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SUPERPROCESS APPROXIMATION FOR A SPATIALLY HOMOGENEOUS BRANCHING WALK

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Abstract
We present an alternative particle picture for super-stable motion. It is based on a non-local branching mechanism in discrete time and only trivial space motion.

1 Introduction

This note is an observation of the limit behavior of a basic branching walk model. We consider systems of particles on the real line which at each discrete time point change state according to the following branching mechanism: Each particle, independently of other, is removed and replaced by a random number of new offspring particles which are spread out at random positions such that, losely speaking, they are centered around their parent. Within the cluster of offspring particles the tail behaviour in the distributions of the number of particles and their positions is parametrized, whereas the dependence structure between these quantities is general. We restrict however to the critical case of unit mean number of offspring.
Our result is that under appropriate scaling of time, space and mass such particle systems converge weakly to the measure-valued branching process known as the super-stable motion. On the particle level we may take the view of having a trivial spatial motion (particles are fixed inbetween the discrete branching points) and non-local branching, whereas after rescaling and letting the mass of a single particle go to zero we obtain a superprocess corresponding to a stable space motion and local branching. Therefore the superprocess particle picture we present here is qualitatively different from the approximations usually applied.

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The higher-dimensional cases of discrete time branching and offspring particles spread around in R^d, $d \geq 2$, can be treated in exactly the same way using obvious modifications. Hence we restrict throughout to the one-dimensional case $d = 1$. The result is realized along a standard log-Laplace functional iteration technique to yield convergence of finite-dimensional distributions, and by applying Aldous' criterion to obtain tightness.

2 Model

The dynamics of the system is determined by only the quantities we denote by

$$N\,; \quad \tau^1, \ldots, \tau^N .$$

Here $N \geq 0$ is an integer-valued random variable which gives the number of offspring at each branching event and $(\tau^j)_{j\geq 1}$ is a sequence of real random variables, such that if $N \geq 1$ then $\tau^1, \ldots, \tau^N$ are the shifts in position of the offspring particles relative to the parent particle (in some arbitrary ordering). We may suppose that N and $(\tau^j)_{j\geq 1}$ are defined on a common probability space equipped with a probability measure P. Introduce the notations

$$N(u) = \sum_{j=1}^{N} 1_{\{\tau^j \leq u\}}\,; \quad A(u) = EN(u)$$

and

$$\bar{N}(u) = N - N(u) + N(-u)\,, \qquad \Phi(z) = E\{(1-z)^N - 1 + Nz\}\,,$$

where E refers to expectation with respect to P.
To justify the interpretations we have given to N and $(\tau^j)_{j\geq 1}$ we may construct the particle system as follows. We let

$$Z_n = \sum_j \delta_{z^j}$$

denote the configuration of particles present at time n. The summation is either void or runs over particles at positions $z^1, z^2, \ldots$. We write $Z_n(dx)$ for the corresponding measure and in the particular case when $Z_0 = \delta_x$ we write $Z_n^x(du)$. By the Ionescu-Tulcea Theorem there exists a probability space on which is defined a probability measure $\mathbf{P}$ such that the sequence Z_n is measurable and satisfies under $\mathbf{P}$, for each $n \geq 0$, the branching property

$$Z_{n+1}^x(du) = \sum_{j=1}^{N} Z_n^{x+\tau^j}(du)\,, \quad Z_0^x = \delta_x\,, \tag{2.1}$$

where the summands are independent given their initial positions. The state space of Z_n may be taken as the set M of finite measures on R. By the extension $Z_t := Z_{[t]}$ for $t \geq 0$, we obtain a continuous time process which for practical purposes is Markov in the sense that we may identify Z_t with the Markov process (Z_t, t). We consider the paths of Z_t as elements in the space $D(I, M)$ of cadlag paths from $I = [0, \infty)$ or $I = [0, T]$ to M. With the Prohorov metric on M we obtain a complete metric space and may furnish $D(I, M)$ with the corresponding Skorokhod topology.
See e.g. Matthes, Kerstan and Mecke (1978), Chapter 12, for generalities on one-dimensional spatially homogeneous branching processes in discrete time with a critical cluster. We are not aware of any limit theorems, however, which are similar to the result presented below.

