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THE SPECTRAL LAWS OF HERMITIAN BLOCK-MATRICES WITH LARGE RANDOM BLOCKS

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Abstract

We are going to study the limiting spectral measure of fixed dimensional Hermitian block-matrices with large dimensional Wigner blocks. We are going also to identify the limiting spectral measure when the Hermitian block-structure is Circulant. Using the limiting spectral measure of a Hermitian Circulant block-matrix we will show that the spectral measure of a Wigner matrix with $k-$weakly dependent entries need not to be the semicircle law in the limit.

1 Preliminaries and main results

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all $n \times n$ matrices with complex-valued entries. Define the normalized trace of a matrix $\mathbf{A} = (A_{ij})_{i,j=1}^{n} \in \mathcal{M}_n(\mathbb{C})$ to be $\mathrm{tr}_n(\mathbf{A}) := \frac{1}{n}\sum_{i=1}^{n} A_{ii}$.

Definition 1. The spectral measure of a Hermitian $n \times n$ matrix $\mathbf{A}$ is the probability measure $\mu_{\mathbf{A}}$ given by

$$\mu_{\mathbf{A}} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j}$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $\mathbf{A}$ and δ_x is the point mass at x.

The weak limit of the spectral measures $\mu_{\mathbf{A}_n}$ of a sequence of matrices $\{\mathbf{A}_n\}$ is called the limiting spectral measure. We will denote the weak convergence of a probability measure μ_n to μ by

$$\mu_n \xrightarrow{\mathcal{D}} \mu \text{ as } n \to \infty.$$

Definition 2. A finite symmetric block-structure $\mathbb{B}_k(a, b, c, \dots)$ (or shortly $\mathbb{B}_k$) over a finite alphabet $\mathcal{K} = \{a, b, c, \dots\}$ is a $k \times k$ symmetric matrix whose entries are elements in $\mathcal{K}$.

If $\mathbb{B}_k$ is a $k \times k$ symmetric block-structure and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ are $n \times n$ Hermitian matrices, then $\mathbb{B}_k(\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots)$ is an $nk \times nk$ Hermitian matrix. One of the interesting block structures is the $k \times k$ symmetric Circulant over $\{a_1, a_2, \dots, a_k\}$ that is defined as

$$\mathbb{C}_k(a_1, a_2, \dots, a_k) = \frac{1}{\sqrt{k}} \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_k \\ a_k & a_1 & a_2 & \dots & a_{k-1} \\ a_{k-1} & a_k & a_1 & \dots & a_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{bmatrix} \tag{1}$$

where $a_j = a_{k-j+2}$ for $j = 2, 3, \dots, k$.
A random matrix $\mathbf{A}$ is a matrix whose entries are random variables. If $\mathbb{B}_k$ is a block-structure and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ are random matrices, then $\mathbb{B}_k(\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots)$ is a random block-matrix.

Definition 3. We call an $n \times n$ Hermitian random matrix $\mathbf{A} = \frac{1}{\sqrt{n}}(X_{ij})_{i,j=1}^n$ a Wigner matrix if $\{X_{ij}; 1 \le i < j\}$ is a family of independent and identically distributed complex random variables such that $E(X_{12}) = 0$ and $E(|X_{12}|^2) = \sigma^2$. In addition, $\{X_{ii}; i \ge 1\}$ is a family of independent and identically distributed real random variables that is independent of the upper-diagonal entries. We will denote all such Wigner matrices of order n by $\mathbf{Wigner}(n, \sigma^2)$.

If $\{\mathbf{A}_n\}$ is a sequence of $\mathbf{Wigner}(n, \sigma^2)$ matrices, then by Wigner's Theorem (*cf.* [2]),

$$\mu_{\mathbf{A}_n} \xrightarrow{\mathcal{D}} \gamma_{0,\sigma^2} \quad \text{as } n \to \infty \qquad a.s.$$

where γ_{α,σ^2} is the semicircle law centered at α and of variance σ^2 which is given as

$$\gamma_{\alpha,\sigma^2}(dx) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - (x-\alpha)^2}\ \mathbf{1}_{[\alpha-2\sigma, \alpha+2\sigma]}(x)dx.$$

Now we are ready to state the main result of this paper.

Theorem 1 (**Existence Theorem**). *Consider a family of independent* $\mathbf{Wigner}(n, 1)$ *matrices* $\left(\{\mathbf{A}_n^{(i)}\} : i = 1, \dots, h\right)$ *for which* $E(|A_{12}^{(i)}|^4) < \infty$ *and* $E(A_{11}^{(i)})^2 < \infty$ *for every* i. *For a fixed* $k \times k$ *symmetric block-structure* $\mathbb{B}_k$, *define*

$$\mathbf{X}_{n,k} := \mathbb{B}_k(\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \dots, \mathbf{A}_n^{(h)}).$$

Then there exists a unique non-random symmetric probability measure $\mu_{\mathbb{B}_k}$ *with a compact support in* $\mathbb{R}$ *which depends only on the block-structure* $\mathbb{B}_k$ *such that*

$$\mu_{\mathbf{X}_{n,k}} \xrightarrow{\mathcal{D}} \mu_{\mathbb{B}_k} \quad \textit{as } n \to \infty \qquad a.s.$$

In [5], Far *et al.* introduced a method to find the limiting spectral measure of random block-matrices with Gaussian blocks and showed how that is applicable to wireless communications. Since our Theorem 1 implies that the law $\mu_{\mathbb{B}_k}$ does not depend on the distribution of the entries of the blocks $\mathbf{A}_n^{(i)}$, the results in [5] have wider applicability than stated there. In particular, they hold for matrices with real Gaussian or non-Gaussian entries. The proof of Theorem 1 relies on free probability theory and will be given in Section 2.2.

Consider the symmetric Circulant block-matrix $\mathbb{C}_k$ defined in (1). If $\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \ldots, \mathbf{A}_n^{(\lfloor \frac{k}{2} \rfloor + 1)}$ are independent $\mathbf{Wigner}(n,1)$ for every n, then Theorem 1 insures the existence of a non-random probability measure ν_k such that

$$\mu_{\mathbb{C}_k(\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \ldots, \mathbf{A}_n^{(\lfloor \frac{k}{2} \rfloor + 1)})} \xrightarrow{\mathcal{D}} \nu_k \text{ as } n \to \infty \qquad a.s.$$

However, Theorem 1 doesn't specify ν_k but we will identify it in the following proposition.

