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STATE TAMENESS: A NEW APPROACH FOR CREDIT CONSTRAINS

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Abstract

We propose a new definition for tameness within the model of security prices as Itô processes that is risk-aware. We give a new definition for arbitrage and characterize it. We then prove a theorem that can be seen as an extension of the second fundamental theorem of asset pricing, and a theorem for valuation of contingent claims of the American type. The valuation of European contingent claims and American contingent claims that we obtain does not require the full range of the volatility matrix. The technique used to prove the theorem on valuation of American contingent claims does not depend on the Doob-Meyer decomposition of supermartingales; its proof is constructive and suggest and alternative way to find approximations of stopping times that are close to optimal.

1 Introduction

In a continuous time setting, where security prices are modeled as Itô processes, the concept of tameness has been introduced as a credit constrain in order to offset the so called "doubling strategies". Harrison and Pliska (1981) and Dybvig and Huang (1988) study the role of this constrain in ruling out doubling strategies. Generally speaking, tameness limits the credit that an agent may have, that is used to offset intermediate losses from trade and consumption. This credit is established in advance in terms of the value of money. Namely, the credit limit is resettled every time to reflect the changes in a bank account. This model is a standard one in financial economics. See Karatzas and Shreve (1998), Karatzas (1996), and Duffie (1996) for some discussion about it. Nonetheless, in order to obtain characterizations of non-arbitrage and completeness, strong technical conditions are made that do not hold for very interesting models in financial economics; see Kreps (1981), Duffie and Huang (1986), Back and Pliska (1991) and Hindy (1995) and more recently Fernholz, Karatzas, and Kardaras (2004). Several approaches have been taken to generalize this model. For example, Levental and Skorohod (1995) study notions of "arbitrage in tame portfolios" and "approximate arbitrage"; Kreps

(1981) and Delbaen and Schachermayer (1994, 1995a, 1995b, 1996, 1997b, 1997a, 1997c, 1998) propose a notion of arbitrage called a "free lunch". However these notions are usually criticized by their lack of economic justification. Loewenstein and Willard (2000) revisit the standard model of security prices as Itô processes, and show that the standard assumptions of positive state prices and existence of an equivalent martingale measure exclude prices which are viable models of competitive equilibrium and are potentially useful for modeling actual financial markets. They propose the concept of "free snacks" for admissible trading strategies. Other references are Stricker (1990), Ansel and Stricker (1992), Delbaen (1992), Schweizer (1992), Clark (1993), Schachermayer (1993), Lakner (1993) and Willard and Dybvig (1999).
In this paper we propose a new definition for tameness. We call it state tameness (see Definition 3.1). Loosely speaking, we call a portfolio $\pi(t)$ a state tame portfolio if the value of its gain process discounted by the so called "state price density process" is bounded below. For a definition of state price density process see equation (2.8). In financial terms, this definition for tameness accounts for constrains on an agent credit that are resettled at all times to reflect the changes in the state of the economy. Let us establish an analogy. In a Poker game, it is natural to assume that the players have credit constrains, depending on the ability of each of them to eventually cover losses. If we think of a particular game for which one player has exhausted his credit, but his stakes of winning are high, it is likely that someone would be willing to take over his risk. If the rules of the game allowed it, this could increase his ability to obtain credit.
We define state arbitrage, see Definition 3.2, and characterize it. As a consequence of Theorem 3.1, our definition of non-arbitrage is an extension of non-arbitrage in the context of standard financial markets. See Karatzas and Shreve (1998). Moreover, whenever equation (2.7) holds and the volatility matrix is invertible, the existence of an equivalent martingale measure implies the non existence of arbitrage opportunities that are state tame, but not conversely; see Remark 3.3. Our definition is weaker that the one proposed by Levental and Skorohod (1995) under the condition that equation (2.7) holds. See Levental and Skorohod (1995)[Theorem 1 and Corollary 1], and Loewenstein and Willard (2000) for the economic meaning of equation (2.7). Our definition of non-state arbitrage is weaker than the one proposed by Delbaen and Schachermayer (1995b); see Remark 3.4. Our definition admits the existence of "free snacks", see e.g., Remark 3.3 and Loewenstein and Willard (2000)[Corollary 2]. See also Loewenstein and Willard (2000)[Corollary 2] and Loewenstein and Willard (2000)[Example 5.3] for the economic viability of those portfolios.
Next, we try to show the usefulness of the concept introduced. This is done by proving two extensions of the second fundamental theorem of asset pricing and a theorem for valuation of contingent claims of the American type suitable for the current context.
The question of completeness is about the ability to replicate or access certain cash flows and not about how these cash flows are valued. Hence, the appropriate measure for formulating the question of completeness is the true statistical probability measure, and not some presumed to exist equivalent martingale probability measure. Jarrow and Madan (1999) elaborate further on this point. We propose a valuation technique that does not require the existence of an equivalent martingale measure and allows for pricing contingent claims, even when the range of the volatility matrix is not maximal. See Theorem 4.1. The standard approach relates the notion of market completeness to uniqueness of the equivalent martingale measure; see Harrison and Kreps (1979), Harrison and Pliska (1981), and Jarrow and Madan (1991). Delbaen (1992) extends the second fundamental theorem for asset prices with continuous sample paths for the case of infinitely many assets. Other extension are Jarrow and Madan (1999), Bättig (1999), and Bättig and Jarrow(1999). The recent paper Fernholz, Karatzas, and Kardaras