3 Assumptions and Result

We will state five assumptions $A1$ to $A5$. First of all we impose the criticality condition

$$EN = 1. \tag{A1}$$

Fix two parameters α and β such that $0 < \alpha \leq 2$ and $0 < \beta \leq 1$. We assume next that there exist functions L_1 and L_2, slowly varying at infinity, such that

$$\Phi(z) = z^{1+\beta} L_1(1/z) \tag{A2}$$

and

$$\int_{-x}^{x} y^2 \, dA(y) = x^{2-\alpha} L_2(x) \,. \tag{A3}$$

Moreover, when $\alpha < 2$ we assume

$$(1 - A(x))/A(-x) \to d \,, \quad x \to \infty \,, \tag{A4}$$

where $0 \leq d \leq \infty$.
We write $x_n \sim y_n$ if $x_n/y_n \to 1$ as $n \to \infty$. Let a_n denote a sequence which satisfies

$$a_n^{\alpha}/L_2(a_n) \sim n \,, \tag{3.1}$$

and define also the sequence

$$T = T_n = n^{\beta}/L_1(n) \,. \tag{3.2}$$

This sequence plays a crucial role in our limit result as the time scaling sequence. Therefore it appears in the sequel mostly as subindex. To simplify notations we use T and T_n interchangeably.
Our final assumption is that for each $\delta > 0$,

$$P(\bar{N}(\delta a_T) > n) = o(1/nT) \,, \quad n \to \infty \,. \tag{A5}$$

(Writing $1/nT = (nT)^{-1}$) We turn to some consequences of and remarks regarding these assumtions. First of all, assumption $A1$ implies that $A(\cdot)$ is a distribution function on R. This allows us to introduce on the probability space where P is defined an auxiliary sequence of i.i.d. random variables $(\ell_j)_{j \geq 1}$ such that

$$P(\ell_1 \leq u) = A(u) \,,$$

and define their partial sums

$$S_n = S_0 + \ell_1 + \ldots + \ell_n \,, \quad S_0 = x \,. \tag{3.3}$$

Now by $A3$ and $A4$, the weak convergence of processes in $D(I, R)$,

$$\frac{1}{a_n} S_{[nt]} \xrightarrow{w} \quad \alpha\text{-stable limit} \quad \xi_t \,, \tag{3.4}$$

holds as $n \to \infty$, see e.g. Gikhman and Skorokhod (1969), Section 9.6.
Regarding the sequence T we see that

$$\Phi(1/n) = 1/nT \,,$$

and we note the tail estimate

$$P(N > n) = \mathcal{O}(1/nT)\,, \tag{3.5}$$

which follows from A2. We remark furthermore that when $\alpha < 2$ then assumption $A3$ is equivalent to

$$E\bar{N}(x) = 1 - A(x) + A(-x) \sim \frac{2-\alpha}{\alpha} x^{-\alpha} L_2(x)\,,$$

thus

$$E\bar{N}(\delta a_T) \sim \frac{2-\alpha}{\alpha}\,\delta^{-\alpha} T^{-1}\,, \tag{3.6}$$

and similarly, in the case $\alpha = 2$,

$$E\bar{N}(\delta a_T) \leq \text{const}\,\delta^{-2} T^{-1}\,, \tag{3.7}$$

which may give some comprehension for assumption $A5$.
We are now ready to state our result. To obtain the limit theorem for Z_t we scale time by T and space by a_T. Disregarding for the moment slowly varying functions, the order of magnitude of the scaling factors are n^β for T and $n^{\beta/\alpha}$ for a_T. The limiting process will be an (α,β)-superprocess realized in $D(I,M)$, which is based on the α-stable process ξ_t in (3.4). It is defined by the log-Laplace functions

$$V_t f(x) := \log E_x e^{-\langle f, X_t\rangle}$$

being for each bounded continuous function f the unique solution of the nonlinear integral equation

$$V_t f(x) = E_x\Big[f(\xi_t) - \int_0^t V_{t-u} f(\xi_u)^{1+\beta}\,du\Big], \tag{3.8}$$

where E_x denotes expectation with respect to ξ_t given $\xi_0 = x$.