Proposition 1. *If* $\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \ldots, \mathbf{A}_n^{(\lfloor \frac{k}{2} \rfloor + 1)}$ *are independent* $\mathbf{Wigner}(n,1)$ *for every* n*, then*

$$\mu_{\mathbb{C}_k(\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \ldots, \mathbf{A}_n^{(\lfloor \frac{k}{2} \rfloor + 1)})} \xrightarrow{\mathcal{D}} \nu_k \textit{ as } n \to \infty \qquad a.s.$$

where

$$\nu_k = \begin{cases} \frac{k-1}{k}\, \gamma_{0, \frac{k-1}{k}} + \frac{1}{k}\, \gamma_{0, \frac{2k-1}{k}}, & \textit{if } k \textit{ is odd;} \\ \\ \frac{k-2}{k}\, \gamma_{0, \frac{k-2}{k}} + \frac{2}{k}\, \gamma_{0, \frac{2k-2}{k}}, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. Since $\mathbf{A}_n^{(j)} = \mathbf{A}_n^{(k-j+2)}$ for $j = 2, 3, \ldots, k$; then [4, Theorem 3.2.2.] implies that $\mathbb{C}_k(\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \ldots, \mathbf{A}_n^{(\lfloor \frac{k}{2} \rfloor + 1)})$ has the same eigenvalues as $\{\mathbf{B}_n^{(j)};\ j = 1, \ldots, k\}$ where

$$\mathbf{B}_n^{(j)} := \frac{1}{\sqrt{k}} [\mathbf{A}_n^{(1)} + 2 \sum_{\ell=2}^{(k+1)/2} \cos(\frac{2\pi(\ell-1)(j-1)}{k}) \mathbf{A}_n^{(\ell)}] \tag{2}$$

if k is odd, and

$$\mathbf{B}_n^{(j)} := \frac{1}{\sqrt{k}} [\mathbf{A}_n^{(1)} + 2 \sum_{\ell=2}^{k/2} \cos(\frac{2\pi(\ell-1)(j-1)}{k}) \mathbf{A}_n^{(\ell)} + \cos((j-1)\pi) \mathbf{A}_n^{(\frac{k}{2}+1)}] \tag{3}$$

if k is even. Hence,

$$\mu_{\mathbb{C}_k(\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \ldots, \mathbf{A}_n^{(\lfloor \frac{k}{2} \rfloor + 1)})} = \frac{1}{k} \sum_{j=1}^{k} \mu_{\mathbf{B}_n^{(j)}}.$$

Using the well known trigonometric sum $\sum_{\ell=0}^{N} \cos(\ell x) = \frac{1}{2}(\frac{\sin((N+\frac{1}{2})x)}{\sin\frac{x}{2}} + 1)$, one can check that

$$\sum_{\ell=0}^{N} \cos^2(\ell x) = \frac{1}{2}(N + \frac{3}{2} + \frac{\sin((2N+1)x)}{\sin x}). \tag{4}$$

Consider the case when k is odd. In Equation (2), for $j \neq 1$, $\mathbf{B}_n^{(j)}$ is a $\mathbf{Wigner}(n, \frac{k-1}{k})$ where the variance of the off-diagonal entries of $\mathbf{B}_n^{(j)}$ is given by $\frac{1}{k}[1 + 4\sum_{\ell=2}^{(k+1)/2} \cos^2(\frac{2\pi(\ell-1)(j-1)}{k})]$ which turns out to be $\frac{k-1}{k}$ by Equation (4). For $j = 1$, $\mathbf{B}_n^{(1)}$ is simply a $\mathbf{Wigner}(n, \frac{2k-1}{k})$. Hence, Wigner's theorem for $\mathbf{B}_n^{(1)}$ and the rest $k-1$ Wigner matrices $\mathbf{B}_n^{(j)}; j = 2, \ldots, k$ finishes the proof of the odd case.
The case when k is even follows from a similar argument by showing that for $j = 1, \frac{k}{2}+1$; $\mathbf{B}_n^{(j)}$ is a $\mathbf{Wigner}(n, \frac{2k-2}{k})$ and for $j \neq 1, \frac{k}{2}+1$; $\mathbf{B}_n^{(j)}$ is a $\mathbf{Wigner}(n, \frac{k-2}{k})$.

$\square$

In [2, p.626], Bai raised the question of whether Wigner's theorem is still holding true when the independence condition in the Wigner matrix is weakened. Schenker and Schulz-Baldes [11] provided an affirmative answer under some dependency assumptions in which the number of correlated entries doesn't grow too fast and the number of dependent rows is finite. After the first draft of the underlying paper was completed, we have learnt that Anderson and Zeitouni [1] showed that it doesn't hold in general and they gave an example in which the limiting spectral distribution is the free multiplicative convolution of the semicircle law and shifted arcsine law. In the rest of this section, we are going to use the following corollary of Proposition 1 to give another example.
Let $\mathbb{W}(a_{11}, a_{12}, \ldots, a_{nn})$ be the Wigner symmetric block-structure, *i.e.*,

$$\mathbb{W}(a_{11}, a_{12}, \ldots, a_{nn}) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{12} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{bmatrix}.$$

Consider the family of $k \times k$ random matrices $\{\mathbf{A}_{ij} : i, j \geq 1\}$ such that $\mathbf{A}_{ij} = \mathbf{A}_{ji}$ and $\mathbf{A}_{ij} = \mathbb{C}_k(a_{ij}, b_{ij}, c_{ij}, \ldots)$ where $\{a_{ij}, b_{ij}, c_{ij}, \ldots : i, j \geq 1\}$ are independent and identically distributed random variables with variance one. Then $\mathbb{K}_{n,k} := \mathbb{W}(\mathbf{A}_{11}, \mathbf{A}_{12}, \ldots, \mathbf{A}_{nn})$ is an $kn \times kn$ symmetric matrix.

Corollary 1. *Fix $k \in \mathbb{N}$. The limiting spectral measure of $\mathbb{K}_{n,k}$ is given by*

$$\mu_{\mathbb{K}_{n,k}} \xrightarrow{\mathcal{D}} \nu_k \text{ as } n \to \infty \qquad a.s.$$

In order to prove this corollary we need the following definitions. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times m$ and $k \times \ell$ matrices, respectively. By $\otimes$ we mean here the Kronecker product for which $\mathbf{A} \otimes \mathbf{B} = (A_{ij}\mathbf{B})_{i=1,\ldots,n;j=1,\ldots,m}$ is an $nk \times m\ell$ matrix. The (p, q)-commutation matrix $\mathbf{P}_{p,q}$ is a $pq \times pq$ matrix defined as

$$\mathbf{P}_{p,q} = \sum_{i=1}^{p} \sum_{j=1}^{q} \mathbf{E}_{ij} \otimes \mathbf{E}_{ij}^T$$

where $\mathbf{E}_{ij}$ is the $p \times q$ matrix whose entries are zero's except the $(i, j)-$entry is 1. It is known that $\mathbf{P}_{p,q}^{-1} = \mathbf{P}_{p,q}^T = \mathbf{P}_{q,p}$ and $\mathbf{P}_{n,k}(\mathbf{A} \otimes \mathbf{B})\mathbf{P}_{\ell,m} = \mathbf{B} \otimes \mathbf{A}$ (*cf.* [7]).