(2004) also extends valuation theory, when an equivalent martingale measures fails to exists; they are motivated by considerations of "diversity"; see Remark 4.1 for a discussion about the connections with this paper.

Last, we formulate an extension of the American contingent claim valuation theory. See Theorem 5.1. We provide a valuation technique of the contingent claims of the American type in a setting that does not require the full range of the volatility matrix. See Theorem 5.1 in conjunction with Theorem 4.1. Our approach is closer in spirit to a computational approach. See Karatzas (1988) and Bensoussan (1984) to review the formal theory of valuation of American contingent claims with unconstrained portfolios; see the survey paper by Myneni (1992) as well as Karatzas and Shreve (1998). Closed form solutions are typically not available for pricing American Options on finite-horizons. Although an extensive literature exist on their numerical computation; interested readers are referred to several survey papers and books such as Broadie and Detemple (1996), Boyle, Broadie, and Glasserman (1997), Carverhill and Webber (1990), Hull (1993), Wilmott, Dewynne, and Howison (1993) for a partial list of fairly recent numerical work on American Options and comparisons of efficiency.

2 The model

In what follows we try to follow as closely as possible the notation in Karatzas and Shreve (1998), and Karatzas(1996). For the sake of completeness we explicitly state all the hypotheses usually used for financial market models with a finite set of continuous assets defined on a Brownian filtration. We assume a d-dimensional Brownian Motion starting at 0 $\{W(t), \mathcal{F}_t; 0 \le t \le T\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where $\{\mathcal{F}_t\}_{0\le t\le T}$ is the $\mathbf{P}$ augmentation by the null sets in $\mathcal{F}^W_T$ of the natural filtration $\mathcal{F}^W_t = \sigma(W(s), 0 \le s \le t)$, $0 \le t \le T$, and $\mathcal{F} = \mathcal{F}_T$.

We assume a risk-free rate process $r(\cdot)$, a n-dimensional mean rate of return process $b(\cdot)$, a n-dimensional dividend rate process $\delta(\cdot)$, a $n \times d$ matrix valued volatility process $(\sigma_{i,j}(\cdot))$; we also assume that $b(t)$, $\delta(t)$, $r(t)$ and $(\sigma_{i,j}(t))$ are progressively measurable processes. Moreover it is assumed that

$$\int_0^T (|r(t)| + \|b(t)\| + \|\delta(t)\| + \sum_{i,j} \sigma^2_{ij}(t))\, dt < \infty$$

As usual we assume a bond price process $B(t)$ that evolves according to the equation

$$dB(t) = B(t)r(t)dt, \qquad B(0) = 1 \tag{2.1}$$

and n stocks whose evolution of the price-per-share process $P_i(t)$ for the i^{th} stock at time t, is given by the stochastic differential equation

$$dP_i(t) = P_i(t)\left[b_i(t)dt + \sum_{1\le j\le d} \sigma_{ij}(t)\, dW_j(t)\right], \quad P_i(0) = p_i \in (0, \infty)$$
$$i = 1, \cdots, n. \tag{2.2}$$

Let $\tau \in \mathcal{S}$ be a stopping time, where $\mathcal{S}$ denotes the set of stopping times $\tau \colon \Omega \mapsto [0, T]$ relative to the filtration $(\mathcal{F}_t)$. We shall say that a stochastic process $X(t)$, $t \in [0, \tau]$ is $(\mathcal{F}_t)$-adapted if $X(t \wedge \tau)$ is $(\mathcal{F}_t)$-adapted, where $s \wedge t = \min\{s, t\}$, for $s, t \in \mathbb{R}$. We consider a portfolio process