Theorem 1 *Suppose the assumptions $A1 - A5$ hold. Let X_0 belong to M and assume that as n, and hence T, tends to infinity, then*

$$Z_0(a_T\,dx) \to X_0(dx)$$

in the sense of weak convergence in M. Then the weak convergence of processes in $D(I,M)$,

$$Z_{Tt}(a_T\,dx) \xrightarrow{w} X_t(dx)\,,$$

holds as $n \to \infty$, where the limit X_t is the (α,β)-superprocess defined by (3.8).

4 Integral equation

Our analysis of the behaviour of Z_n under the scaling in the theorem will be based on an integral equation for the function defined as

$$Q_n f(x) = 1 - Ee^{-\langle f, Z_n^x\rangle}\,, \quad n \geq 0\,,$$

where f is a nonnegative test function. For functions g on R with $0 \leq g(x) \leq 1$, define

$$\Psi[g](x) = E\Big\{\prod_{j=1}^{N}(1 - g(x+\tau^j)) - 1 + \sum_{j=1}^{N} g(x+\tau^j)\Big\}\,, \tag{4.1}$$

interpreting the expression inside brackets as zero if $N = 0$. The quantity $\Psi[g](x)$ is a functional of the collection $(\tau^j)_{j\geq 1}$ conditional upon them being centered around x.
By (2.1),

$$1 - Q_{n+1}f(x) = E\prod_{j=1}^{N}\big(1 - Q_n f(x+\tau^j)\big)\,.$$

Hence

$$Q_{n+1}f(x) = E\sum_{j=1}^{N} Q_n f(x+\tau^j) - \Psi\big[Q_n f\big](x)\,.$$

However,

$$\begin{aligned} E\sum_{j=1}^{N} Q_n f(x+\tau^j) &= E\int_R Q_n f(x+u)\,dN(u) \\ &= \int_R Q_n f(x+u)\,dA(u) = EQ_n f(x+\ell_1) \end{aligned}$$

so we have

$$Q_{n+1}f(x) = EQ_n f(x+\ell_1) - \Psi\big[Q_n f\big](x)\,.$$

An iteration of the preceeding relation yields

$$Q_{n+1}f(x) = EQ_{n-1}f(x+\ell_1+\ell_2) - E\Psi\big[Q_{n-1}f\big](x+\ell_1) - \Psi\big[Q_n f\big](x)$$

and after n such steps

$$Q_{n+1}f(x) = E_x\Big\{Q_0 f(S_{n+1}) - \sum_{k=1}^{n+1}\Psi\big[Q_{n+1-k}f\big](S_{k-1})\Big\}\,,$$

where the sequence S_n is defined in (3.3) and E_x denotes the conditional expectation given that $S_0 = x$. Of course, $Q_0 f(x) = 1 - e^{-f(x)}$.
In order to write this relation in integral form introduce

$$H(t) = \sum_{j=1}^{\infty} 1_{\{j\leq t\}}\,, \quad t\geq 0\,,$$

and make the extensions to continuous time: $S_t = S_{[t]}$ and $Q_t f(x) = Q_{[t]}f(x)$. Then we have

$$Q_t f(x) = E_x\Big\{1 - e^{-f(S_t)} - \int_0^t \Psi\big[Q_{t-u}f\big](S_{u-})\,dH(u)\Big\}\,. \tag{4.2}$$

5 Three Technical Lemmas

We introduce a cut-off version of the functional $\Psi[g](x)$ defined in (4.1) as follows,

$$g\mapsto \Psi^u[g](x) := E\Big\{\prod_{|\tau_j|\leq u}(1 - g(x+\tau^j)) - 1 + \sum_{|\tau_j|\leq u} g(x+\tau^j)\Big\}\,. \tag{5.1}$$

Lemma 1 *If the function f satisfies $0 \le f \le f_0 \le 1$ for some constant f_0, then for all real x and all nonnegative u, c_1, c_2 with $c_1 \ge c_2$,*

$$0 \le \Psi[f](x) - \Psi^u[f](x) \le \Delta(u, f_0, c_1, c_2)$$

with

$$\Delta(u, f_0, c_1, c_2) = 2c_1 f_0^2 E\bar{N}(u) + 2P(\bar{N}(u) > c_2) + c_2 f_0 P(N > c_1)\,.$$

