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AN EXPONENTIAL MARTINGALE EQUATION

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Abstract

We prove an existence of a unique solution of an exponential martingale equation in the class of BMO martingales. The solution is used to characterize optimal martingale measures.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\mathbf{F} = (\mathcal{F}_t, t \in [0,T])$. We assume that all local martingales with respect to $\mathbf{F}$ are continuous. Here T is a fixed time horizon and $\mathcal{F} = \mathcal{F}_T$. Let $\mathcal{M}$ be a stable subspace of the space of square integrable martingales H^2. Then its ordinary orthogonal $\mathcal{M}^\perp$ is a stable subspace and any element of $\mathcal{M}$ is strongly orthogonal to any element of $\mathcal{M}^\perp$ (see, e.g. [3], [8]).
We consider the following exponential equation

$$\mathcal{E}_T(m)\mathcal{E}_T^\alpha(m^\perp) = c\exp\{\eta\}, \tag{1}$$

where η is a given F_T-measurable random variable and α is a given real number. A solution of equation (1) is a triple $(c, m, m^\perp)$, where c is strictly positive constant, $m \in \mathcal{M}$ and $m^\perp \in \mathcal{M}^\perp$. Here $\mathcal{E}(X)$ is the Doleans-Dade exponential of X.
It is evident that if $\alpha = 1$ then equation (1) admits an "explicit" solution. E.g., if $\alpha = 1$ and η is bounded, then using the unique decomposition of the martingale $E(\exp\{\eta\}/F_t)$

$$E(\exp\{\eta\}/F_t) = E\exp\{\eta\} + m_t(\eta) + m_t^\perp(\eta), \quad m(\eta) \in \mathcal{M}, \quad m^\perp(\eta) \in \mathcal{M}^\perp, \tag{2}$$

it is easy to verify that the triple $c = \frac{1}{E\exp\{\eta\}}$,

$$m_t = \int_0^t \frac{1}{E(\exp\{\eta\}/F_s)} dm_s(\eta), \quad m_t^\perp = \int_0^t \frac{1}{E(\exp\{\eta\}/F_s)} dm_s^\perp(\eta)$$

satisfies equation (1). Note also that if $\alpha = 0$ then a solution of (1) does not exist in general. In particular, in this case equation (1) admits a solution only if η satisfies (2) with $m^\perp(\eta) = 0$. Other cases are much more involved and (1) is equivalent to solve a certain martingale backward equation with square generator.

Equations of such type are arising in mathematical finance and they are used to characterize optimal martingale measures (see, Biagini, Guasoni and Pratelli (2000), Mania and Tevzadze (2000), (2003),(2005)). Note that equation (1) can be applied also to the financial market models with infinitely many assets (see M. De Donno, P. Guasoni, M. Pratelli (2003)). In Biagini at al (2000) an exponential equation of the form

$$\frac{\mathcal{E}_T(\int_0^\cdot h_s dW_s)}{\mathcal{E}_T(\int_0^\cdot k_s dM_s)} = c\exp\{\int_0^T \lambda_s^2 ds\} \tag{3}$$

was considered (which corresponds to the case $\alpha = -1$). Assuming that any element of H^2 is representable as a sum of stochastic integrals $H\cdot W + K\cdot M$, where W is a Brownian motion and M is a martingale (not necessarily continuous) orthogonal to W, they identified the variance-optimal martingale measure as the solution of equation (3). In so-called extreme cases (already studied in Pham et al. (1998), Laurent and Pham (1999) using different methods), when the market price of risk λ is measurable with respect to the σ-algebras generated by W and M respectively, they gave explicit solutions of (3)) providing an explicit form for the density of the variance-optimal martingale measure. These extreme cases correspond in our setting to the following conditions on the random variable η:

$$\exp\{\eta\} = c(\eta) + m_T(\eta), \quad \text{for a constant} \quad c(\eta) \quad \text{and} \quad m_T(\eta) \in \mathcal{M}, \tag{4}$$

$$\exp\{\frac{1}{\alpha}\eta\} = c(\eta)^\perp + m_T(\eta)^\perp, \ \text{ for a constant } \ c(\eta)^\perp \ \text{ and } \ m(\eta)^\perp \in \mathcal{M}^\perp \tag{5}$$

respectively. It is easy to see that if (4) is satisfied then the triple $c = \frac{1}{c(\eta)}$, $m = (c(\eta) + m(\eta))^{-1} \cdot m(\eta)$, $m^\perp = 0$ solves equation (1) and under condition (5) equation (1) is satisfied if $m = 0$, $m^\perp = (c(\eta)^\perp + m(\eta)^\perp)^{-1} \cdot m(\eta)^\perp$ and $c = E^{-\alpha}\exp\{\frac{1}{\alpha}\eta\}$.