Proof. Since $\mathbb{K}_{n,k} = \sum_{i,j=1}^{n} \widetilde{\mathbf{E}}_{ij} \otimes \mathbf{A}_{ij}$ where $\widetilde{\mathbf{E}}_{ij}$ is the $n \times n$ matrix whose entries are zero's except the $(i, j)-$entry is 1. Hence

$$\begin{aligned} \mathbf{P}_{k,n}\mathbb{K}_{n,k}\mathbf{P}_{n,k} &= \textstyle\sum_{i,j=1}^{n} \mathbf{A}_{ij} \otimes \widetilde{\mathbf{E}}_{ij} \\ &= \textstyle\sum_{i,j=1}^{n} \mathbb{C}_k(a_{ij}, b_{ij}, c_{ij}, \ldots) \otimes \widetilde{\mathbf{E}}_{ij} \\ &= \textstyle\sum_{i,j=1}^{n} \mathbb{C}_k(a_{ij}\widetilde{\mathbf{E}}_{ij}, b_{ij}\widetilde{\mathbf{E}}_{ij}, c_{ij}\widetilde{\mathbf{E}}_{ij}, \ldots) \\ &= \mathbb{C}_k(\textstyle\sum_{i,j=1}^{n} a_{ij}\widetilde{\mathbf{E}}_{ij}, \sum_{i,j=1}^{n} b_{ij}\widetilde{\mathbf{E}}_{ij}, \sum_{i,j=1}^{n} c_{ij}\widetilde{\mathbf{E}}_{ij}, \ldots) \\ &= \mathbb{C}_k(\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \ldots) \end{aligned}$$

where $\mathbf{A}_n = (a_{ij})_{i,j=1}^n$, $\mathbf{B}_n = (b_{ij})_{i,j=1}^n$, $\mathbf{C}_n = (c_{ij})_{i,j=1}^n$, ... are independent $\mathbf{Wigner}(n, 1)$ matrices. Therefore, $\mathbb{K}_{n,k}$ and $\mathbb{C}_k(\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \ldots)$ are similar to each other and so have the same eigenvalues. Thus the result follows. ∎

Now, we define the distance on $\mathbb{N}^2$ by $d\left((i,j),(i',j')\right) = max\{|i-i'|,|j-j'|\}$ and for $S,T \subset \mathbb{N}^2$; $d\left(S,T\right) = min\{d\left((i,j),(i',j')\right) : (i,j) \in S,\ (i',j') \in T\}$. We say the random field $\{X_{ij} : (i,j) \in \mathbb{N}^2_{\leq}\}$ is $(k-1)$-dependent if the σ-fields $\mathcal{F}_S = \sigma(\{X_{ij} : (i,j) \in S\})$ and $\mathcal{F}_T = \sigma(\{X_{ij} : (i,j) \in T\})$ are independent for all $S,T \subset \mathbb{N}^2_{\leq}$ such that $d\left(S,T\right) > k-1$.
The matrix $\mathbb{K}_{n,k} = \mathbb{W}(\mathbf{A}_{11}, \mathbf{A}_{12}, \ldots, \mathbf{A}_{nn})$, defined in Corollary 1, is an $kn \times kn$ matrix with $(k-1)$-dependent entries, up to symmetry. That is, if we write $\mathbb{K}_{n,k} = (X_{ij})_{i,j=1}^{nk}$, then $\{X_{ij} : (i,j) \in \mathbb{N}^2_{\leq}\}$ is a $(k-1)$-dependent random field. However, the limiting spectral measure of $\mathbb{K}_{n,k}$ is not the semicircle law but rather a mixture of two semicircle laws due to Corollary 1.
Our example violates the conditions imposed on the Wigner matrix by Schenker and Schulz-Baldes in [11] in both the number of correlated entries and the number of dependent rows grow as $O(n^2)$ and not $o(n^2)$.
Unfortunately, $\{X_{ij} : (i,j) \in \mathbb{N}^2_{\leq}\}$, in our example, is not strictly stationary as the distributions remain the same only when shifts are made by multiple of k.

2 Proofs

In order to prove Theorem 1 we need to introduce some definitions from free probability theory.
A noncommutative probability space $(\mathcal{A}, \tau)$ is a pair of a unital algebra $\mathcal{A}$ with a unit element $\mathbb{I}$ and a linear functional τ, called the state, for which $\tau(\mathbb{I}) = 1$. We call an element $\mathbf{a} \in \mathcal{A}$ a noncommutative random variable and call $\tau(\mathbf{a}^n)$ its n^{th} moment. We say that $\mathcal{A}$ is a *-algebra if the involution * is defined on $\mathcal{A}$. In addition, we assume that $\tau(\mathbf{a}^*) = \overline{\tau(\mathbf{a})}$ and $\tau(\mathbf{a}^*\mathbf{a}) \geq 0$. Henceforth, we will consider only *-algebras. We say that $\mathbf{a} \in \mathcal{A}$ is selfadjoint if $\mathbf{a}^* = \mathbf{a}$.
Fix a noncommutative probability space $(\mathcal{A}, \tau)$. For each selfadjoint $\mathbf{a} \in \mathcal{A}$ there exists a probability measure $\mu_{\mathbf{a}}$ on $\mathbb{R}$ such that

$$\tau(\mathbf{a}^n) = \int_{\mathbb{R}} x^n \mu_{\mathbf{a}}(dx)$$

for all $n \geq 1$, see [9, p.2]. The probability measure $\mu_{\mathbf{a}}$ is unique if $|\tau(\mathbf{a}^n)| \leq M^n$ for some $M > 0$ and for all $n \geq 1$.