Proof:
The first inequality follows from the more general monotonicity property

$$0 \le f \le g \le 1 \quad \Longrightarrow \quad 0 \le \Psi[f](x) \le \Psi[g](x) \le 1\,. \tag{5.2}$$

To see this, observe that for constants $0 \le f_j \le 1$,

$$\sum_{j=1}^{n} f_j - 1 + \prod_{j=1}^{n}(1 - f_j) = \sum_{j=1}^{n-1} f_j \left\{1 - (1 - f_{j+1}) \ldots (1 - f_n)\right\}, \quad n \ge 2\,, \tag{5.3}$$

which is monotone in each f_j. By A1, $\Psi[1](x) = E[N - 1; N > 0] = P(N = 0) \le 1$.
Next,

$$0 \le \Psi[f](x) - \Psi^u[f](x) = \Psi_1^u[f](x) + \Psi_2^u[f](x)\,,$$

where

$$\begin{aligned}
\Psi_1^u[f](x) &= E\Big\{1 - \prod_{|\tau^j|>u} (1 - f(x + \tau^j))\Big\}\Big\{1 - \prod_{|\tau^j|\le u} (1 - f(x + \tau^j))\Big\}, \\
\Psi_2^u[f](x) &= E\Big\{\prod_{|\tau^j|>u} (1 - f(x + \tau^j)) - 1 + \sum_{|\tau^j|>u} f(x + \tau^j)\Big\}.
\end{aligned}$$

By straightforward estimates,

$$\begin{aligned}
\Psi_1^u[f](x) &\le E\Big\{1 - (1 - f_0)^{\bar{N}(u)}\Big\}\Big\{1 - (1 - f_0)^{N - \bar{N}(u)}\Big\} \\
&\le f_0^2 E\bar{N}(u)(N - \bar{N}(u))1_{\{N \le c_1\}} + f_0 E\bar{N}(u)1_{\{N > c_1,\, \bar{N}(u) \le c_2\}} \\
&\qquad + P(N > c_1, \bar{N}(u) > c_2) \\
&\le c_1 f_0^2 E\bar{N}(u) + c_2 f_0 P(N > c_1) + P(\bar{N}(u) > c_2)\,.
\end{aligned} \tag{5.4}$$

It is not difficult to prove by induction from (5.3)

$$\prod_{j=1}^{n}(1 - f_j) - 1 + \sum_{j=1}^{n} f_j \le \Big[\sum_{j=1}^{n} f_j\Big]^2 \le f_0^2 n^2$$

(the simpler bound $\le f_0 n$ is not appropriate for our purpose). Hence

$$\Psi_2^u[f](x) \le \max\left\{1, f_0^2 E\bar{N}(u)^2\right\},$$

and so

$$\begin{aligned}
\Psi_2^u[f](x) &\le P(\bar{N}(u) > c_1) + f_0^2 E\big[\bar{N}(u)^2, \bar{N}(u) \le c_1\big] \\
&\le P(\bar{N}(u) > c_2) + c_1 f_0^2 E\bar{N}(u)\,.
\end{aligned} \tag{5.5}$$

The estimates in (5.4) and (5.5) are both independent of x and add up to $\Delta(u, f_0, c_1, c_2)$. ■

Recall the sequences a_n and $T = T_n$ defined in (3.1) and (3.2). Along with the previously introduced functionals $\Psi[g]$ in (4.1) and $\Psi^u[g]$ in (5.1) we consider the following scaled version,

$$\Psi_T[g](x) = E\Big\{ \prod_{j=1}^{N}(1 - g(x + \frac{\tau^j}{a_T}) - 1 + \sum_{j=1}^{N} g(x + \frac{\tau^j}{a_T})\Big\} . \tag{5.6}$$

Lemma 2 *For any bounded function f such that*

$$h(\delta) := \sup_x \sup_{|y|\le\delta} \big|f(x+y) - f(x)\big| \to 0 , \quad \textit{as} \quad \delta \to 0+, \tag{5.7}$$

we have

$$\lim_{n\to\infty} nT\,\Psi_T[f/n](x) = f(x)^{1+\beta} \quad \textit{uniformly in } x .$$