Our aim is to prove the existence of a unique solution of equation (1) for arbitrary $\alpha \neq 0$ and η of a general structure, assuming that it satisfies the following boundedness condition:

B) η is an F_T-measurable random variable of the form

$$\eta = \bar\eta + \gamma A_T, \tag{6}$$

where $\bar\eta \in L^\infty$, γ is a constant and $A = (A_t, t \in [0,T])$ is a continuous F-adapted increasing process such that

$$E(A_T - A_\tau/F_\tau) \le C$$

for all stopping times τ for a constant $C > 0$.

The main statement of the paper is the following

Theorem 1. Let condition B) be satisfied. Then there is a constant $\gamma_0 > 0$ such that for any $|\gamma| \le \gamma_0$ there exists a unique triple $(c, m, m^\perp)$, where $c \in R_+, m \in BMO \cap \mathcal{M}, m^\perp \in BMO \cap \mathcal{M}^\perp$, that satisfies equation (1).

In case $\alpha = -1$ and $\gamma = 0$ this theorem was proved in [14].
In P. Grandits and T. Rheinländer (2002), T. Rheinländer (2005) and D. Hobson (2004) minimal entropy and q-optimal martingale measures are studied using a fundamental representation equation of type

$$\frac{q}{2}K_T = M_T - \frac{q-1}{2}\langle M\rangle_T + L_T + \frac{1}{2} + c',$$

where $K_T = \int_0^T \lambda_t^2 dt$, $M_t = \int_0^t \varphi_u(dB_u + q\lambda_u du)$,$L_t = \int_0^t \xi_u dZ_u$, B and Z are independent Brownian motions and λ is fixed process. This equation is equivalent to (1) for suitable α, η and for the probability measure $d\tilde{P} = \mathcal{E}_T(-q\int_0^\cdot \lambda_s dM_s)dP$, since $\tilde{B}_t = B_t + q\int_0^t \lambda_s ds$ and Z_t are independent Brownian motions w.r.t. $\tilde{P}$. Assumption B) guarantees the solvability of the equation either if $\operatorname{ess\,sup}_\tau \tilde{E}\left(\int_\tau^T q^2\lambda_u^2 du|\mathcal{F}_\tau\right)$ is small enough or if K_T is bounded.
One can show that equation (1) is equivalent to the following semimartingale backward equation with the square generator

$$Y_t = Y_0 - \frac{\gamma}{2}A_t - \langle L\rangle_t - \frac{1}{\alpha}\langle L^\perp\rangle_t + L_t + L_t^\perp, \quad Y_T = \frac{1}{2}\bar{\eta}. \tag{7}$$

We show that there exists a unique triple $(Y, L, L^\perp)$, where Y is a bounded continuous semimartingale, $L \in BMO\cap\mathcal{M}$, $L^\perp \in BMO\cap\mathcal{M}^\perp$, that satisfies equation (7). If the filtration F is generated by a multidimensional Brownian motion and if A_T is bounded, the existence of a solution of equation (7) follows from the results of M. Kobylanski (2000) and J.P. Lepeltier and J. San Martin (1998), where the BSDEs (Backward Stochasic Differential Equations) with generators satisfying the square growth conditions were considered. We prove existence and uniqueness of (7) (or (1)) by different methods.
In section 2, using the BMO norm for martingales (L and $L^\perp$) and the $L^\infty([0,T]\times\Omega)$ norm for semimartingales Y, we apply the fixed-point theorem to show an existence of a solution first in case when L^∞ norm of $\bar{\eta}$ and constant γ are sufficiently small. Then we construct the solution for an arbitrary bounded $\bar{\eta}$.
In section 3 we construct a solution of equation (1) using the value process of a certain optimization problem, for some values of the parameter α. We give also a necessary condition for equation (1) to admit a solution in the class BMO (or a bounded solution Y for equation (7)). This condition shows that we can't expect an existence of a bounded solution of (7) for arbitrary γ.

2 Proof of the main Theorem

We recall the definition of BMO-martingales and of a similar notion for the processes of finite variation.
The square integrable continuous martingale M belongs to the class BMO if there is a constant $C > 0$ such that

$$E^{\frac{1}{2}}(< M >_T - < M >_\tau |\mathcal{F}_\tau) \le C, \quad P-a.s.$$

for every stopping time τ. The smallest constant with this property (or $+\infty$ if it does not exist) is called the BMO norm of M and is denoted by $||M||_{BMO}$.
For the process of finite variation A we denote by $\mathrm{var}_s^t(A)$ the variation on the segment $[s,t]$. If

