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WHEN DOES A RANDOMLY WEIGHTED SELF–NORMALIZED SUM CONVERGE IN DISTRIBUTION?

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Abstract
We determine exactly when a certain randomly weighted self–normalized sum converges in distribution, partially verifying a 1965 conjecture of Leo Breiman, and then apply our results to characterize the asymptotic distribution of relative sums and to provide a short proof of a 1973 conjecture of Logan, Mallows, Rice and Shepp on the asymptotic distribution of self–normalized sums in the case of symmetry.

0.1 A conjecture of Breiman

Throughout this paper $\{Y_i\}_{i\geq 1}$ will denote a sequence of i.i.d. Y random variables, where Y is non–negative with distribution function G. Let $Y \in D(\alpha)$, with $0 < \alpha \leq 2$, denote that Y is in the domain of attraction of a stable law of index α. We shall use the notation $Y \in D(0)$ to mean that $1 - G$ is a slowly varying function at infinity. Now let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. X random variables independent of $\{Y_i\}_{i\geq 1}$, where X satisfies

$$E|X| < \infty \text{ and } EX = 0. \tag{1}$$

Consider the randomly weighted self–normalized sum

$$R_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}.$$

(Here and elsewhere we define $0/0 = 0$.) In a beautiful paper, Breiman (1965) proved the following result characterizing when R_n converges in distribution to a non–degenerate law.

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Theorem 1 *Suppose for each such sequence $\{X_i\}_{i\geq 1}$ of i.i.d X random variables, independent of $\{Y_i\}_{i\geq 1}$, the ratio R_n converges in distribution, and the limit law of R_n is non–degenerate for at least one such sequence $\{X_i\}_{i\geq 1}$. Then $Y \in D(\alpha)$, with $0 \leq \alpha < 1$.*

Theorem 1 is a restatement of his Theorem 4. At the end of his 1965 paper Breiman conjectured that the conclusion of Theorem 1 remains true as long as there exist one i.i.d. X sequence $\{X_i\}_{i\geq 1}$, satisfying (1), such that R_n converges in distribution to a non–degenerate law. We shall provide a partial solution to his conjecture (we assume $E|X|^p < \infty$ for some $p > 2$) and at the same time give a new characterization for a non-negative random variable $Y \in D(\alpha)$, with $0 \leq \alpha < 1$.

Before we do this, let us briefly describe and comment upon Breiman's proof of Theorem 1. Let

$$D_n^{(1)} \geq \cdots \geq D_n^{(n)} \geq 0 \tag{2}$$

denote the order values of $Y_j / \sum_{i=1}^n Y_i$, $j = 1, \ldots, n$. Clearly along subsequences $\{n'\}$ of $\{n\}$, the ordered random variables $D_n^{(i)}$, $i = 1, \ldots, n$, converge in distribution to random sequences $\{D_i\}_{i\geq 1}$ satisfying $D_i \geq 0$, $i \geq 1$, and $\sum_{i=1}^\infty D_i = 1$. From this one readily concludes that the limit laws of R_n are of the form

$$\sum_{i=1}^\infty X_i D_i.$$

Breiman argues in his proof that if $\{D_i'\}_{i\geq 1}$ is any other random sequence satisfying $D_i' \geq 0$, $i \geq 1$, $\sum_{i=1}^\infty D_i' = 1$ and

$$\sum_{i=1}^\infty X_i D_i' =_d \sum_{i=1}^\infty X_i D_i, \tag{3}$$

for all sequences $\{X_i\}_{i\geq 1}$ of i.i.d. X random variables independent of $\{Y_i\}_{i\geq 1}$ satisfying (1), then

$$\{D_i'\}_{i\geq 1} =_d \{D_i\}_{i\geq 1}. \tag{4}$$

This implies that along the full sequence $\{n\}$,

$$\max_{1\leq j\leq n} Y_j / \sum_{i=1}^n Y_i \to_d D_1, \tag{5}$$

where D_1 is either non–degenerate or $D_1 = 1$. Breiman proves that when $D_1 = 1$, $Y \in D(0)$, and when D_1 is non–degenerate necessarily $Y \in D(\alpha)$, with $0 < \alpha < 1$.