$(\pi_0(t), \pi(t))$, $t \in [0, \tau]$ to be a $(\mathcal{F}_t)$-progressively measurable $\mathbb{R} \times \mathbb{R}^n$ valued process, such that

$$\int_0^\tau |\sum_{0 \le i \le n} \pi_i(t)||r(t)|\,dt + \int_0^\tau |\pi\prime(t)(b(t) + \delta(t) - r(t)\mathbf{1}_n)|\,dt$$
$$+ \int_0^\tau \|\sigma\prime(t)\pi(t)\|^2\,dt < \infty \tag{2.3}$$

holds almost surely, with $\|x\| = (x_1^2 + \cdots + x_d^2)^{1/2}$ for $x \in \mathbb{R}^d$, and $\mathbf{1}'_n = (1, \cdots, 1) \in \mathbb{R}^n$. A $(\mathcal{F}_t)$-adapted process $\{C(t), 0 \le t \le \tau\}$ with increasing , right continuous paths, $C(0) = 0$, and $C(\tau) < \infty$ almost surely (a.s.) is called a *cumulative consumption process.* Following the standard literature (see e.g.: Karatzas and Shreve (1998), Karatzas(1996)) for a given $x \in \mathbb{R}$ and (π_0, π, C) as above, the process $X(t) \equiv X^{x,\pi,C}(t)$, $0 \le t \le \tau$ given by the equation

$$\gamma(t)X(t) = x - \int_{(0,t]} \gamma(s)\,dC(s)$$
$$+ \int_0^t \gamma(s)\pi'(s)\,[\sigma(s)\,dW(s) \ + (b(s) + \delta(s) - r(s)\mathbf{1}_n))\,ds] \tag{2.4}$$

where $\gamma(t)$ is defined as

$$\gamma(t) \triangleq \frac{1}{B(t)} = \exp\left(-\int_0^t r(s)\,ds\right), \tag{2.5}$$

is the *wealth process* associated with the initial capital x, portfolio π, and cumulative consumption process C.

Remark 2.1. *Let us observe that the condition defined by equation* (2.3) *is slightly different from the condition that defines a portfolio in the standard setting where the terminal time is not random. In fact, only the former condition is needed in order to obtain a well defined wealth process as defined by equation* (2.4).

We define a progressively measurable *market price of risk* process $\theta(t) = (\theta_1(t), \cdots, \theta_d(t))$ with values in $\mathbb{R}^d$ for $t \in [0, T]$ as the unique process $\theta(t) \in \ker^{\perp}(\sigma(t))$, the orthogonal complement of the kernel of $\sigma(t)$, such that

$$b(t) + \delta(t) - r(t)\mathbf{1}_n - proj_{\ker(\sigma'(t))}(b(t) + \delta(t) - r(t)\mathbf{1}_n) = \sigma(t)\theta(t) \qquad \text{a.s.} \tag{2.6}$$

(See Karatzas and Shreve (1998) for a proof that $\theta(\cdot)$ is progressively measurable.) Moreover, we assume that $\theta(\cdot)$ satisfies the mild condition

$$\int_0^T \|\theta(t)\|^2\,dt < \infty \qquad \text{a.s.} \tag{2.7}$$

We define a *state price density process* by

$$H_0(t) = \gamma(t)Z_0(t) \tag{2.8}$$

where

$$Z_0(t) = \exp\left\{-\int_0^t \theta'(s)\,dW(s) \ - \frac{1}{2}\int_0^t \|\theta(s)\|^2\,ds\right\}. \tag{2.9}$$

The name "state price density process" is usually given to the process defined by equation (2.8) when the market is a standard financial market; see Karatzas and Shreve (1998). In that case the process $Z_0(t)$ is a martingale and $Z_0(T)$ is indeed a *state price density.* However, in our setting we allow the possibility that $\mathbf{E}Z_0(T) < 1$.

3 State tameness and state arbitrage. Characterization

We propose the following definition for tameness.

Definition 3.1. Given a stopping time $\tau \in \mathcal{S}$, a self-financed portfolio process $(\pi_0(t), \pi(t))$, $t \in [0, \tau]$ is said to be *state-tame*, if the discounted gain process $H_0(t)G(t)$, $t \in [0, \tau]$ is bounded below, where $G(t) = G^\pi(t)$ is the gain process defined as

$$G(t) = \gamma^{-1}(t) \int_0^t \gamma(s)\pi'(s) \left[\sigma(s)\, dW(s) \ + (b(s) + \delta(s) - r(s)\mathbf{1}_n))\, ds\right]. \tag{3.1}$$

Definition 3.2. A self finance state-tame portfolio $\pi(t)$, $t \in [0, T]$ is said to be a *state arbitrage opportunity* if

$$\mathbf{P}\left[H_0(T)G(T) \geq 0\right] = 1, \qquad \text{and} \qquad \mathbf{P}\left[H_0(T)G(T) > 0\right] > 0 \tag{3.2}$$

where $G(t)$ is the gain process that corresponds to $\pi(t)$. We say that a market $\mathcal{M}$ is state-arbitrage-free if no such portfolios exist in it.

