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INEQUALITY OF TWO CRITICAL PROBABILITIES FOR PERCOLATION

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Abstract
For a locally finite, connected graph G with distinguished vertex 0, let $p_c(G)$ be the usual critical probability for bond percolation on G, and

$$p_{\rm cut}(G) = \sup\{p : \inf_{\Pi} \mathsf{E}_p|C(0) \cap \Pi| = 0\} \ (\leq p_c),$$

where Π ranges over cutsets (sets of vertices "separating 0 from ∞"), E_p refers to (Bernoulli bond) percolation with p the probability that an edge is open, and $C(0)$ is the open cluster containing 0.(The definition is easily seen to be independent of the choice of distinguished vertex.)
We disprove a conjecture of Russ Lyons stating that $p_{\rm cut}(G) = p_c(G)$ for every G, and propose a possible alternative.

1 Introduction

We consider the usual (Bernoulli) bond percolation on a locally finite, connected graph $G = (V, E)$: each $e \in E$ is "open" with probability p, these choices made independently. We write H for the resulting random subgraph of G and $\Pr_p$ for the law of H. (For percolation background, see e.g. [2] or [5].)
For $X, Y \subseteq V$ we denote by $\{X \longleftrightarrow Y\}$ the event that some $x \in X$ and $y \in Y$ are in the same component of H (of course $\{x \longleftrightarrow y\}$ means $\{\{x\} \longleftrightarrow \{y\}\}$), and by $\{x \longleftrightarrow \infty\}$ the event that the component of x in H—denoted $C(x)$—is infinite.
The standard *critical probability* for bond percolation is

$$p_c = p_c(G) = \sup\{p : \Pr_p(H \text{ contains an infinite component}) = 0\}.$$

(For connected G this is easily seen to coincide with $\sup\{p : \Pr_p(|C(x)| = \infty) = 0\}$ for any $x \in V$.) Our focus here is on a second critical probability, introduced by Russ Lyons in [3].

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Given $0 \in V$, say $\Pi \subseteq V$ is a *cutset* for 0 if the component of 0 in $G - \Pi$ is finite ("Π separates 0 from infinity"), and set

$$p_{\text{cut}}(G) = \sup\{p : \inf_{\Pi} \mathsf{E}_p|C(0) \cap \Pi| = 0\}.$$

Again this is easily seen not to depend on our choice of the distinguished vertex. Here we will keep 0 fixed throughout, and Π will always range over cutsets for 0.
Note $p_{\text{cut}} \leq p_c$ is trivial because of the "first moment method" bound

$$\Pr(0 \longleftrightarrow \infty) = \inf_{\Pi} \Pr(0 \longleftrightarrow \Pi) \leq \inf_{\Pi} \mathsf{E}|C(0) \cap \Pi|. \tag{1}$$

Lyons proved $p_{\text{cut}} = p_c$ for trees [3] and for certain "tree-like" graphs [4], and suggested ([3], [4]) the possibility that p_{cut} and p_c coincide for *every* G. That equality also holds for the Euclidean cubic lattices follows from [1], [6].
Here we give a simple family of graphs for which equality does not hold, and, in Section 3, suggest a possible salvage.

2 Example

A graph G is said to be *spherically symmetric* about $0 \in V$ if $(\mathrm{Aut}(G))_0$ is transitive on

$$\Pi_n := \{v : d(0, v) = n\}$$

for each n. For such G, dealing with p_{cut} is simplified by the following observation.

Lemma 2.1 *If G is spherically symmetric then (for any p)*

$$\inf_{\Pi} \mathsf{E}_p|C(0) \cap \Pi| = \inf_{n} \mathsf{E}_p|C(0) \cap \Pi_n|.$$

Proof. Fix a cutset Π, say with $\Pi \subseteq \Pi_0 \cup \cdots \cup \Pi_m$ (of course it is enough to consider minimal Π's, which are necessarily finite).
Let P be chosen uniformly from the set of $(0, \Pi_m)$-paths of length m. By spherical symmetry we have $\Pr(v \in P) = |\Pi_k|^{-1}$ for each $k \leq m$ and $v \in \Pi_k$.
So setting $\alpha_k = |\Pi \cap \Pi_k|/|\Pi_k|$, we have

$$1 = \Pr(P \cap \Pi \neq \emptyset) \leq \sum_{k \leq m} \mathsf{E}|P \cap \Pi \cap \Pi_k| = \sum_{k \leq m} \alpha_k,$$

and

$$\mathsf{E}|C(0) \cap \Pi| = \sum_{k} \alpha_k \mathsf{E}|C(0) \cap \Pi_k| \geq \inf_{n} \mathsf{E}|C(0) \cap \Pi_n|.$$

Construction
Set $n_0 = 1$ and let $n_1, n_2, \ldots$ be integers with $n_i \geq 5n_{i-1}$ for $i \geq 1$. Set $m_i = n_1 + \cdots + n_i$. The m_ith levels play a special role, and we abbreviate $\Pi_{m_i} = \Sigma_i$. The structure of G is as follows.
For each i, Σ_i is a clique of size 2^{n_i}, and $G[\cup\{\Pi_n : m_{i-1} \leq n \leq m_i\}]$ consists of the cliques at levels m_{i-1} and m_i together with, for each $v \in \Sigma_{i-1}$, $2^{n_i - n_{i-1}}$ paths of length n_i joining v to Σ_i, the resulting 2^{n_i} paths being disjoint except where they share an end in Σ_{i-1}.