At first glance it may seem reasonable that it would be enough for (3) to hold for some i.i.d. X sequence $\{X_i\}_{i\geq 1}$ satisfying (1) in order to conclude (4). In fact, consider a sequence $\{s_i\}_{i\geq 1}$ of independent Rademacher functions and let $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 1}$ be two sequences of non-increasing non-negative constants summing to 1. (By Rademacher we mean that $P\{s_i = 1\} = P\{s_i = -1\} = 1/2$, for each $i \geq 1$.) A special case of a result of Marcinkiewicz, see Theorem 5.1.5 in Ramachandran and Lau (1991), says that

$$\sum_{i=1}^\infty s_i a_i =_d \sum_{i=1}^\infty s_i b_i$$

if and only if $\{a_i\}_{i\geq 1} = \{b_i\}_{i\geq 1}$.

However, Jim Fill has shown that there exist two non–identically distributed random sequences $\{D'_i\}_{i\geq 1}$and $\{D_i\}_{i\geq 1}$ such that $\sum_{i=1}^{\infty} s_i D'_i =_d \sum_{i=1}^{\infty} s_i D_i$. Here is his example. Let $\{D'_i\}_{i\geq 1}$ equal to $(1, 0, 0, \ldots,)$ with probability $1/5$ and $(1/4, 1/4, 1/4, 1/4, 0, \ldots)$ with probability $4/5$ and let $\{D_i\}_{i\geq 1}$ equal to $(1/2, 1/2, 0, \ldots)$ with probability $1/5$ and $(1/2, 1/4, 1/4, 0, \ldots)$ with probability $4/5$. Clearly $\{D'_i\}_{i\geq 1}$and $\{D_i\}_{i\geq 1}$ are not equal in distribution. Whereas, calculation verifies that (3) holds.

This indicates that one must look for another way to try to establish Breiman's conjecture, than merely to refine his original proof. Our partial solution to Breiman's conjecture is contained in the following theorem.

Theorem 2 *Suppose that* $\{X_i\}_{i\geq 1}$ *is a sequence of i.i.d.* X *random variables independent of* $\{Y_i\}_{i\geq 1}$ *, where* X *satisfies* $E|X|^p < \infty$ *for some* $p > 2$ *and* $EX = 0$*, then the ratio* R_n *converges in distribution to a non-degenerate random variable* R *if and only if* $Y \in D(\alpha)$*, with* $0 \leq \alpha < 1$.

The proof of Theorem 1 will follow readily from the following characterization of when $Y \in D(\alpha)$, with $0 \leq \alpha < 1$. We shall soon see that whether $Y \in D(\alpha)$, with $0 \leq \alpha < 1$, or not depends on the limit of ET_n^2, where

$$T_n := \frac{\sum_{i=1}^{n} s_i Y_i}{\sum_{i=1}^{n} Y_i}, \tag{6}$$

with $\{s_i\}_{i\geq 1}$ being a sequence of independent Rademacher random variables independent of $\{Y_i\}_{i\geq 1}$.

Proposition 3 *We have* $Y \in D(\alpha)$*, with* $0 \leq \alpha < 1$ *if and only if*

$$nE\left(\frac{Y_1}{\sum_{i=1}^{n} Y_i}\right)^2 \to 1 - \alpha. \tag{7}$$

Remark 1 It can be inferred from Theorems 1, 2 and Proposition 1 of Breiman (1965) that the limit in (7) is equal to zero if and only if

$$\max_{1\leq j\leq n} Y_j / \sum_{i=1}^{n} Y_i \to_p 0 \tag{8}$$

if and only if there exist constants $B_n \nearrow$ such that

$$\sum_{i=1}^{n} Y_i / B_n \to_p 1. \tag{9}$$

Remark 2 Proposition 1 should be compared to a result of Bingham and Teugels (1981), which says

$$E\left(\frac{\sum_{i=1}^{n} Y_i}{\max_{1\leq i\leq n} Y_i}\right) \to \rho, \tag{10}$$

where $\infty > \rho > 1$ if and only if $Y \in D(\alpha)$, where $0 < \alpha < 1$, with $\alpha = (\rho - 1)/\rho$.