Theorem 3.1. *A market $\mathcal{M}$ is state-arbitrage-free if and only if the process $\theta(t)$ satisfies*

$$b(t) + \delta(t) - r(t)\mathbf{1} = \sigma(t)\theta(t) \qquad 0 \leq t \leq T \ \ a.s. \tag{3.3}$$

Remark 3.1. *We observe that if $\theta(t)$ satisfies equation* (3.3) *then for any initial capital x, and consumption process $C(t)$,*

$$\begin{aligned} &H_0(t)X(t) + \int_{(0,t]} H_0(s)\, dC(s) \\ &= x + \textstyle\int_0^t H_0(s) \left[\sigma'(s)\pi_(s) - X(s)\theta(s)\right]' dW(s) \quad . \end{aligned} \tag{3.4}$$

Proof of Theorem 3.1. First, we prove necessity. For $0 \leq t \leq T$ we define

$$\begin{aligned} p(t) &= proj_{\ker(\sigma'(t))}(b(t) + \delta(t) - r(t)\mathbf{1}_n) \\ \pi(t) &= \begin{cases} \|p(t)\|^{-1} p(t) & \text{if } p(t) \neq 0, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and define $\pi_0(t) = G(t) - \pi'(t)\mathbf{1}_n$ where $G(t)$ is the gain process defined by equation (2.4) with zero initial capital, and zero cumulative consumption process. It follows that $(\pi_0(t), \pi(t))$ is a self-financed portfolio with gain process

$$G(t) = \gamma^{-1}(t) \int_0^t \|p(s)\|\, \gamma(s)\mathbf{1}_{p(s)\neq 0}\, ds.$$

Since $H_0(t)G(t) \geq 0$, the non-state-arbitrage hypothesis implies the desired result. To prove sufficiency, assume that $\theta(t)$ satisfies equation (3.3), $\pi(t)$ is a self-financed portfolio and $G(t)$ is the gain process that corresponds to $\pi(t)$ as in Definition 3.1. Remark 3.1 implies that $H_0(t)G(t)$ is a local-martingale. By state-tameness it is also bounded below. Fatou's lemma implies that $H_0(t)G(t)$ is a super-martingale. The result follows. $\square$

Remark 3.2. *We can extend the definition of state arbitrage opportunity to state tame portfolios defined on a random time. It is worth to mentioning that Theorem 3.1 remains true even with this apparently stronger definition.*

Remark 3.3. *It is well known that absence of arbitrage opportunities on tame portfolios is implied by the existence of an equivalent martingale measure under which discounted prices (by the bond price process) plus discounted cumulative dividends become martingales; see e.g., Duffie (1996)[Chapter 6]. If the volatility matrix* $\sigma(\cdot)$ *is invertible and equation (2.7) holds, it is known that the non existence of arbitrage opportunities in tame portfolios is equivalent to* $\mathbf{E}Z_0(T) = 1.$*, see e.g., Levental and Skorohod (1995)[Corollary 1]. Our framework allows for the possibility that* $\mathbf{E}Z_0(T) < 1$*, as is the case of, for instance, Levental and Skorohod (1995)[Example 1]. Therefore, in the cited example, any arbitrage opportunity that is a tame portfolio, would not be a state tame portfolio.*

Remark 3.4. *It is known that the non existence of arbitrage opportunities in tame portfolios implies that equation (3.3) holds a.s. for Lebesgue-almost-every* $t \in [0, T]$*; see e.g. Karatzas and Shreve (1998)[Theorem 4.2]. At the same time, by Theorem 3.1, non existence of arbitrage opportunities in state-tame portfolios is equivalent to assuming that equation (3.3) holds a.s. for Lebesgue-almost-every* $t \in [0, T]$*. Under a more general setting, Delbaen and Schachermayer(1994) have proved that the existence of an equivalent martingale measure is equivalent to a property called "no free lunch with vanishing risk" (NFLVR). It is also known that the concept of NFLVR is stronger that the non existence of arbitrage opportunities in tame portfolios; see e.g., Delbaen and Schachermayer(1995b)[Theorem 1.3]. It follows that our definition of non-state-arbitrage is weaker that NFLVR.*