$$E(\mathrm{var}_\tau^T(A)|\mathcal{F}_\tau) \le C, \quad P-\text{a.s.}$$

for every stopping time τ, let us denote by $|A|_\omega$ the smallest constant with this property.
We say that the process B strongly dominates the process A and write $A \prec B$, if the difference $B - A \in \mathcal{A}^+_{\mathrm{loc}}$, i.e., is a locally integrable increasing process. We shall use also the notation $\varphi \cdot X$ for the stochastic integral with respect to the semimartingale X.
Let $N \in BMO(P)$ and $dQ = \mathcal{E}_T(N)dP$. Then Q is a probability measure equivalent to P by Theorem 2.3 Kazamaki 1994). Denote by $\psi = \psi(X) = \langle X, N\rangle - X$ the Girsanov's transformation. It is well known that (see Kazamaki 1994) both H^2 and BMO are invariant under transformation ψ. Let $\mathcal{M}(Q)$ and $\mathcal{M}^\perp(Q)$ be images of the mapping ψ for $\mathcal{M}$ and $\mathcal{M}^\perp$ respectively, . Note that $\mathcal{M}(Q)$ and $\mathcal{M}^\perp(Q)$ are stable orthogonal subspaces of the space $H^2(Q)$ of square integrable martingales with respect to Q.
We shall need the following lemma to switch solutions of equation (1) for different final random variables.
Lemma 1. Let there exists $m_1, m_1^\perp \in BMO$, $m_1 \in \mathcal{M}, m_1^\perp \in \mathcal{M}^\perp$ such that

$$\mathcal{E}_T(m_1)\mathcal{E}^\alpha_T(m_1^\perp) = c_1 \exp\{\eta_1\}. \tag{8}$$

Let Q be a probability measure defined by

$$dQ = \mathcal{E}_T(m_1 + m_1^\perp)dP$$

and assume that there exists $m_2, m_2^\perp \in BMO(Q)$, $m_2 \in \mathcal{M}(Q), m_2^\perp \in \mathcal{M}^\perp(Q)$ such that

$$\mathcal{E}_T(m_2)\mathcal{E}^\alpha_T(m_2^\perp) = c_2 \exp\{\eta_2\}. \tag{9}$$

Then there exists a solution of equation

$$\mathcal{E}_T(m)\mathcal{E}^\alpha_T(m^\perp) = c \exp\{\eta_1 + \eta_2\}. \tag{10}$$

Proof. Note that

$$\frac{dP}{dQ} = \mathcal{E}^{-1}_T(m_1 + m_1^\perp) = \mathcal{E}_T(\tilde m_1 + \tilde m_1^\perp),$$

where $\tilde m_1 = \langle m_1\rangle - m_1$ and $\tilde m_1^\perp = \langle m_1^\perp\rangle - m_1^\perp$ are BMO martingales under Q.
By Girsanov's theorem m_2 and $m_2^\perp$ are special semimartingales under P with the decomposition

$$m_2 = \hat m_2 + \langle m_2, \tilde m_1\rangle, \quad m_2^\perp = \hat m_2^\perp + \langle m_2^\perp, \tilde m_1^\perp\rangle$$

where $\hat m_2 = m_2 - \langle m_2, \;\; \tilde m_1\rangle$ and $\hat m_2^\perp = m_2^\perp - \langle m_2^\perp, \tilde m_1^\perp\rangle$ are $BMO(P)$-martingales according to Theorem 3.6 of Kazamaki (1994).
Girsanov's theorem and the uniqueness of the canonical decomposition of special semimartingales imply that

$$\langle \hat m_2, m_1\rangle = -\langle m_2, \tilde m_1\rangle, \quad \langle \hat m_2^\perp, m_1^\perp\rangle = -\langle m_2^\perp, \tilde m_1^\perp\rangle. \tag{11}$$

Multiplying now equations (2.1) and (2.2) and using Yor's formula we obtain

$$\mathcal{E}_T(m_1 + m_2 + \langle \hat m_2, m_1\rangle)\mathcal{E}^\alpha_T(m_1^\perp + m_2^\perp + \langle \hat m_2^\perp, m_1^\perp\rangle) = c_1c_2 \exp\{\eta_1 + \eta_2\}. \tag{12}$$

By equality (11) and Theorem 3.6 of Kazamaki (1994) $m_2 + \langle \hat m_2, m_1\rangle$ and $m_2^\perp + \langle \hat m_2^\perp, m_1^\perp\rangle$ are $BMO(P)$ martingales. It is easy to see that these martingales are strongly orthogonal to each other. Therefore, $c = c_1c_2$, $\;m = m_1 + m_2 + \langle \hat m_2, m_1\rangle$ and $m^\perp = m_1^\perp + m_2^\perp + \langle \hat m_2^\perp, m_1^\perp\rangle$ satisfy equation (2.3).

Now we shall show the uniqueness of a solution of equation (1) in the class BMO.

PROPOSITION 1. Let η be an F_T-measurable random variable. If there exists a triple $(c, m, m^{\perp})$, where $c \in R_+, m \in BMO \cap \mathcal{M}, m^{\perp} \in BMO \cap \mathcal{M}^{\perp}$ satisfying equation (1), then such solution is unique.

