Corollary 4.8.7 of [4] with $G_N := K_N \cap \tilde{E}_N$ and $f_N := f|_{\tilde{E}_N}$. Furthermore, notice that $C_c^2(E)$ is an algebra that separates points and E is complete and separable. Now Corollary 4.8.16 of Ethier and Kurtz [4] implies the assertion. □

Open Question: Athreya and Swart [1] prove a self-duality of the resem-process given by (1). We were not able to establish a graphical representation for this duality. Thus, the question whether our technique also works in this case yet waits to be answered.

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