Proof:
We apply the monotonicity relation (5.2) and Lemma 1 to $\Psi_T[f]$, assuming n is large enough that we can take $f_0 = \|f\|/n < 1$. Then for any $u > 0$ and $c_1 \ge c_2 > 0$,

$$\begin{aligned} \Psi_T[f/n](x) &\le \Psi_T^u[f/n](x) + \Delta(u, \|f\|n^{-1}, c_1, c_2) \\ &\le \Phi[(f(x) + h(u/a_T))/n] + \Delta(u, \|f\|n^{-1}, c_1, c_2) . \end{aligned}$$

Choose, for $\delta > 0$ and $0 < \varepsilon \le 1$, $u = \delta a_T$, $c_1 = \varepsilon n$ and $c_2 = \varepsilon^{2+\beta} n$. Then

$$\Psi_T[f/n](x) \le \Phi[(f(x) + h_x(\delta))/n] + \Delta(\delta a_T, \|f\|n^{-1}, \varepsilon n, \varepsilon^{2+\beta} n) .$$

Similarly, again working with $u = \delta a_T$,

$$\Psi_T[f/n](x) \ge \Psi_T^u[f/n](x) \ge \Phi[(f(x) - h(\delta))^+/n] .$$

Hence,

$$\begin{aligned} \Phi[(f(x) - h(\delta))^+/n] &\le \Psi_T[f/n](x) \\ &\le \Phi[(f(x) + h(\delta))/n] + \Delta(\delta a_T, \|f\|n^{-1}, \varepsilon n, \varepsilon^{2+\beta} n) . \end{aligned}$$

By (3.6–7), assumption A5 and (3.5), the error term Δ is uniform in x and satisfies

$$\begin{aligned} \Delta(\delta a_T, \|f\|n^{-1}, \varepsilon n, \varepsilon^{2+\beta} n) &= 2\varepsilon\|f\|^2 n^{-2} E\bar{N}(\delta a_T) + 2P(\bar{N}(\delta a_T) > \varepsilon^{2+\beta} n) \\ &\qquad + \varepsilon^{2+\beta}\|f\| P(N > \varepsilon n) \\ &= \varepsilon\,\mathcal{O}(1/nT) + o(1/nT) , \end{aligned}$$

so

$$nT\Delta(\delta a_T, \|f\|n^{-1}, \varepsilon, \varepsilon^{2+\beta} n) = o(1) + \varepsilon\,\mathcal{O}(1) .$$

Therefore

$$\begin{aligned} 0 \vee (f(x) - h(\delta))^{1+\beta} L_-^{(n)} &\le nT\Phi_T[f/n](x) \\ &\le (f(x) + h(\delta))^{1+\beta} L_+^{(n)} + o(1) + \varepsilon\,\mathcal{O}(1) , \end{aligned}$$

where $L_\pm^{(n)} \sim 1$. Take $n \to \infty$, $\delta \to 0$ and $\varepsilon \to 0$ to finish the proof of the lemma. ■

Lemma 3 *For each pair of bounded functions f, g, there is n_0 and a constant C such that*

$$nT\big\|\Psi_T[f/n] - \Psi_T[g/n]\big\| \le C\|f-g\|, \quad n \ge n_0.$$

Proof:
Note first that the expression on the left side of the inequality is bounded by $\|f\|^{1+\beta} + \|g\|^{1+\beta}$ times some constant related to L_1. We prove the lemma for $\beta < 1$. The case $\beta = 1$ is simpler. The lemma holds with identical proof also for $\Psi[f]$. In what follows we hence drop the subscript T.
We may represent $\Psi[f]$ in terms of stochastic integrals with respect to the point process $N(u)$ as

$$\Psi[f](x) = E\Big\{e^{\int \log(1-f(x+s))\,N(ds)} - 1 + \int f(x+s)\,N(ds)\Big\}.$$