Definition 4 ([8]). A family of subalgebras $(\mathcal{A}_j; j \in J)$ of $\mathcal{A}$, which contain $\mathbb{I}$, is said to be free with respect to τ if for every $k \geq 1$ and $j_1 \neq j_2 \neq \ldots \neq j_k \in J \subset \mathbb{N}$

$$\tau(\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_k) = 0$$

for all $\mathbf{a}_i \in \mathcal{A}_{j_i}$ whenever $\tau(\mathbf{a}_i) = 0$ for every $1 \leq i \leq k$.

Random variables in a noncommutative probability space $(\mathcal{A}, \tau)$ are said to be free if the subalgebras generated by them and $\mathbb{I}$ are free.

Definition 5. We say that a family of sequences of random matrices $(\{\mathbf{A}_n^{(l)}\}; l = 1, \ldots, m)$ is almost surely asymptotically free (*cf.* [8]) if for every noncommutative polynomial p in m variables

$$\mathrm{tr}_n\left(p(\mathbf{A}_n^{(1)}, \ldots, \mathbf{A}_n^{(m)})\right) \xrightarrow{n\to\infty} \tau\left(p(\mathbf{a}_1, \ldots, \mathbf{a}_m)\right) \qquad a.s.$$

where $(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ is a family of free noncommutative random variables in some noncommutative probability space $(\mathcal{A}, \tau)$.

In [3], Capitaine and Donati-Martin showed the asymptotic freeness for independent Wigner matrices when the distribution of the entries is symmetric and satisfies Poincaré inequality. Recently, Guionnet [6] gave a proof where she assumes that all the moments of the entries exist. Szarek [12] showed us a proof for symmetric and non-symmetric matrices with uniformly bounded entries. Szarek's proof, in brief, is based on concentration inequalities and some tools of operator theory. In this paper, we give a combinatorial proof of the almost sure asymptotic freeness for Wigner matrices under the assumption of finite variance and fourth moment of the entries.

Theorem 2. *If* $(\{\mathbf{A}_n^{(l)}\}; l = 1, \ldots, m)$ *is a family of independent* $\mathbf{Wigner}(n, 1)$ *matrices for which* $E(|A_{12}^{(l)}|^4) < \infty$ *and* $E(A_{11}^{(l)})^2 < \infty$*, then* $(\{\mathbf{A}_n^{(l)}\}; l = 1, \ldots, m)$ *is almost surely asymptotically free.*

2.1 Proof of Theorem 2

The Schatten p-norm of a matrix $\mathbf{A}$ is defined as $\|\mathbf{A}\|_p := (\mathrm{tr}_n|\mathbf{A}|^p)^{\frac{1}{p}}$ whenever $1 \leq p < \infty$, where $|\mathbf{A}| = (\mathbf{A}^T\mathbf{A})^{\frac{1}{2}}$. The operator norm is defined as $\|\mathbf{A}\| := \max_{1\leq i\leq n} |\lambda_i|$ where λ_i; $i = 1, 2, \ldots, n$ are the eigenvalues of $\mathbf{A}$. The following three inequalities hold true;

1. Domination inequality [8, p.154]

$$|\mathrm{tr}_n(\mathbf{A})| \leq \|\mathbf{A}\|_1 \leq \|\mathbf{A}\|_p \leq \|\mathbf{A}\| \tag{5}$$

2. Hölder's inequality [8, p.154]

$$\|\mathbf{AB}\|_r \leq \|\mathbf{A}\|_p\|\mathbf{B}\|_q \tag{6}$$

whenever $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for $p, q > 1$ and $r \geq 1$.

3. Generalized Hölder's inequality

$$\|\mathbf{A}^{(1)}\mathbf{A}^{(2)}\cdots\mathbf{A}^{(m)}\|_1 \leq \|\mathbf{A}^{(1)}\|_{p_1}\|\mathbf{A}^{(2)}\|_{p_2}\cdots\|\mathbf{A}^{(m)}\|_{p_m} \tag{7}$$

where $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(m)}$ are $n \times n$ matrices and $\sum_{i=1}^m \frac{1}{p_i} = 1$. This inequality follows from (6) by induction.

Let $\mathbf{A} = \frac{1}{\sqrt{n}}(X_{ij})_{i,j=1}^n$ be a $\mathbf{Wigner}(n, 1)$ matrix. We define $\widetilde{\mathbf{A}}(c) = \frac{1}{\sqrt{n}}(\widetilde{X}_{ij}(c))_{i,j=1}^n$ to be the matrix whose off-diagonal entries are those of $\mathbf{A}$ truncated by $c/\sqrt{n}$ and standardized. We will also assume that the diagonal entries of $\widetilde{\mathbf{A}}(c)$ are equal to zero. In other words,

$$\widetilde{X}_{ij}(c) = \begin{cases} \frac{1}{\sigma(c)}\left[X_{ij}\mathbf{1}_{(|X_{ij}|\leq c)} - E(X_{ij}\mathbf{1}_{(|X_{ij}|\leq c)})\right], & \text{for } i < j; \\ 0, & \text{for } i = j \end{cases}$$

where $\mathbf{1}_{(|X_{ij}|\leq c)}$ is equal to one if $|X_{ij}| \leq c$ and zero otherwise; and

$$\sigma^2(c) = E \mid X_{ij}\mathbf{1}_{(|X_{ij}|\leq c)} - E(X_{ij}\mathbf{1}_{(|X_{ij}|\leq c)}) \mid^2 \leq 1.$$

We would choose sufficiently large c so that $\sigma(c) > 0$ and $\widetilde{X}_{ij}(c)$ would be well defined. Note that $\sigma^2(c) \to 1$ as $c \to \infty$ and $\mathrm{Var}(X_{12}\mathbf{1}_{(|X_{12}|>c)}) \leq 1 - \sigma^2(c)$.
The proof of Theorem 2 resembles the proof of Wigner's theorem given in [2]. We will split it into a number of lemmas.

Lemma 1. *Let* $\left(\{\mathbf{A}_n^{(l)}\} : l = 1, \ldots, m\right)$ *be a family of independent sequences of* $\mathbf{Wigner}(n, 1)$ *matrices for which* $E(|A_{12}^{(l)}|^4) < \infty$ *and* $E(A_{11}^{(l)})^2 < \infty$ *for every* l*. Then, for any* $\epsilon > 0$ *there exists* $M \in (0, \infty)$ *such that for every* $c \geq M$*:*

$$\limsup_{n\to\infty} |\mathrm{tr}_n \left(\mathbf{A}_n^{(1)}\mathbf{A}_n^{(2)} \cdots \mathbf{A}_n^{(m)}\right) - \mathrm{tr}_n(\widetilde{\mathbf{A}}_n^{(1)}(c)\widetilde{\mathbf{A}}_n^{(2)}(c) \cdots \widetilde{\mathbf{A}}_n^{(m)}(c))| < \epsilon$$

with probability one.