4 State European Contingent Claims. Valuation

Throughout the rest of the paper we assume that equation (3.3) is satisfied.
A $(\mathcal{F}_t)$-progressively measurable semi-martingale $\Gamma(t), 0 \leq t \leq \tau$, where $\tau \in \mathcal{S}$ is a stopping time is called a *cumulative income process for the random time interval* $(0, \tau]$. Let $X(t)$ defined by

$$\gamma(t)X(t) = x + \int_{(0,t]} \gamma(s)\, d\Gamma(s) + \int_0^t \gamma(s)\pi'(s)\left[\sigma(s)\, dW(s) \; + (b(s) + \delta(s) - r(s)\mathbf{1}_n))\, ds\right], \tag{4.1}$$

where $\pi(t)$, $t \in [0, \tau]$, is a $\mathbb{R}^n$ valued $(\mathcal{F}_t)$-progressively measurable process such that

$$\int_0^\tau \left(|\pi'(t)(b(t) + \delta(t) - r(t)\mathbf{1}_n)| + \|\sigma'(t)\pi(t)\|^2\right)\, dt < \infty.$$

It follows that $X(t)$ defines a wealth associated with the initial capital x and cumulative income process $\Gamma(t)$. Namely, if $\pi_0(t) = X(t) - \pi'(t)\mathbf{1}_n$, (π_0, π) defines a portfolio process whose wealth process is $X(t)$ and cumulative income process is $\Gamma(t)$. Moreover, it follows that

$$H_0(t)X(t) - \int_{(0,t]} H_0(s)\, d\Gamma(s) = x + \int_0^t H_0(s)\left[\sigma'(s)\pi_(s) - X(s)\theta(s)\right]'\, dW(s) \quad . \tag{4.2}$$

We say that the portfolio is *state Γ-tame* if the process $H_0(t)X(t)$ is (uniformly) bounded below.
We propose to extend the concepts of European contingent claim, financiability and completeness. Let $Y(t)$ $t \in [0,\tau]$ be a cumulative income process with $Y(0)=0$. Assume that Y has a decomposition $Y(t) = Y_{loc}(t) + Y_{fv}(t)$, as a sum of a local martingale and a process of finite variation. Let $Y_{fv}(t) = Y_{fv}^+(t) - Y_{fv}^-(t)$ be the representation of $Y_{fv}(t)$ as the difference of two non decreasing RCLL progressively measurable processes with $Y_{fv}^+(0) = Y_{fv}^-(0) = 0$, where $Y_{fv}^+(t)$ and $Y_{fv}^-(t)$ are the positive and negative variation of $Y_{fv}(t)$ in the interval $[0,t]$ respectively. We denote by $|Y_{fv}|(t) = Y_{fv}^+ + Y_{fv}^-(t)$ the total variation of $Y_{fv}(t)$ on the interval $[0,t]$. We also denote Y^- the process defined as $Y^-(t) = Y_{loc}(t) - Y_{fv}^-(t)$.

Definition 4.1. Given a stopping time $\tau \in \mathcal{S}$, we shall call *state European contingent claim (SECC) with expiration date* τ any progressively measurable semi-martingale $Y(t)$, $t \in [0,\tau]$, with $Y(0)=0$, such that $-\int_0^\tau H_0(t)\,dY_{fv}^-(t)$ is bounded below and

$$\mathbf{E}\left[\int_0^\tau H_0^2(t)\,d\langle Y\rangle(t)\right] + \mathbf{E}\left[\int_0^\tau H_0(t)\,d\,|Y_{fv}|\,(t)\right] < \infty. \tag{4.3}$$

Here $\langle Y\rangle(t)$ stands for the quadratic variation process of the semi-martingale $Y(t)$. We define u_e by the formula

$$u_e = \mathbf{E}\int_0^\tau H_0(t)\,dY. \tag{4.4}$$

Definition 4.2. A state European contingent claim $Y(t)$ with expiration date τ is called *attainable* if there exist a state $(-Y)$-tame portfolio process $\pi(t)$, $t \in [0,\tau]$ with

$$X^{u_e,\pi,-Y}(\tau^-) = Y(\tau), \qquad \text{a.s.} \tag{4.5}$$

The market model $\mathcal{M}$ is called *state complete* if every state European contingent claim is attainable. Otherwise it is called *state incomplete.*

For the following theorem we assume $\{i_1 < \cdots < i_k\} \subseteq \{1,\cdots,d\}$ is a set of indexes and let $\{i_{k+1} < \cdots < i_d\} \subseteq \{1,\cdots,d\}$ be its complement. Let $\sigma_i(t)$, $1 \le i \le k$, be the i^{th} column process for the matrix valued process $(\sigma_{i,j}(t)), 0 \le t \le T$. Namely, $\sigma_i(t)$, $1 \le i \le k$, is the $\mathbb{R}^n$-valued progressively measurable process whose j^{th}, $1 \le j \le d$ entry agrees with $\sigma_{i,j}(t)$, for $0 \le t \le T$. We denote by $\sigma_{i_1,\cdots,i_k}(t)$, $0 \le t \le T$ the $n \times k$ matrix valued process whose j^{th} column process agrees with $\sigma_{i_j}(t)$, $0 \le t \le T$ for $1 \le j \le k$. We shall denote as $\{\mathcal{F}_t^{i_1,\cdots,i_k}, 0 \le t \le T\}$ the $\mathbf{P}$ augmentation by the null sets of the natural filtration $\{\sigma(W_{i_1}(s),\cdots,W_{i_k}(s), 0 \le s \le t), 0 \le t \le T\}$.