Hence

$$\begin{aligned}\Big|\Psi[f/n](x) - \Psi[g/n](x)\Big| &\le E\Big|\int_{\int -\log(1-g(x+s)/n)\,N(ds)}^{\int -\log(1-f(x+s)/n)\,N(ds)} (1-e^{-\lambda})\,d\lambda\Big| \\ &\quad + E\int \Big|H[f/n](x+s) - H[g/n](x+s)\Big|N(ds), \qquad (5.8)\end{aligned}$$

where

$$H[f](x) = -\log(1-f(x)) - f(x).$$

For $n \ge 2(\|f\| + \|g\|)$ we have

$$\|\log(1-f/n) - \log(1-g/n)\| \le 2\|f-g\|/n$$

and therefore the width of the λ-interval of integration in the first term on the right side of (5.8) is bounded by $2\|f-g\|N/n$. It is then easy to see, also using the monotonicity of the integrand $1-e^{-\lambda}$, that

$$\begin{aligned}E\Big|&\int_{-\int \log(1-g(x+s)/n)\,N(ds)}^{-\int \log(1-f(x+s)/n)\,N(ds)} (1-e^{-\lambda})\,d\lambda\Big| \\ &\le E\int_{2(\|f\|+\|g\|)N/n}^{2(\|f\|+\|g\|+\|f-g\|)N/n} (1-e^{-\lambda})\,d\lambda \\ &= \int_0^\infty (1-e^{-\lambda})P(d_1N < \lambda n \le d_2N)\,d\lambda,\end{aligned}$$

where

$$d_1 = 2(\|f\| + \|g\|), \quad d_2 = 2(\|f\| + \|g\| + \|f-g\|).$$

By (3.5) and the property of the slowly varying function L_1,

$$nT(\lambda n/d_i)^{1+\beta}P(N > \lambda n/d_i) = \mathcal{O}(1), \quad n \to \infty.$$

Hence

$$\begin{aligned}nT&\int_0^\infty (1-e^{-\lambda})\,P(d_1N < \lambda n \le d_2N)\,d\lambda \\ &\le \text{const}\int_0^\infty \lambda^{-(1+\beta)}(1-e^{-\lambda})\,d\lambda\,(d_2^{1+\beta} - d_1^{1+\beta}) \\ &\le \text{const}\,\|f-g\|.\end{aligned}$$

This together with the estimate

$$nT\|H[f/n]-H[g/n]\| \le \frac{2T}{n}(\|f\|+\|g\|)\,\|f-g\|,$$

which again holds for $n \ge 2(\|f\|+\|g\|)$, and (5.8) shows that

$$nT\|\Psi[f/n]-\Psi[g/n]\| \le \text{const}\,\|f-g\|$$

for n sufficiently large, which is the statement of the lemma. ∎

6 Integral Equation for the Scaled Process

We investigate the limit behaviour in a system of particles where individual particle mass is scaled by $1/n$, the overall density is preserved and the space-time scaling (a_T, T) is applied. Put

$$f^{(n)}(x) = f(x/a_T).$$

We are led to study the function

$$V_t^{(n)} f(x) := nQ_{Tt}(f^{(n)}/n)(a_T x)\,.$$

By (4.2),

$$\begin{aligned} nQ_{Tt}(f^{(n)}/n)(a_T x) &= E_{a_T x}\Big\{n(1-e^{-f(S_{Tt}/a_T)/n}) \\ &\qquad - \int_0^{Tt} n\Psi\big[Q_{tT-u}(f^{(n)}/n)\big](S_{u-})\,dH(u)\Big\}\,. \end{aligned}$$

Put

$$\xi_t^{(n)} = a_T^{-1} S_{Tt}\,,$$

and write now, with slight abuse of notation, E_x for conditional expectation given $\xi_0^{(n)} = x$. Then

$$\begin{aligned} nQ_{Tt}(f^{(n)}/n)(a_T x) &= E_x\Big\{n(1-e^{-f(\xi_t^{(n)})/n}) \\ &\qquad - \int_0^{t} n\Psi\big[Q_{T(t-u)}(f^{(n)}/n)\big](a_T \xi_{u-}^{(n)})\,dH(uT)\Big\}\,. \end{aligned}$$

However,

$$\Psi\big[Q_{T(t-u)}(f^{(n)}/n)\big](a_T x) = \Psi_T\big[n^{-1}V_{t-u}^{(n)} f\big](x).$$