Proof. Let $(c, m, m^{\perp})$ and $(c', l, l^{\perp})$ be two solutions of (1) from the class BMO. Then (1) implies that

$$c' \frac{\mathcal{E}_T(m)}{\mathcal{E}_T(l)} = c \frac{\mathcal{E}_T^{\alpha}(l^{\perp})}{\mathcal{E}_T^{\alpha}(m^{\perp})}. \tag{13}$$

Recall that if M and N are continuous local martingales then (see, e.g., Jacod 1979)

$$\frac{\mathcal{E}_t(M)}{\mathcal{E}_t(N)} = \mathcal{E}_t(M - N - \langle M - N, N\rangle), \quad \mathcal{E}_t^{\alpha}(M) = \mathcal{E}_t(\alpha M + \frac{\alpha(\alpha-1)}{2}\langle M\rangle).$$

Therefore, from (13) we have that

$$c'\mathcal{E}_T(m - l - \langle m - l, l\rangle) = c\mathcal{E}_T^{\alpha}(l^{\perp} - m^{\perp} - \langle l^{\perp} - m^{\perp}, m^{\perp}\rangle) =$$

$$= c\mathcal{E}_T(\alpha(l^{\perp} - m^{\perp}) - \langle \alpha(l^{\perp} - m^{\perp}), \frac{\alpha+1}{2}m^{\perp} - \frac{\alpha-1}{2}l^{\perp}\rangle). \tag{14}$$

Let Q be a measure defined by

$$dQ = \mathcal{E}_T(l + \frac{\alpha+1}{2}m^{\perp} - \frac{\alpha-1}{2}l^{\perp})dP.$$

Since $l + \frac{\alpha+1}{2}m^{\perp} - \frac{\alpha-1}{2}l^{\perp} \in BMO(P)$ the processes $m - l - \langle m - l, l\rangle$ and $\alpha(l^{\perp} - m^{\perp}) - \langle \alpha(l^{\perp} - m^{\perp}), \frac{\alpha+1}{2}m^{\perp} - \frac{\alpha-1}{2}l^{\perp}\rangle$ are BMO-martingales with respect to the measure Q according to Theorem 3.6 of Kazamaki (1994) and corresponding Doleans-Dade exponentials are uniformly integrable Q-martingales. Therefore equality (14) holds for all $t \in [0, T]$, which implies that $c = c'$ and

$$m - l - \langle m - l, l\rangle = \alpha(l^{\perp} - m^{\perp}) - \langle \alpha(l^{\perp} - m^{\perp}), \frac{\alpha+1}{2}m^{\perp} - \frac{\alpha-1}{2}l^{\perp}\rangle.$$

Hence $m - l = \alpha(l^{\perp} - m^{\perp})$ and since $m - l$ is orthogonal to $l^{\perp} - m^{\perp}$, we obtain that $m = l$ and $m^{\perp} = l^{\perp}$.

PROPOSITION 2A. Let η be a bounded F_T-measurable random variable. Then there exists a triple $(c, m, m^{\perp})$, where $c \in R_+, m \in BMO \cap \mathcal{M}, m^{\perp} \in BMO \cap \mathcal{M}^{\perp}$, that satisfies equation (1).

Proof. It is evident that equation (1) is equivalent to the following martingale equation

$$-\ln c - \frac{1}{2}\langle m\rangle_T - \frac{\alpha}{2}\langle m^{\perp}\rangle_T + m_T + \alpha m_T^{\perp} = \eta. \tag{15}$$

Denoting $c' = -\frac{1}{2}\ln c,\ L = \frac{1}{2}m,\ L^{\perp} = \frac{\alpha}{2}m^{\perp},\ \xi = \frac{1}{2}\eta$ one can write this equation in the form

$$c' - \langle L\rangle_T - \frac{1}{\alpha}\langle L^{\perp}\rangle_T + L_T + L_T^{\perp} = \xi. \tag{16}$$

where $\alpha \neq 0$. The latter equation can be also written in the following equivalent semimartingale form as a BSDE

$$Y_t = Y_0 - \langle L + L^{\perp}, L + \frac{1}{\alpha}L^{\perp}\rangle_t + L_t + L_t^{\perp}, \quad Y_T = \xi. \tag{17}$$

Let first show that there exists a solution $(c, m, m^{\perp})$ of equation (16) if $|\xi|_\infty$ is small enough. For brevity we shall use the notation $\langle m\rangle_{tT} = \langle m\rangle_T - \langle m\rangle_t$ for the square characteristic of a martingale m.
Let consider the mapping

$$L_t + L_t^{\perp} = E(\xi + \langle l + l^{\perp}, l + \frac{1}{\alpha}\, l^{\perp}\rangle_T/\mathcal{F}_t) - \tag{18}$$

$$-E(\xi + \langle l + l^{\perp}, l + \frac{1}{\alpha}\, l^{\perp}\rangle_T),$$

$$Y_t = E(\xi + \langle l + l^{\perp}, l + \frac{1}{\alpha}\, l^{\perp}\rangle_{tT}/\mathcal{F}_t), \tag{19}$$