Theorem 4.1. *Assume that $\theta_i(t) = 0$ for $i \notin \{i_1,\cdots,i_k\}$, where $\theta(t) = (\theta_1(t),\cdots,\theta_d(t))$ is the market price of risk. Assume that $\sigma_{i_1,\cdots,i_k}(t)$ is a $\mathcal{F}_t^{i_1,\cdots,i_k}$-progressively measurable matrix valued process such that Range$(\sigma_{i_{k+1},\cdots,i_d}(t))$ = Range$^\perp(\sigma_{i_1,\cdots,i_k}(t))$ almost surely for Lebesgue-almost-every t. In addition assume that the interest rate process γ is $\mathcal{F}_t^{i_1,\cdots,i_k}$-progressively measurable. Then, any $\mathcal{F}_t^{i_1\cdots,i_k}$-progressively measurable state European contingent claim is attainable if and only if Rank$(\sigma_{i_1,\cdots,i_k}(t)) = k$ a.s. for Lebesgue-almost-every t. In particular, a financial market $\mathcal{M}$ is state complete if and only if $\sigma(t)$ has maximal range a.s. for Lebesgue-almost-every t, $0 \le t \le T$.*

Proof of sufficiency. Let $Y(t)$, $t \in [0,\tau]$, be a $\mathcal{F}^{i_1,\cdots,i_k}(t)$-progressively measurable SECC with $\tau \in \mathcal{S}$. Define

$$X(t) = H_0^{-1}(t)\mathbf{E}\left[\int_{(t,\tau]} H_0(s)\,dY(s) \mid \mathcal{F}^{i_1,\cdots,i_k}(t)\right] \qquad \text{for } t \in [0,\tau]. \tag{4.6}$$

From the representation of Brownian martingales as stochastic integrals it follows that there exist a progressively measurable $\mathbb{R}^d$-valued process $\varphi'(t) = (\varphi_1(t),\cdots,\varphi_d(t))$, $t \in [0,\tau]$, such that

$$H_0(t)X(t) + \int_{(0,t]} H_0(s)\,dY(s) \;= u_e + \int_0^t \varphi'(s)\,dW(s) \tag{4.7}$$

where $\varphi_i(t) = 0$ for $i \notin \{i_1,\cdots,i_k\}$. Define $\pi_e(t)$, $t \in [0,\tau]$, as the unique $\mathbb{R}^n$-valued progressively measurable process such that

$$\sigma'(t)\pi_e(t) = H_0^{-1}(t)\varphi(t) + X(t)\theta(t). \tag{4.8}$$

The existence and uniqueness of such a portfolio follows from the hypotheses (see Lemma 1.4.7 in Karatzas and Shreve (1998)). Define $(\pi_e)_0(t) = X(t) - \pi(t)'\mathbf{1}_n$. It follows using Itô's formula that $X(t)$ defines a wealth process with cumulative income process $-Y(t)$, with the desired characteristics. (To prove the state $-Y(t)$ tameness of the portfolio $\pi_e(t)$, let u_e^- be the constant defined by the equation (4.4) corresponding to the SECC $Y^-(t)$. Let $X^-(t)$, $\varphi^-(t)$, and $\pi_e^-(t)$ be the processes defined by equations (4.6), (4.7), (4.8) respectively corresponding to the SECC $Y^-(t)$; it follows that $X(t) \geq X^-(t)$, $0 \leq t \leq \tau$. The $-Y(t)$ tameness of $\pi_e(t)$ is implied by the $-Y^-(t)$ tameness of $\pi_e^-(t)$. The latter follows by the definition of SECC.) ☐