Thus

$$\begin{aligned} V_t^{(n)} f(x) &= E_x\Big\{n(1-e^{-f(\xi_t^{(n)})/n}) \\ &\qquad - \int_0^{t} nT\,\Psi_T\big[n^{-1}V_{t-u}^{(n)} f\big](\xi_{u-}^{(n)})\,T^{-1}dH(uT)\Big\}. \end{aligned} \tag{6.1}$$

Proof of the Theorem

The main point in the proof is to show that $V_t^{(n)}f$ converges uniformly to V_tf in the sense

$$\sup_{t\le t_0}\left\|V_t^{(n)}f - V_tf\right\| \to 0, \quad n\to\infty. \tag{6.2}$$

This proves the convergence of one-dimensional distributions. The uniformity in x accomodates for a general initial measure X_0, see e.g. Kaj and Sagitov (1998) for details in a similar case. The extension to finite-dimensional distributions is straightforward using recursion relations in the familiar manner for Markov branching processes. We omit this part of the proof.
The reason for requiring that the limit (6.2) holds uniformly in t is that it enables us to prove tightness by means of Aldous criterion, e.g. along the lines of Dawson (1993), Section 4.6, Sagitov (1994) or Kaj and Sagitov (1998), Section 4.4. We do not repeat in this proof the very similar arguments needed for the present case. We do point out, however, that it suffices to prove (6.2) for some $t_0>0$. This follows easily from the semigroup properties of V_tf and $-n\log(1-n^{-1}V_t^{(n)}f)$ together with

$$\|-n\log(1-n^{-1}V_t^{(n)}f)-V_t^{(n)}f\|\le\|f\|^2/n$$

for large n.
Consider $t_0>0$, let $t\in[0,t_0]$ and let f denote a uniformly continuous function on R. By (3.8) and (6.1),

$$\left|V_t^{(n)}f(x)-V_tf(x)\right|\le I_1+I_2+I_3+I_4\,,$$

where

$$\begin{aligned}
I_1 &= \left|E_x\Big\{n\big(1-e^{-f(\xi_t^{(n)})/n}\big)-f(\xi_t)\Big\}\right|,\\
I_2 &= E_x\int_0^t\left|nT\Psi_T\big[n^{-1}V_{t-s}^{(n)}f\big](\xi_{s-}^{(n)})-nT\Psi_T\big[n^{-1}V_{t-s}f\big](\xi_{s-}^{(n)})\right|T^{-1}\,dH(sT),\\
I_3 &= E_x\int_0^t\left|nT\Psi\big[n^{-1}V_{t-s}f\big](\xi_{s-}^{(n)})-V_{t-s}f(\xi_{s-}^{(n)})^{1+\beta}\right|T^{-1}\,dH(sT),\\
I_4 &= \left|E_x\int_0^t V_{t-s}f(\xi_{s-}^{(n)})^{1+\beta}\,T^{-1}\,dH(sT)-\int_0^t V_{t-s}f(\xi_s)^{1+\beta}\,ds\right|.
\end{aligned}$$

For convenience we use at some occasions below the alternate semigroup notation $S_tf(x)=E_xf(\xi_t)$, and note the strong continuity

$$\lim_{t\to0}\left\|S_tf-f\right\|=0, \tag{6.3}$$

valid since all stable processes have the Feller property.
For $n\ge2\|f\|$,

$$I_1\le\frac{1}{n}\|f\|^2+\left|E_x[f(\xi_t^{(n)})-f(\xi_t)]\right|.$$

Let $\Delta_c(x)$ denote the standard modulus of continuity in the trajectory space $D(I,R)$ for $\xi_t^{(n)}$ and ξ_t, see e.g. Gikhman and Skorokhod (1969), Section 9.5. Then

$$\begin{aligned}
\left|E_x[f(\xi_t^{(n)})-f(\xi_t)]\right| &\le \left|E_x[f(\xi_t^{(n)})-f(x)]\right|+\left|S_tf(x)-f(x)\right|\\
&\le \sup_{t\le t_0}\sup_x\left|E_x[f(\xi_t^{(n)})-f(x)\,;\,\Delta_{t_0}(\xi^{(n)})\le\varepsilon]\right|\\
&\qquad +2\|f\|\limsup_{n\to\infty}P\big(\Delta_{t_0}(\xi^{(n)})>\varepsilon\big)+\sup_{t\le t_0}\left\|S_tf-f\right\|.
\end{aligned}$$