Proof. First, we can write the difference between products of matrices as a telescopic sum, *i.e.*,

$$\mathbf{A}_n^{(1)}\mathbf{A}_n^{(2)} \cdots \mathbf{A}_n^{(m)} - \widetilde{\mathbf{A}}_n^{(1)}(c)\widetilde{\mathbf{A}}_n^{(2)}(c) \cdots \widetilde{\mathbf{A}}_n^{(m)}(c) = \sum_{j=1}^{m} \prod_{k=1}^{j-1} \widetilde{\mathbf{A}}_n^{(k)}(c)(\mathbf{A}_n^{(j)} - \widetilde{\mathbf{A}}_n^{(j)}(c)) \prod_{l=j+1}^{m} \mathbf{A}_n^{(l)}$$

with the convention that $\prod_{k=1}^{0} \widetilde{\mathbf{A}}_n^{(k)}(c) = \prod_{l=m+1}^{m} \mathbf{A}_n^{(l)} = \mathbf{I}_n$. But,
$|\mathrm{tr}_n \left(\prod_{k=1}^{j-1} \widetilde{\mathbf{A}}_n^{(k)}(c)\ (\mathbf{A}_n^{(j)} - \widetilde{\mathbf{A}}_n^{(j)}(c))\ \prod_{l=j+1}^{m} \mathbf{A}_n^{(l)}\right)| =$

$$\begin{aligned}
&= |\mathrm{tr}_n \left(\prod_{l=j+1}^{m} \mathbf{A}_n^{(l)} \prod_{k=1}^{j-1} \widetilde{\mathbf{A}}_n^{(k)}(c)\ (\mathbf{A}_n^{(j)} - \widetilde{\mathbf{A}}_n^{(j)}(c))\right)| \\
&\leq \| \prod_{l=j+1}^{m} \mathbf{A}_n^{(l)} \prod_{k=1}^{j-1} \widetilde{\mathbf{A}}_n^{(k)}(c)\|_2 \cdot \|\mathbf{A}_n^{(j)} - \widetilde{\mathbf{A}}_n^{(j)}(c)\|_2 \\
&\leq \prod_{l=j+1}^{m} \|\mathbf{A}_n^{(l)}\|_{2(m-1)} \cdot \prod_{k=1}^{j-1} \|\widetilde{\mathbf{A}}_n^{(k)}(c)\|_{2(m-1)} \cdot \|\mathbf{A}_n^{(j)} - \widetilde{\mathbf{A}}_n^{(j)}(c)\|_2
\end{aligned}$$

for all $1 \leq j \leq m$ with the convention that $\prod_{k=1}^{0} \|\widetilde{\mathbf{A}}_n^{(k)}(c)\|_p = \prod_{l=m+1}^{m} \|\mathbf{A}_n^{(l)}\|_p = 1$. The last two inequalities are due to the generalized Hölder's inequality (7).
It is known that if $E(|X_{12}^{(l)}|^4) < \infty$ and $E(X_{11}^{(l)})^2 < \infty$ for every l, then

$$\lim_{n\to\infty} \|\mathbf{A}_n^{(l)}\| = 2$$

almost surely (*cf.* [2, Theorem 2.12]). Similarly, we can see that

$$\lim_{n\to\infty} \|\widetilde{\mathbf{A}}_n^{(l)}(c)\| = 2$$

almost surely for each l. By the domination inequality (5)

$$\|\mathbf{A}_n^{(l)}\|_{2(m-1)} \leq \|\mathbf{A}_n^{(l)}\| \text{ and } \|\widetilde{\mathbf{A}}_n^{(k)}(c)\|_{2(m-1)} \leq \|\widetilde{\mathbf{A}}_n^{(k)}(c)\|.$$

Therefore,

$$\limsup_{n\to\infty} \prod_{l=j+1}^{m} \|\mathbf{A}_n^{(l)}\|_{2(m-1)} \cdot \prod_{k=1}^{j-1} \|\widetilde{\mathbf{A}}_n^{(k)}(c)\|_{2(m-1)} \leq 2^{m-1}.$$

Now, let $\widehat{\mathbf{A}}_n^{(j)}(c) := \mathbf{A}_n^{(j)} - \sigma_j(c)\widetilde{\mathbf{A}}_n^{(j)}(c)$ or $\widehat{X}_{rs}^{(j)}(c) := X_{rs}^{(j)} - \sigma_j(c)\widetilde{X}_{rs}^{(j)}(c)$ for every r and s. Thus,

$$\|\mathbf{A}_n^{(j)} - \widetilde{\mathbf{A}}_n^{(j)}(c)\|_2 \leq \|\widehat{\mathbf{A}}_n^{(j)}(c)\|_2 + |1 - \sigma_j(c)|\ \|\widetilde{\mathbf{A}}_n^{(j)}(c)\|_2.$$

By definition,

$$\|\widehat{\mathbf{A}}_n^{(j)}(c)\|_2^2 = \frac{1}{n^2}\sum_{r=1}^{n}\sum_{s=1}^{n}|\widehat{X}_{rs}^{(j)}(c)|^2 = \frac{1}{n^2}\sum_{r=1}^{n}|\widehat{X}_{rr}^{(j)}(c)|^2 + \frac{1}{n^2}\sum_{r\neq s}|\widehat{X}_{rs}^{(j)}(c)|^2.$$

Note that

$$\widehat{X}_{rs}^{(j)}(c) = \begin{cases} X_{rs}^{(j)}\mathbf{1}_{(|X_{rs}^{(j)}|>c)} - E(X_{rs}^{(j)}\mathbf{1}_{(|X_{rs}^{(j)}|>c)}), & \text{for } r<s; \\ X_{rr}^{(j)}, & \text{for } r=s. \end{cases}$$

Since $E(X_{11}^{(j)})^2 < \infty$ then $\lim_{n\to\infty}\frac{1}{n^2}\sum_{r=1}^{n}(X_{rr}^{(j)})^2 = 0$ almost surely due to the Strong Law of Large Numbers ($SLLN$). Once more the $SLLN$ implies that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{r\neq s}|\widehat{X}_{rs}^{(j)}(c)|^2 = \mathrm{Var}(X_{12}(j)\mathbf{1}_{(|X_{12}^{(j)}|>c)}) \quad a.s.$$