which transforms BMO-martingales l and $l^{\perp}$ into a triple $(Y, L, L^{\perp})$, where L and $L^{\perp}$ are BMO-martingales and Y is a semimartingale. Using $L^{\infty}([0,T]\times\Omega)$ norm for semimartingales and BMO norms for martingales, we shall show that if the norm $|\xi|_\infty$ is sufficiently small, then there exists $r > 0$ such that mapping (18) is a contraction in the ball

$$\mathcal{B}_r = \{(l, l^{\perp}), ||l + l^{\perp}||_{\mathrm{BMO}} \le r\}$$

Using the Ito formula for $Y_T^2 - Y_t^2$ and (18),(19) we have

$$Y_t^2 - Y_T^2 = -2\int_t^T Y_s d(L_s + L_s^{\perp}) +$$

$$2\int_t^T Y_s d\langle l + l^{\perp}, l + \frac{1}{\alpha}\, l^{\perp}\rangle_s - \langle L + L^{\perp}\rangle_{tT}. \tag{20}$$

For any $l, l^{\perp} \in$ BMO by condition B) and (19) Y is bounded and L and $L^{\perp}$ are square integrable martingales. Therefore, the stochastic integral $Y\cdot(L + L^{\perp})$ is a martingale and taking conditional expectations in (20) we have

$$Y_t^2 + E(\langle L + L^{\perp}\rangle_{tT}/\mathcal{F}_t) = E(\xi^2/\mathcal{F}_t) + 2E(\int_t^T Y_s d\langle l + l^{\perp}, l + \frac{1}{\alpha}\, l^{\perp}\rangle_s/\mathcal{F}_t).$$

Since $\langle l + l^{\perp}, l + \frac{1}{\alpha}\, l^{\perp}\rangle \prec (|\,\frac{1}{\alpha}\,| \vee 1)\langle l + l^{\perp}\rangle$ and $\frac{1}{2}|Y|_\infty^2 + \frac{1}{2}|L + L^{\perp}|_{\mathrm{BMO}}^2 \le \operatorname{esssup}(Y_t^2 + E(\langle L + L^{\perp}\rangle_{tT}/\mathcal{F}_t))$, using the notation $\beta = 2(|\,\frac{1}{\alpha}\,| \vee 1)$ we get

$$Y_t^2 + E(\langle L + L^{\perp}\rangle_{tT}/\mathcal{F}_t) \le$$

$$\le |\xi|_\infty^2 + \beta|Y|_\infty E(\langle l + l^{\perp}\rangle_{tT}/\mathcal{F}_t) \le \tag{21}$$

$$\le |\xi|_\infty^2 + \frac{1}{2}|Y|_\infty^2 + \frac{1}{2}\beta^2 E^2(\langle l + l^{\perp}\rangle_{tT}/\mathcal{F}_t)$$

and hence

$$|L + L^{\perp}|_{\mathrm{BMO}}^2 \le 2|\xi|_\infty^2 + \beta^2|l + l^{\perp}|_{\mathrm{BMO}}^4.$$

If $2|\xi|_\infty \le \frac{1}{2\beta}$, then there exists r satisfying the inequality

$$r^2 \ge 4|\xi|_\infty^2 + \beta^2 r^4. \tag{22}$$

Therefore $|l + l^{\perp}|_{\mathrm{BMO}} \le r$ implies $|L + L^{\perp}|_{\mathrm{BMO}}^2 \le 4|\xi|_\infty^2 + \beta^2 r^4 \le r^2$.

Now we shall show that the mapping (18) is a contraction on the ball $\mathcal{B}_r$ from the space BMO. Let $Y_i, L_i, L_i^\perp, i = 1,2$ correspond to $l_i, l_i^\perp, i = 1,2$ by transformation (18),(19). Since $Y_1(T) - Y_2(T) = 0$ applying the Ito formula for $(Y_1 - Y_2)^2$ similarly to (21) we obtain

$$(Y_1(t) - Y_2(t))^2 + E(\langle L_1 - L_2 + L_1^\perp - L_2{}^\perp\rangle_{tT}/\mathcal{F}_t) \le \tag{23}$$

$$2E(\int_t^T |Y_1(s) - Y_2(s)| d\mathrm{var}_t^s(\langle l_1 + l_1^\perp, l_1 + \frac{1}{\alpha}\, l_1^\perp\rangle - \langle l_2 + l_2^\perp, l_2 + \frac{1}{\alpha}\, l_2^\perp\rangle)/\mathcal{F}_t) \le$$

$$\le |Y_1 - Y_2|^2_\infty + E^2(\mathrm{var}_t^T(\langle l_1 + l_1^\perp, l_1 + \frac{1}{\alpha}\, l_1^\perp\rangle - \langle l_2 + l_2^\perp, l_2 + \frac{1}{\alpha}\, l_2^\perp\rangle)/\mathcal{F}_t).$$