Analysis
It is easy to see that $p_c(G) \geq 1/2$: with $B_i = \{\Sigma_{i-1} \longleftrightarrow \Sigma_i\}$ we have

$$\Pr_p(0 \longleftrightarrow \infty) < \Pr_p(B_i) < |\Sigma_i| p^{n_i} = (2p)^{n_i},$$

which for $p < 1/2$ tends to zero as $i \to \infty$.
(Actually $p_c(G) = \exp_2[-\liminf n_i/m_i]$ ($= 1/2$ if $n_i/m_i \to 1$): as shown in [3], for spherically symmetric *trees* T one has

$$p_c(T) = [\liminf_n |\Pi_n|^{1/n}]^{-1}$$

(the expression in brackets is the *growth number* of T); and in our case this gives ($p_c(G) \leq$) $p_c(T) = \exp_2[-\liminf n_i/m_i]$ for $T = (V(G), E')$, where E' consists of those edge of G not lying in any Σ_i.)
We assert, on the other hand, that

$$p_{\rm cut}(G) \leq .4. \tag{2}$$

(The bound is by no means best possible, but there seems no point in trying to optimize either the construction or the analysis, and we will always opt for crude but convenient estimates in what follows.)
According to Lemma 2.1, (2) can be established by showing that for

$$p = (1-\varepsilon)/2 \quad \text{with} \quad \varepsilon \leq 1/5 \tag{3}$$

and any n,

$$\mathsf{E}_p|C(0) \cap \Pi_n| > 1. \tag{4}$$

Proof. Fix ε, p as in (3), and write $\Pr$ and E for $\Pr_p$ and E_p. For B_i as above,

$$\Pr(B_i) > |\Sigma_i| p^{n_i} - \binom{|\Sigma_i|}{2} p^{2n_i} > \frac{1}{2}(1-\varepsilon)^{n_i}. \tag{5}$$

Let $C_i = \{H[\Sigma_i]\ \text{ is connected}\}$ and $\xi_i = 1 - \Pr(C_i)$. A routine calculation gives (say)

$$\xi_i < (1+o(1))|\Sigma_i|((1+\varepsilon)/2)^{|\Sigma_i|-1} < (3/4)^{2^{n_i}}. \tag{6}$$

Set $D_i = \{0 \longleftrightarrow x \quad \forall x \in \Sigma_i\}$. Since $D_i \supseteq \cap_{j=1}^i (B_j \cap C_j)$ and the B_j's and C_j's are independent, (5) gives

$$\Pr(D_i) > 2^{-i}(1-\varepsilon)^{m_i} \prod_{j=1}^{i} (1-\xi_j).$$

Now let $n = m_i - k$ with $0 \leq k < n_i$. For $v \in \Pi_n$,

$$\Pr(0 \longleftrightarrow v) > \Pr(D_{i-1}) \max\{p^{n_i-k}, \Pr(B_i)(1-\xi_i)p^k\} =: \beta_n. \tag{7}$$

The first expression in brackets is of course the probability that the "direct" (i.e. unique shortest) path from Σ_{i-1} to v is open. The second is a lower bound on $\Pr(B_i \cap C_i \cap F_v)$, where F_v is the event that the direct path from Σ_i to v is open (note $\Pr(F_v | B_i \cap C_i) \geq \Pr(F_v)$).
Then (for instance) $\beta_n > (1/2)(1-\varepsilon)^{4n_i/3} 2^{-n_i/2}$ (since $\Pr(D_{i-1}) > (1-\varepsilon)^{n_i/3}$ and the maximum in (7) is at least $(p^{n_i} \Pr(B_i)(1-\xi_i))^{1/2} > (1/2)(1-\varepsilon)^{n_i} 2^{-n_i/2}$), yielding, finally,

$$\mathsf{E}|C(0) \cap \Pi_n| > |\Pi_n| \beta_n = 2^{n_i} \beta_n > (1/2)(1-\varepsilon)^{4n_i/3} 2^{n_i/2} > 1.$$

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Remark. As Russ Lyons points out, our construction can easily be modified to have bounded degree; roughly: replace the cliques by expanders and, for each $v \in \Sigma_{i-1}$, replace the paths from v to Σ_i by a tree of the same depth and with the same leaves, having large (but constant) degrees near the root and paths following this initial stage. We omit the details.

3 A possible substitute

Following [5], we regard equality of $p_{\text{cut}}(G)$ and $p_c(G)$ as saying that the first moment bound (1) correctly identifies the critical probability for bond percolation on G. The construction of Section 2 violates this in the natural way, by arranging that the events $\{0 \longleftrightarrow v\}$, v in some cutset Π, are heavily dependent, so that $|C(0) \cap \Pi| = \sum_{v \in \Pi} \mathbf{1}_{\{0 \longleftrightarrow v\}}$, while usually zero, has a large expectation. This is achieved by making connections *within* Π easy, so that once we do reach at least one $v \in \Pi$ (an unlikely event), we then typically reach the rest of Π *via* v. So a natural way to try to salvage the conjecture is to disallow the effects of such "internal" connections, as follows.

For Π a cutset and $v \in \Pi$, let $A(v, \Pi)$ be the event that there is an open $(0, v)$-path disjoint from $\Pi \setminus \{v\}$. Then we again have a first moment bound,

$$\Pr(0 \longleftrightarrow \infty) < \Pr(0 \longleftrightarrow \Pi) \leq \sum_{v \in \Pi} \Pr(A(v, \Pi))$$

for each Π, implying

$$p_c \geq p'_{\text{cut}} := \sup\{p : \inf_{\Pi} \sum_{v \in \Pi} \Pr(A(v, \Pi)) = 0\}.$$

(Of course $p'_{\text{cut}} \geq p_{\text{cut}}$.)

Question 3.1 *Could it be that $p'_{\text{cut}}(G) = p_c(G)$ for all G?*

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