Hence, $\lim_{n\to\infty}\|\widehat{\mathbf{A}}_n^{(j)}(c)\|_2^2 = \mathrm{Var}(X_{12}^{(j)}\mathbf{1}_{(|X_{12}^{(j)}|>c)})$ almost surely. It is also evident that $\lim_{n\to\infty}\|\widetilde{\mathbf{A}}_n^{(j)}(c)\|_2 = 1$ almost surely. Therefore,

$$\limsup_{n\to\infty}\|\mathbf{A}_n^{(j)} - \widetilde{\mathbf{A}}_n^{(j)}(c)\|_2 \leq \sqrt{1-\sigma_j^2(c)} + |1-\sigma_j(c)|$$

almost surely. Consequently, for all $c>0$

$$\begin{aligned}&\limsup_{n\to\infty}\left|\mathrm{tr}_n\left(\mathbf{A}_n^{(1)}\mathbf{A}_n^{(2)}\cdots\mathbf{A}_n^{(m)}\right) - \mathrm{tr}_n(\widetilde{\mathbf{A}}_n^{(1)}(c)\widetilde{\mathbf{A}}_n^{(2)}(c)\cdots\widetilde{\mathbf{A}}_n^{(m)}(c))\right| \\ &\leq 2^{m-1}\sum_{j=1}^{m}(\sqrt{1-\sigma_j^2(c)} + |1-\sigma_j(c)|)\end{aligned}$$

with probability one. But since $\sigma_j(c)\to 1$ as $c\to\infty$, then for any $\epsilon_1>0$, there exists $M\in(0,\infty)$ such that $\sqrt{1-\sigma_j^2(c)} + |1-\sigma_j(c)| < \epsilon_1$ for every $c\geq M$ and for all j. Hence,

$$\begin{aligned}&\limsup_{n\to\infty}\left|\mathrm{tr}_n\left(\mathbf{A}_n^{(1)}\mathbf{A}_n^{(2)}\cdots\mathbf{A}_n^{(m)}\right) - \mathrm{tr}_n(\widetilde{\mathbf{A}}_n^{(1)}(c)\widetilde{\mathbf{A}}_n^{(2)}(c)\cdots\widetilde{\mathbf{A}}_n^{(m)}(c))\right| \\ &\leq m2^{m-1}\epsilon_1 = \epsilon\end{aligned}$$

with probability one. □

Lemma 2 ([10]). *If $\left(\{\widetilde{\mathbf{A}}_n^{(l)}\}; l=1,\ldots m\right)$ is a family of independent sequences of* $\mathbf{Wigner}(n,1)$ *matrices whose entries are bounded, then*

$$\lim_{n\to\infty}E\left(\mathrm{tr}_n\left(\widetilde{\mathbf{A}}_n^{(1)}\widetilde{\mathbf{A}}_n^{(2)}\cdots\widetilde{\mathbf{A}}_n^{(m)}\right)\right) = \tau\left(\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_m\right) \tag{8}$$

where $\mathbf{a}_i$'s are some free noncommutative random variables in $(\mathcal{A},\tau)$ such that $\mathbf{a}_i$ has the semicircle law $\gamma_{0,1}$ for all i.

We say that a partition $\pi = \{B_1,\ldots,B_p\}$ of a set of integers is non-crossing if $a<b<c<d$ is impossible for $a,c\in B_i$ and $b,d\in B_j$ when $i\neq j$. We denote the family of all non-crossing

partitions of $\{1,\ldots,k\}$ by $\mathrm{NC}(k)$. Also let $\mathrm{NC}_2(k)$ be the family of all non-crossing pair partitions which is empty unless k is even. The Catalan number

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

is equal to the size of $\mathrm{NC}(k)$ and also the size of $\mathrm{NC}_2(2k)$.
If $(\mathbf{a}_l; l=1,\ldots m)$ is a family of free semicircular random variables which have mean zero and variance one, then (*cf.* [10, Equation (8)])

$$\tau\left(\mathbf{a}_{i_1}\mathbf{a}_{i_2}\cdots\mathbf{a}_{i_k}\right) = \begin{cases} \sum_{\pi\in \mathrm{NC}_2(k)} \prod_{\{p,q\}\in\pi} \mathbf{1}_{i_p=i_q}, & \text{if } k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

for any $i_1,\ldots,i_k \in \{1,\ldots,m\}$.

Lemma 3. *If* $\left(\{\widetilde{\mathbf{A}}_n^{(l)}\}; l=1,\ldots m\right)$ *is a family of independent sequences of* $\mathbf{Wigner}(n,1)$ *matrices whose entries are bounded, then*

$$\sum_{n=1}^{\infty} \mathrm{Var}\left(\mathrm{tr}_n\left(\prod_{i=1}^{m}\widetilde{\mathbf{A}}_n^{(l_i)}\right)\right) < \infty$$

for all $l_1,\ldots,l_m \geq 1$ *with possible repetitions among them.*

Proof. It is enough to show that

$$\mathrm{Var}\left(\mathrm{tr}_n\left(\prod_{i=1}^{m}\widetilde{\mathbf{A}}_n^{(l_i)}\right)\right) = O(n^{-2}).$$

We will denote the number of distinct integers among $(i_1,\ldots,i_m)$ by $\langle i_1,\ldots,i_m\rangle$. Let $\overline{z}$ be the complex conjugate of z.
First,

$$\mathrm{Var}\left(\mathrm{tr}_n\left(\prod_{i=1}^{m}\widetilde{\mathbf{A}}_n^{(l_i)}\right)\right) = E\left(\mid \mathrm{tr}_n\left(\prod_{i=1}^{m}\widetilde{\mathbf{A}}_n^{(l_i)}\right)\mid^2\right) - \mid E\left(\mathrm{tr}_n\left(\prod_{i=1}^{m}\widetilde{\mathbf{A}}_n^{(l_i)}\right)\right)\mid^2 =$$

$$= \frac{1}{n^{m+2}}\sum [E\left(\prod_{r=1}^{m}\widetilde{X}_{i_r\,i_{r+1}}^{(l_r)}\prod_{s=1}^{m}\overline{\widetilde{X}}_{j_s\,j_{s+1}}^{(l_s)}\right) - E\left(\prod_{r=1}^{m}\widetilde{X}_{i_r\,i_{r+1}}^{(l_r)}\right)E\left(\prod_{s=1}^{m}\overline{\widetilde{X}}_{j_s\,j_{s+1}}^{(l_s)}\right)]$$

where the sums are running over $(i_1,\ldots,i_m)$ and $(j_1,\ldots,j_m)$ in $\{1,\ldots,n\}^m$ and such that $i_{m+1}=i_1$ and $j_{m+1}=j_1$. The term under summation is zero unless:

1. Each one of the unordered pairs $(\{i_1,i_2\},\ldots,\{i_m,i_1\},\{j_1,j_2\},\ldots,\{j_m,j_1\})$ appears at least twice.