Proof of necessity. Let us assume that any $\mathcal{F}_t^{i_1,\cdots,i_k}$-progressively measurable SECC is attainable. Let $f : L(\mathbb{R}^k;\mathbb{R}^n) \mapsto \mathbb{R}^k$ be a bounded measurable function such that: $f(\sigma) \in Kernel(\sigma)$ and $f(\sigma) \neq \mathbf{0}$ if $Kernel(\sigma) \neq \{\mathbf{0}\}$, hold for every $\sigma \in L(\mathbb{R}^k;\mathbb{R}^n)$. (See Karatzas (1996), p. 9). Let us define $\psi(t)$ to be the bounded, $\mathcal{F}_t^{i_1,\cdots,i_k}$-progressively measurable process such that $\psi_{i_1,\cdots,i_k} = f(\sigma_{i_1,\cdots,i_k}(t))$ and $\psi_j(t) = 0$ for $j \notin \{i_1,\cdots,i_k\}$. We define the $\mathcal{F}^{i_1,\cdots,i_k}$-progressively measurable SECC by

$$Y(t) = \int_0^t \frac{1}{H_0(s)}\psi'(s)\,dW(s) \qquad \text{for } 0 \leq t \leq \tau. \tag{4.9}$$

Let π_e be the $-Y$ state tame portfolio with wealth process $X^{u_e,\pi_e,-Y}$ as in equation (4.5) and u_e defined by equation (4.4). It follows that

$$H_0(t)X^{u_e,\pi_e,-Y}(t) + \int_{(0,t]} H_0(s)\,dY(s) = u_e + \int_0^t \psi'(s)\,dW(s) \tag{4.10}$$

is a martingale. Using equation (4.2), and the representation of Brownian martingales as stochastic integrals we obtain

$$\begin{aligned}\psi_{i_1,\cdots,i_k}(t) &= \sigma'_{i_1,\cdots,i_k}(t)\pi_e(t) - X(t)\theta_{i_1,\cdots,i_k}(t)\\ &\in Kernel^{\perp}(\sigma_{i_1,\cdots,i_k}(t) \cap Kernel(\sigma_{i_1,\cdots,i_k}(t)) = \{\mathbf{0}\}\end{aligned} \tag{4.11}$$

a.s. for Lebesgue-almost-every t, $0 \leq t \leq \tau$. The result follows. ☐

Remark 4.1. *Fernholz, Karatzas, and Kardaras (2004) are able to hedge contingent claims of European type when a martingale measure fails to exists. The framework of their paper is the same as ours, namely, the model of security prices as Itô processes. In addition they assume that the eigenvalues of the stochastic $n \times n$-matrix of variation-covariation rate processes $\sigma(t)\sigma'(t), t \in [0,T]$ are uniformly bounded away from zero. This latter condition implies that equation (3.3) holds; as a consequence their results on valuation are implied by Theorem 4.1.*

5 State American Contingent Claims. Valuation.

Definition 5.1. Let $(\Gamma(t), L(t))$, $0 \leq t \leq \tau$, a couple of RCLL progressively measurable semi-martingales where $\Gamma(t)$, $t \in [0,\tau]$, is a cumulative income process with $\Gamma(0) = 0$. Assume that the process

$$Y(t) = \int_{(0,t]} H_0(s)\, d\Gamma(s) + L(t)H_0(t) \qquad \text{for } 0 \leq t \leq \tau, \tag{5.1}$$

is a continuous semi-martingale such that Y and $L(t)H_0(t)$, $0 \leq t \leq \tau$, are uniformly bounded below. We shall call a *state American contingent claim (SACC)* a couple of processes as above such that

$$u_a = \sup_{\tau' \in \mathcal{S}(\tau)} \mathbf{E}[Y(\tau')] < \infty, \tag{5.2}$$

where $\mathcal{S}(\tau) = \{\tau' \in \mathcal{S}; \tau' \leq \tau\}$. We shall call the process $Y(t)$ the *discounted payoff process*, $L(t)$ the *lump-sum settlement process* and u_a the *value of the state American contingent claim.*

Theorem 5.1. *Let $\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, d\}$ be a set of indexes. Assume the hypotheses of theorem 4.1. If $(\Gamma(t), L(t))$ is a state American contingent claim where the discounted payoff process is $\mathcal{F}_t^{i_1,\cdots,i_k}$-progressively measurable then there exist a $-\Gamma(t)$ state tame portfolio π_a such that*

$$X^{u_a,\pi_a,-\Gamma}(t) \geq L(t) \qquad a.s. \text{ for } 0 \leq t \leq \tau. \tag{5.3}$$

Indeed,

$$\begin{aligned} u_a = \inf\{u \in \mathbb{R} \mid \quad & \text{there exist a } -\Gamma(t) \text{ state tame portfolio} \\ & \pi \text{ with } X^{u,\pi,-\Gamma}(t) \geq L(t) \text{ a.s. for } 0 \leq t \leq \tau\}. \end{aligned} \tag{5.4}$$