On the other hand using the Kunita-Watanabe inequality, elementary inequalities $(a+b)^2 \le 2(a^2+b^2)$ and $\langle l + l^\perp, l + \frac{1}{\alpha}\, l^\perp\rangle \ \prec \frac{1}{2}\beta\langle l + l^\perp\rangle$ we get

$$E^2(\mathrm{var}_t^T(\langle l_1 + l_1^\perp, l_1 + \frac{1}{\alpha}\, l_1^\perp\rangle - \langle l_2 + l_2^\perp, l_2 + \frac{1}{\alpha}\, l_2^\perp\rangle)/\mathcal{F}_t) \le$$

$$\le 2E^2(\mathrm{var}_t^T\langle l_1 - l_2 + l_1^\perp - l_2^\perp, l_1 + \frac{1}{\alpha}\, l_1^\perp\rangle/\mathcal{F}_t) +$$

$$2E^2(\mathrm{var}_t^T\langle l_1 + l_2{}^\perp, l_1 - l_2 + \frac{1}{\alpha}(l_1^\perp - l_2^\perp)\rangle/\mathcal{F}_t) \le \tag{24}$$

$$\le 2E(\langle l_1 - l_2 + l_1^\perp - l_2^\perp\rangle_{tT}/\mathcal{F}_t)E(\langle l_1 + \frac{1}{\alpha}\, l_1^\perp\rangle_{tT}/\mathcal{F}_t) +$$

$$2E(\langle l_2 + l_2{}^\perp\rangle_{tT}/\mathcal{F}_t)E(\langle l_1 - l_2 + \frac{1}{\alpha}(l_1^\perp - l_2^\perp)\rangle_{tT}/\mathcal{F}_t) \le$$

$$\le \beta^2 E(\langle l_1 - l_2 + l_1^\perp - l_2^\perp\rangle_{tT}/\mathcal{F}_t)\times$$

$$[E(\langle l_1 + l_1^\perp\rangle_{tT}/\mathcal{F}_t) + E(\langle l_2 + l_2{}^\perp\rangle_{tT}/\mathcal{F}_t)].$$

If $l_1, l_2 \in \mathcal{B}_r$, the relations (23) and (24) imply the inequality

$$|L_1 - L_2 + L_1^\perp - L_2{}^\perp|_{\mathrm{BMO}} \le r\beta|l_1 - l_2 + l_1^\perp - l_2^\perp|_{\mathrm{BMO}}.$$

Finally we remark that if $|\xi|_\infty \le \frac{1}{4\beta}$ and $\frac{1}{8\beta^2} \le r^2 < \frac{1}{4\beta^2}$ then the inequalities (22) and $r\beta < 1$ are satisfied simultaneously. Thus we obtain that if $|\xi|_\infty$ is small enough, then the mapping (18) is a contraction and by fixed point theorem equation (17) (and hence equation (1)) admits a unique solution. In particular if $|\xi|_\infty \le \frac{1}{4\beta}$ then the BMO-norm of the solution is less than $\frac{1}{\beta}$.

To get rid of the assumption that $|\xi|_\infty$ should be small enough, let us use the Lemma 1. Let us take an integer $n \ge 1$ so that equation

$$c_1\mathcal{E}_T(m)\mathcal{E}_T^\alpha(m^\perp) = \exp\{\frac{1}{n}\xi\} \tag{25}$$

admits a solution. Let $dQ = \mathcal{E}_T(m_1 + m_1^\perp)dP$, where $(m_1, m_1^\perp) \in BMO(P)$ is a solution of (25). Since the norm $|\xi|_\infty$ is invariant with respect to an equivalent change of measure and since the Girsanov transformation is an isomorphism of $BMO(P)$ onto $BMO(Q)$, similarly as above one can show that there exists a pair $m_2, m_2^\perp \in BMO(Q)$ that satisfies equation (25). Therefore by Lemma 1, there exists a solution of equation

$$c_2\mathcal{E}_T(m)\mathcal{E}_T^\alpha(m^\perp) = \exp\{\frac{2}{n}\xi\}. \tag{26}$$

Using now Lemma 1 to equation (26) by induction we obtain that there exists a solution of equation (1).

Remark. For the solution $(Y, m, m^{\perp})$ of (17) the following estimate is true

$$|Y_t| \leq |\xi|_{\infty} \quad \text{for all} \quad t \in [0,T].$$

Indeed, since $m, m^{\perp} \in BMO$, the process Y is a uniformly integrable martingale under Q, where $dQ = \mathcal{E}_T(L + \alpha L^{\perp})dP$ then (17). Therefore

$$Y_t = E^Q(\xi/F_t) \leq |\xi|_{\infty}.$$

Let us consider now the case, where the final random variable is of the form $\eta = \gamma A_T$ for a constant γ and an increasing process $(A_t, t \in [0,T])$.