2. At least one of the unordered pairs $(\{i_1,i_2\},\ldots,\{i_m,i_1\})$ is identical to one of the unordered pairs $(\{j_1,j_2\},\ldots,\{j_m,j_1\})$.

The first condition implies that $\langle\langle i_1,\ldots,i_m,j_1,\ldots,j_m\rangle\rangle \leq m+2$. Adding the second condition forces at least two more integers to be replications which implies that $\langle\langle i_1,\ldots,i_m,j_1,\ldots,j_m\rangle\rangle \leq m$. Since $|\widetilde{X}_{i\,j}^{(l)}|$ are bounded for every i, j and l, then

$$\operatorname{Var}\left(\operatorname{tr}_n\left(\prod_{i=1}^{m}\widetilde{\mathbf{A}}_n^{(l_i)}\right)\right)\leq\frac{C}{n^2}.$$

□

Concusion of the proof of Theorem 2. Any noncommutative polynomial p can be written as a linear combination of noncommutative monomials, *i.e.*,

$$p\left(\mathbf{A}_n^{(1)},\ldots,\mathbf{A}_n^{(m)}\right)=\sum u_{i_1,\ldots,i_k}\mathbf{A}_n^{(i_1)}\mathbf{A}_n^{(i_2)}\cdots\mathbf{A}_n^{(i_k)}$$

where the sum runs over $i_1,\ldots,i_k\in\{1,\ldots,m\}$ for $k\geq 1$ and $u_{i_1,\ldots,i_k}\in\mathbb{C}$ are constants. Note that p has a finite number of terms. Therefore,

$$\begin{aligned}&|\operatorname{tr}_n p\left(\mathbf{A}_n^{(1)},\mathbf{A}_n^{(2)},\ldots,\mathbf{A}_n^{(m)}\right)-\operatorname{tr}_n p(\widetilde{\mathbf{A}}_n^{(1)}(c),\widetilde{\mathbf{A}}_n^{(2)}(c),\ldots,\widetilde{\mathbf{A}}_n^{(m)}(c))|\\&\leq\sum|u_{i_1,\ldots,i_k}|\;|\operatorname{tr}_n\left(\mathbf{A}_n^{(i_1)}\mathbf{A}_n^{(i_2)}\cdots\mathbf{A}_n^{(i_k)}\right)-\operatorname{tr}_n(\widetilde{\mathbf{A}}_n^{(i_1)}(c)\widetilde{\mathbf{A}}_n^{(i_2)}(c)\cdots\widetilde{\mathbf{A}}_n^{(i_k)}(c))|\end{aligned}$$

Hence, given $\epsilon>0$

$$\begin{aligned}&\limsup_{n\to\infty}|\operatorname{tr}_n p\left(\mathbf{A}_n^{(1)},\mathbf{A}_n^{(2)},\ldots,\mathbf{A}_n^{(m)}\right)-\operatorname{tr}_n p(\widetilde{\mathbf{A}}_n^{(1)}(c),\widetilde{\mathbf{A}}_n^{(2)}(c),\ldots,\widetilde{\mathbf{A}}_n^{(m)}(c))|\\&\leq\epsilon\sum|u_{i_1,\ldots,i_k}|\end{aligned}\tag{10}$$

almost surely, for sufficiently large c.
Due to Lemma 2

$$\lim_{n\to\infty}E\left(\operatorname{tr}_n(\widetilde{\mathbf{A}}_n^{(i_1)}(c)\widetilde{\mathbf{A}}_n^{(i_2)}(c)\cdots\widetilde{\mathbf{A}}_n^{(i_k)}(c))\right)=\tau\left(\mathbf{a}_{i_1}\mathbf{a}_{i_2}\cdots\mathbf{a}_{i_k}\right)\tag{11}$$

for all $i_1,\ldots,i_k\in\{1,\ldots,m\}$ and $k\geq 1$ where $\mathbf{a}_i$'s are some free noncommutative random variables in some noncommutative probability space $(\mathcal{A},\tau)$ such that $\mathbf{a}_i$ has the semicircle law $\gamma_{0,1}$ for all i. Lemma 3 implies that the limit in (11) is holding true in the almost sure sense due to Borel-Cantelli lemma. Consequently, for every $c>0$

$$\lim_{n\to\infty}\operatorname{tr}_n p(\widetilde{\mathbf{A}}_n^{(1)}(c),\widetilde{\mathbf{A}}_n^{(2)}(c),\ldots,\widetilde{\mathbf{A}}_n^{(m)}(c))=\tau\left(p(\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_m)\right)\tag{12}$$

almost surely, since τ is a linear functional.
Equation(10) implies that

$$\begin{aligned}&\limsup_{n\to\infty}|\operatorname{tr}_n p\left(\mathbf{A}_n^{(1)},\mathbf{A}_n^{(2)},\ldots,\mathbf{A}_n^{(m)}\right)-\tau\left(p(\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_m)\right)|\\&\leq\limsup_{n\to\infty}|\operatorname{tr}_n p\left(\mathbf{A}_n^{(1)},\mathbf{A}_n^{(2)},\ldots,\mathbf{A}_n^{(m)}\right)-\operatorname{tr}_n p(\widetilde{\mathbf{A}}_n^{(1)}(c),\widetilde{\mathbf{A}}_n^{(2)}(c),\ldots,\widetilde{\mathbf{A}}_n^{(m)}(c))|\\&+\limsup_{n\to\infty}|\operatorname{tr}_n p(\widetilde{\mathbf{A}}_n^{(1)}(c),\widetilde{\mathbf{A}}_n^{(2)}(c),\ldots,\widetilde{\mathbf{A}}_n^{(m)}(c))-\tau\left(p(\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_m)\right)|\\&\leq\epsilon_1\end{aligned}\tag{13}$$

almost surely, for sufficiently large $c>0$ and arbitrary $\epsilon_1>0$.