Lemma 1. *Given $\tau_1, \tau_2 \in \mathcal{S}(\tau)$, there exist $\tau' \in \mathcal{S}(\tau)$ with*

$$u_a \geq \mathbf{E}[Y(\tau')] \geq \max\{\mathbf{E}[Y(\tau_1)], \mathbf{E}[Y(\tau_2)]\}$$

such that

$$\mathbf{E}[Y(\tau') \mid \mathcal{F}_t] \geq \max\{\mathbf{E}[Y(\tau_1) \mid \mathcal{F}_t], \mathbf{E}[Y(\tau_2) \mid \mathcal{F}_t]\} \text{ for all } t \in [0,\tau].$$

Proof. Define

$$\begin{aligned} \tau' = {} & \tau_1 \wedge \tau_2 \mathbf{1}_{\mathbf{E}[Y(\tau_1 \vee \tau_2)|\mathcal{F}_t](\tau_1\wedge\tau_2) < Y(\tau_1\wedge\tau_2)} \\ & + \tau_1 \vee \tau_2 \mathbf{1}_{\mathbf{E}[Y(\tau_1 \vee \tau_2)|\mathcal{F}_t](\tau_1\wedge\tau_2) \geq Y(\tau_1\wedge\tau_2)} \end{aligned}$$

where $s \vee t = \max\{s,t\}$, and $s \wedge t = \min\{s,t\}$. Then τ' has the required properties. $\square$

Proof of Theorem 5.1 . Let $Y(t)$, $0 \leq t \leq \tau$, be the discounted payoff process. There exist a sequence of stopping times (σ_n) in $\mathcal{S}(\tau)$ such that $\mathbf{E}[Y(\sigma_n)] \uparrow u_a$, $\mathbf{E}[Y(\sigma_{n+1}) \mid \mathcal{F}_t] \geq \mathbf{E}[Y(\sigma_n) \mid \mathcal{F}_t]$ for $t \in [0,\tau]$, with the property that for any rational $q \in \mathbb{Q} \cap [0,T]$, there exist $N_q \in \mathbb{N}$ such that $\mathbf{E}[Y(\sigma_n) \mid \mathcal{F}_t](q \wedge \tau) \geq Y(q \wedge \tau)$. The latter follows by lemma 1. By Doob's inequality, $\mathbf{E}[Y(\sigma_n) \mid \mathcal{F}_t]$ is a Cauchy sequence in the sense of uniform convergence in probability. By completeness of the space of local-martingales, there exist a local-martingale $\overline{Y}(t)$, $t \in [0,\tau]$, such that $\mathbf{E}[Y(\sigma_n) \mid \mathcal{F}_t] \to \overline{Y}(t)$, $t \in [0,\tau]$, uniformly in probability. It follows by continuity that $\overline{Y}(t) \geq Y(t)$ for $t \in [0,\tau]$, and clearly $\overline{Y}(0) = u_a$. Define τ_n to be the first hitting time of $\overline{Y}(t)$, $t \in [0,\tau]$, to the set $[-n,n]^c$. From the representation of Brownian martingales as stochastic integrals it follows that there exist a progressively measurable $\mathbb{R}^d$-valued process $\varphi'(t) = (\varphi_1(t), \cdots, \varphi_d(t))$, $t \in [0,\tau_n]$, such that

$$\overline{Y}(t) = u_a + \int_0^t \varphi'(s)\, dW(s) \tag{5.5}$$

where $\varphi_i(t) = 0$ for $i \notin \{i_1, \cdots, i_k\}$. Define $X(t)$, $t \in [0,\tau]$, by

$$H_0(t)X(t) + \int_{(0,t]} H_0(s)\, d\Gamma(s) \ = \overline{Y}(t).$$

Define $\pi_a(t)$, $t \in [0,\tau]$, as the unique $\mathbb{R}^n$-valued progressively measurable process such that

$$\sigma'(t)\pi_a(t) = H_0^{-1}(t)\varphi(t) + X(t)\theta(t).$$

The existence and uniqueness of such a portfolio follows by the hypotheses (see Lemma 1.4.7 in Karatzas and Shreve (1998)). Define $(\pi_a)_0(t) = X(t) - \pi_a(t)'\mathbf{1}_n$. It follows using Itô's formula that $X(t)$ defines a wealth process with cumulative income process $-\Gamma(t)$, $t \in [0,\tau]$, with the desired characteristics. Equation (5.4) is a consequence to the fact that the discounted payoff process is a super-martingale. □

Remark 5.1. *Let us observe that it is not possible to obtain optimal stopping times for the version of the theorem for valuation of American contingent claims that we presented. Nonetheless, it is worth to point out that the conditions of the Theorem 5.1, are probably the weakest possible.*

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