Proposition 2b. Assume that $\eta = \gamma A_T$, $|A|_{\omega} < \infty$ and $|\gamma| < \frac{|\alpha| \wedge 1}{2|A|_{\omega}}$. Then there exists a unique triple $(c, m, m^{\perp})$, where $c \in R_+, m \in BMO \cap \mathcal{M}, m^{\perp} \in BMO \cap \mathcal{M}^{\perp}$, that satisfies equation (1).

Proof. The proof is similar to the proof of Proposition 2a. The only difference is that in equation (19) ξ is replaced by $\bar{\gamma}(A_T - A_t)$, where $\bar{\gamma} = \frac{1}{2}\gamma$ and equation (17) is replaced by the BSDE

$$Y_t = Y_0 - \bar{\gamma} A_t - \langle L + L^{\perp}, L + \frac{1}{\alpha} L^{\perp} \rangle_t + L_t + L_t^{\perp}, \quad Y_T = 0. \tag{27}$$

Therefore, applying the Itô formula for Y_t^2 and after using the same arguments we obtain

$$\begin{aligned} Y_t^2 + E(\langle L + L^{\perp} \rangle_{tT}/\mathcal{F}_t) \leq \\ \leq 2|Y|_{\infty}\big[\bar{\gamma} E(A_T - A_t/\mathcal{F}_t) + \beta E(\langle l + l^{\perp} \rangle_{tT}/\mathcal{F}_t)\big] \leq \\ \leq |Y|_{\infty}^2 + 2\bar{\gamma}^2 E^2(A_T - A_t/\mathcal{F}_t) + 2\beta^2 E^2(\langle l + l^{\perp} \rangle_{tT}/\mathcal{F}_t) \end{aligned} \tag{28}$$

and hence

$$|L + L^{\perp}|_{\mathrm{BMO}}^2 \leq 2\bar{\gamma}^2 |A|_{\omega}^2 + 2\beta^2 |l + l^{\perp}|_{\mathrm{BMO}}^4.$$

If $|\bar{\gamma}||A|_{\omega} < \frac{1}{4\beta}$ (or equivalently $|\gamma| < \frac{|\alpha| \wedge 1}{2|A|_{\omega}}$) we can take r satisfying the inequalities

$$r^2 \geq 2\bar{\gamma}^2 |A|_{\omega}^2 + 2\beta^2 r^4 \text{ and } 2\beta r < 1. \tag{29}$$

Therefore $|l + l^{\perp}|_{\mathrm{BMO}} \leq r$ implies $|L + L^{\perp}|_{\mathrm{BMO}}^2 \leq 2\bar{\gamma}^2 |A|_{\omega} + 2\beta^2 r^4 \leq r^2$.

The contraction property of the mapping is proved similarly to Proposition 2a and the Lipschitz constant is again $2\beta r$.

The proof of Theorem 1. The uniqueness is proved in Proposition 1. An existence is a consequence of Proposition 2a, Proposition 2b and Lemma 1.

3 Solution as a value process of an optimization problem

In this section we shall construct the solution of equation (1) using the value process of a certain optimization problem for some values of parameter α.

Let $A = (A_t, t \in [0,T])$ be a continuous F-adapted increasing process. Assume that

D) there exists $m^* \in \mathcal{M}^\perp \cap BMO$ such that

$$E(e^{A_T - A_\tau}\mathcal{E}_{tT}^{1-\alpha}(m^*)/F_\tau) \le C \tag{30}$$

for all stopping times τ, for some $C > 0$, where $\mathcal{E}_{tT}(m^*) = \frac{\mathcal{E}_T(m^*)}{\mathcal{E}_t(m^*)}$.
Remark. It is evident that condition D) is satisfied if

$$E(e^{A_T - A_\tau}/F_\tau) \le C \tag{31}$$

for all stopping times τ, for some $C > 0$ and vice versa if condition D) and $\alpha\bar{\in}(0,1)$ are satisfied then there exists the measure $dQ = \mathcal{E}_T(m^{**})dP,\ \ m^{**} \in \mathcal{M}^\perp \cap BMO$ such that

$$E^Q(e^{A_T - A_\tau}/F_\tau) \le C.$$

Let us introduce the value process

$$V_t = V_t(\alpha) = \operatorname*{ess\,inf}_{m^\perp \in \mathcal{M}^\perp} E(e^{A_T - A_t}\mathcal{E}_{tT}^{1-\alpha}(m^\perp)|F_t). \tag{32}$$