□

2.2 Proof of Theorem 1

Fix $k \geq 1$ and a symmetric block-structure $\mathbb{B}_k$. Let us introduce the noncommutative probability space $(\mathcal{A} \bigotimes \mathcal{M}_k(\mathbb{C}), \tau \bigotimes \mathrm{tr}_k)$, where $\bigotimes$ stands for the tensor product. A typical element in $\mathcal{A} \bigotimes \mathcal{M}_k(\mathbb{C})$ is a $k \times k$ matrix whose entries are noncommutative random variables in $\mathcal{A}$. For example, $\mathbb{B}_k(\mathbf{a}_1, \ldots, \mathbf{a}_h) \in \mathcal{A} \bigotimes \mathcal{M}_k(\mathbb{C})$ for any $\mathbf{a}_1, \ldots, \mathbf{a}_h \in \mathcal{A}$. The state $\tau \bigotimes \mathrm{tr}_k$ is defined by $\tau \bigotimes \mathrm{tr}_k(\mathbf{A}) = \frac{1}{k}\sum_{i=1}^{k} \tau(A_{ii})$ for any $\mathbf{A} \in \mathcal{A} \bigotimes \mathcal{M}_k(\mathbb{C})$. The involution is given by the *-transpose, *i.e.*, for any $\mathbf{A} = (\mathbf{a}_{ij})_{i,j=1,\ldots,k} \in \mathcal{A} \bigotimes \mathcal{M}_k(\mathbb{C})$ the involution of $\mathbf{A}$ is given by $(\mathbf{a}_{ij}^*)_{i,j=1,\ldots,k}^T$.

The proof of Theorem 1 is based on the method of moments. First, we are going to show that for every $s \in \mathbb{N}$, the limit of $\mathrm{tr}_{nk}\left(\mathbb{B}_k\left(\mathbf{A}_n^{(1)}, \ldots, \mathbf{A}_n^{(h)}\right)^s\right)$ exists as $n \to \infty$, almost surely.

Fix $s \geq 1$. We can see that the trace for the s-power of $\mathbf{X}_{n,k} = \mathbb{B}_k\left(\mathbf{A}_n^{(1)}, \ldots, \mathbf{A}_n^{(h)}\right)$ is the trace of some noncommutative polynomials in the matrices $\mathbf{A}_n^{(1)}, \ldots, \mathbf{A}_n^{(h)}$. In other words,

$$\mathrm{tr}_{nk}\left(\mathbf{X}_{n,k}^s\right) = \frac{1}{k}\sum_{i=1}^{k} \mathrm{tr}_n\left(p_i\left(\mathbf{A}_n^{(1)}, \ldots, \mathbf{A}_n^{(h)}\right)\right)$$

for some noncommutative polynomial p_i and $1 \leq i \leq k$. Theorem 2 implies that for each i

$$\mathrm{tr}_n\left(p_i\left(\mathbf{A}_n^{(1)}, \ldots, \mathbf{A}_n^{(h)}\right)\right) \to \tau\left(p_i\left(\mathbf{a}_1, \ldots, \mathbf{a}_h\right)\right) \text{ as } n \to \infty \qquad a.s.$$

where $\{\mathbf{a}_l : l = 1, \ldots, m\}$ is a family of free noncommutative random variables that have the semicircle law with variance equals to one. Therefore

$$\mathrm{tr}_{nk}\left(\mathbb{B}_k\left(\mathbf{A}_n^{(1)}, \ldots, \mathbf{A}_n^{(h)}\right)^s\right) \to \frac{1}{k}\tau\left(\sum_{i=1}^{k} p_i\left(\mathbf{a}_1, \ldots, \mathbf{a}_h\right)\right) \text{ as } n \to \infty \qquad a.s.$$

Thus,

$$\mathrm{tr}_{nk}\left(\mathbf{X}_{n,k}^s\right) \to \tau \bigotimes \mathrm{tr}_k\left(\mathbb{B}_k\left(\mathbf{a}_1, \ldots, \mathbf{a}_h\right)^s\right) \text{ as } n \to \infty \qquad a.s.$$

Since $\mathbb{B}_k\left(\mathbf{a}_1, \ldots, \mathbf{a}_h\right)$ is self-adjoint, then there exists a probability measure $\mu_{\mathbb{B}_k}$ such that

$$\tau \bigotimes \mathrm{tr}_k\left(\mathbb{B}_k\left(\mathbf{a}_1, \ldots, \mathbf{a}_h\right)^s\right) = \int_{\mathbb{R}} x^s \mu_{\mathbb{B}_k}(dx).$$

Note that if s is an odd integer then $\tau \bigotimes \mathrm{tr}_k\left(\mathbb{B}_k\left(\mathbf{a}_1, \ldots, \mathbf{a}_h\right)^s\right)$ is zero by Equation (9). This implies that $\mu_{\mathbb{B}_k}$ is a symmetric probability measure.

To complete the proof, we need to prove that $\mu_{\mathbb{B}_k}$ is unique and has a compact support in $\mathbb{R}$. Both follows by showing that there exist $M > 0$ and $C > 0$ such that $\tau \bigotimes \mathrm{tr}_k\left(\mathbb{B}_k\left(\mathbf{a}_1, \ldots, \mathbf{a}_h\right)^{2s}\right) \leq C\, M^{2s}$ for all $s \geq 1$. But for a fixed $s \geq 1$

$$\tau \bigotimes \mathrm{tr}_k\left(\mathbb{B}_k\left(\mathbf{a}_1, \ldots, \mathbf{a}_h\right)^{2s}\right) = \sum_{\mathbb{J}(2s,k)} \tau(B_{j_1 j_2} B_{j_2 j_3} \cdots B_{j_{2s} j_1})$$

where $B_{ij} \in \{\mathbf{a}_1, \ldots, \mathbf{a}_h\}$ and $\mathbb{J}(m,k) := \{(j_1, \ldots, j_m) : 1 \leq j_1, \ldots, j_m \leq k\}$. But again by Equation (9),

$$\sum_{\mathbb{J}(2s,k)} \tau(B_{j_1 j_2} B_{j_2 j_3} \cdots B_{j_{2s} j_1}) \leq k^{2s} C_s = (2k)^{2s}$$

where the Catalan number C_s is at most 4^s.
Therefore, there exists a unique non-random symmetric probability measure $\mu_{\mathbb{B}_k}$ with a compact support in $\mathbb{R}$ that has the moments $\tau \bigotimes \mathrm{tr}_k \left(\mathbb{B}_k \left(\mathbf{a}_1, \ldots, \mathbf{a}_h \right)^s \right)$, for every $s \geq 1$, such that

$$\mu_{\mathbf{X}_{n,k}} \xrightarrow{\mathcal{D}} \mu_{\mathbb{B}_k} \text{ as } n \to \infty \qquad a.s.$$

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