We now conclude from the tightness property in (3.4), the uniform continuity of f and (6.3) that we can find n_0 so large and t_0 so small that the right side in the above inequality, and hence I_1, is bounded uniformly in x and t as $n \to \infty$.
By Lemma 3,

$$I_2 \leq C \int_0^t \left\| V_{t-s}^{(n)} f - V_{t-s} f \right\| T^{-1} dH(sT) \leq \frac{C}{2T} + C \int_0^t \left\| V_s^{(n)} f - V_s f \right\| ds.$$

In order to apply Lemma 2 for the purpose of estimating I_3, we must verify that $V_t f(x)$ is uniformly continuous, i.e. satisfies (5.4). However, it follows from (3.8) that

$$\begin{aligned} |V_t f(x+y) - V_t f(x)| &\leq |S_t f(x+y) - S_t f(x)| \\ &+ \int_0^t |V_{t-u} f(x+y+\xi_u - \xi_0)^{1+\beta} - V_{t-u} f(x+\xi_u-\xi_0)^{1+\beta}| \, du, \end{aligned}$$

and hence that the function

$$H_t(\delta) := \sup_x \sup_{|y| \leq \delta} |V_t f(x+y) - V_t f(x)|$$

satisfies

$$\begin{aligned} H_t(\delta) &\leq \sup_x \sup_{|y|\leq\delta} |S_t f(x+y) - S_t f(x)| \\ &\quad + \int_0^t \sup_x \sup_{|y|\leq\delta} |V_t f(x+y)^{1+\beta} - V_t f(x)^{1+\beta}| \, du \\ &\leq \sup_x \sup_{|y|\leq\delta} |S_t f(x+y) - S_t f(x)| + (2\|f\|)^\beta \int_0^t H_{t-s}(\delta) \, du. \end{aligned}$$

Since ξ has independent increments and f is uniformly continuous the property (5.4) holds for $S_t f$. But it is then immediate from the previous inequality and Gronwall's Lemma that (5.4) also holds for $V_t f$.
Inspecting the proof of Lemma 2 we see moreover that the error terms only depend on g via its norm $\|g\|$. Since $V_t f \leq \|f\|$ for all t we obtain

$$\begin{aligned} I_3 &\leq \int_0^t \left\| nT\Psi\big[n^{-1} V_{t-s} f\big] - V_{t-s} f^{1+\beta} \right\| T^{-1} \, dH(sT) \\ &\leq t_0 \sup_{s \leq t_0} \|nT\Psi\big[n^{-1} V_s f\big] - V_s f^{1+\beta}\| \to 0, \quad n \to \infty. \end{aligned}$$

Finally, for the term I_4 we use

$$I_4 \leq \|f\|^{1+\beta} \frac{1}{2T} + \left| E_x \int_0^t V_{t-s} f(\xi_s^{(n)})^{1+\beta} - V_{t-s} f(\xi_s)^{1+\beta} \, ds \right|.$$

¿From the just proven uniform continuity of $V_t f$ and the fact that

$$\xi \mapsto \int_0^t V_{t-s} f(\xi_s)^{1+\beta} \, ds$$

is a continuous map we conclude that I_4 can be made uniformly small.

Summing up, given $\varepsilon > 0$, we have found a function $\delta_{t_0}^{(n)} \to 0$, $n \to \infty$ uniformly in x, such that for all $t \le t_0$

$$\left|V_t^{(n)} f(x) - V_t f(x)\right| \le \delta_{t_0}^{(n)} + C \int_0^t \|V_s^{(n)} f - V_s f\|\, ds,$$

and therefore

$$\sup_{t \le t_0} \|V_t^{(n)} f - V_t f\| \le \delta_{t_0}^{(n)} / (1 - Ct_0) < \varepsilon$$

for appropriately small t_0 and sufficiently large n, which finishes the proof of (6.2) and hence of the theorem. ■

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