Note that V_t is value process of an optimization problem, which contains the problem of finding of the q-optimal martingale measure dual to the power utility maximization problem.
Theorem 2. Let $\alpha < 0$ and let η be an F_T-measurable random variable of the form $\eta = A_T$, where $A = (A_t, t \in [0,T])$ is a continuous F-adapted increasing process. If condition D) is satisfied, then the triple

$$(c, m, m^\perp) = (\frac{1}{V_0}, \frac{1}{V}\cdot L, \frac{1}{\alpha V}\cdot L^\perp) \in R_+ \times (BMO \cap \mathcal{M}) \times (BMO \cap \mathcal{M}^\perp),$$

where $L + L^\perp$ is the martingale part of V, satisfies equation (1).
Moreover $\ln V_t$ is a unique bounded solution of (17).
If $E(A_T - A_\tau/F_\tau) \le C$ for every stopping time τ and if there exists a triple $(c, m, m^\perp)$, $c \in R_+, m \in BMO \cap \mathcal{M}, m^\perp \in BMO \cap \mathcal{M}^\perp$, satisfying equation (1), then condition D) is fulfilled.
Proof. It is evident that

$$1 \le V_t \le C, \tag{33}$$

where the first inequality follows from Jensen's inequality and the second follows from condition D). Similarly to Theorem 1 of [15] one can show that V_t defined by (32) is a special semimartingale which is a unique bounded solution of the BSDE

$$V_t = -\int_0^t V_s dA_s - \frac{1-\alpha}{2\alpha}\int_0^t \frac{1}{V_s} d < L^\perp > + L_t + L_t^\perp,\ V_T = 1. \tag{34}$$

and $L, L^\perp \in BMO$.
Since $V_t \ge 1$, the martingales $m = \frac{1}{V}\cdot L$ and $m^\perp = \frac{1}{\alpha V}\cdot L^\perp$ also belong to the class BMO and it is not difficult to see that the triple $(1/V_0, m, m^\perp)$ satisfies equation (1). Indeed, applying the Itô formula for lnV_t from equation (34) we have

$$lnV_t = lnV_0 - A_t - \frac{1}{2} < \frac{1}{V}\cdot L >_t + (\frac{1}{V}\cdot L)_t +$$

$$-\frac{\alpha}{2} < \frac{1}{\alpha V}\cdot L^\perp >_t + \alpha(\frac{1}{\alpha V}\cdot L^\perp)_t. \tag{35}$$

Using the boundary condition and (34) we obtain the equality (1).
Assume now that there exists a triple $c \in R_+, m \in BMO \cap \mathcal{M}, m^\perp \in BMO \cap \mathcal{M}^\perp$ that satisfies equation (1). Consider the process

$$Y_t = ce^{-A_t}\mathcal{E}_t(m)\mathcal{E}_t^\alpha(m^\perp).$$

Equality (1) implies that $Y_T = 1$, hence

$$Y_t = e^{A_T - A_t}\mathcal{E}_{tT}^{-1}(m)\mathcal{E}_{tT}^{-\alpha}(m^\perp) \tag{36}$$

and

$$lnY_t = E(A_T - A_t/F_t) + \frac{1}{2}E(< m >_{tT} /F_t) + \frac{\alpha}{2}E(< m^\perp >_{tT} /F_t).$$

Since m and $m^\perp$ belong to BMO and $E(A_T - A_t/F_t) \le const$, the process lnY_t is bounded. Since Y is strictly positive, this implies that the process Y satisfies the two-sided inequality

$$0 < c \le Y_t \le C. \tag{37}$$

(in particular, $Y_t = V_t$ by the uniqueness of (34))
Therefore (36) and (37) imply that

$$E(e^{A_T - A_t}\mathcal{E}_{tT}^{1-\alpha}(m^\perp)/F_t) = Y_t E(\mathcal{E}_{tT}(m)\mathcal{E}_{tT}(m^\perp)/F_t) = Y_t \le C.$$

Remark. If $0 < \alpha < 1$ and there is a triple $(c, m, m^\perp)$, where $c \in R_+, m \in BMO \cap \mathcal{M}, m^\perp \in BMO \cap \mathcal{M}^\perp$ satisfying equation (1), then there exists $C > 0$ such that

$$E(e^{A_T - A_\tau}/F_\tau) \le C$$

for all stopping times τ.
Indeed, similarly as above, the process Y defined by (36) satisfies inequality (37) and from (36) we have

$$E(e^{A_T - A_t}/F_t) = Y_t E(\mathcal{E}_{tT}(m)\mathcal{E}_{tT}^\alpha(m^\perp)/F_t).$$

Since $0 < \alpha < 1$ and since $\mathcal{E}_t(m^\perp)$ is a martingale under the measure $dQ = \mathcal{E}_T(m)dP$, using the Hölder inequality we obtain from the latter equality that

$$E(e^{A_T - A_t}/F_t) = Y_t E^Q(\mathcal{E}_{tT}^\alpha(m^\perp)/F_t) \le$$

$$\le C(E^Q(\mathcal{E}_{tT}(m^\perp)/F_t)^\alpha = C.$$

This remark and Theorem 2 show that if γ is large enough so that condition $E(e^{\gamma(A_T - A_\tau)}/F_\tau) \le$ *const* is not satisfied then the bounded solution of the equation (27) does not exist.

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