PROOF. First assume that $Y \in D(\alpha)$, where $0 < \alpha < 1$. Notice that with T_n defined as in (6),

$$ET_n^2 = nE\left(\frac{Y_1}{\sum_{i=1}^n Y_i}\right)^2.$$

By Corollary 1 of Le Page, Woodroofe and Zinn (1981),

$$T_n \to_d T := \frac{\sum_{i=1}^\infty s_i (\Gamma_i)^{-1/\alpha}}{\sum_{i=1}^\infty (\Gamma_i)^{-1/\alpha}}, \tag{11}$$

where $\Gamma_i = \sum_{j=1}^i \xi_j$, with $\{\xi_j\}_{j\ge 1}$ being a sequence of i.i.d. exponential random variables with mean 1 independent of $\{s_i\}_{i\ge 1}$. Since clearly $|T_n| \le 1$, we can infer by (11) that for any $Y \in D(\alpha)$, with $0 < \alpha < 1$,

$$ET_n^2 \to ET^2. \tag{12}$$

We shall prove that

$$ET^2 = -\int_0^\infty s\omega''(s) \exp(-\omega(s))\, ds, \tag{13}$$

where

$$\omega(s) = \int_0^\infty \left[1 - \exp\left(-sx^{-1/\alpha}\right)\right] dx = \alpha \int_0^\infty (1 - \exp(-sy))\, y^{-1-\alpha} dy$$
$$= s\int_0^\infty \exp(-sy)\, y^{-\alpha} dy = s^\alpha \Gamma(1-\alpha).$$

From this one gets from (13) after a little calculus that

$$ET^2 = \alpha(1-\alpha)\Gamma(1-\alpha)\int_0^\infty s^{\alpha-1}\exp(-s^\alpha\Gamma(1-\alpha))\, ds = 1-\alpha.$$

We get

$$nE\left(\frac{Y_1}{\sum_{i=1}^n Y_i}\right)^2 = n\int_0^\infty tE\left(Y_1^2 \exp(-t(Y_1+\cdots+Y_n))\right) dt$$
$$= n\int_0^\infty tE\left(Y_1^2\exp(-tY_1)\right) E\exp(-t(Y_2+\cdots+Y_n))\, dt$$
$$= n\int_0^\infty tE\left(Y_1^2\exp(-tY_1)\right)(E\exp(-tY_1))^{n-1}\, dt. \tag{14}$$

Now for any fixed $0<\alpha<1$ the limit in (11) remains the same for any $Y \in D(\alpha)$. Therefore for convenience we can and shall choose $Y = U^{-1/\alpha}$, where U is Uniform $(0,1)$. Therefore we can write the expession in (14) as

$$\int_0^\infty t\int_0^n \left(\left(\frac{x}{n}\right)^{-2/\alpha}\exp\left(-t\left(\frac{x}{n}\right)^{-1/\alpha}\right)dx\right)$$
$$\times\left(1-\frac{1}{n}\int_0^n\left[1-\exp\left(-t\left(\frac{x}{n}\right)^{-1/\alpha}\right)\right]dx\right)^{n-1} dt,$$

which by the change of variables $t = s/n^{1/\alpha}$,

$$
\begin{aligned}
= \quad & \int_0^\infty s \int_0^n \left(x^{-2/\alpha} \exp\left(-sx^{-1/\alpha}\right) dx \right) \\
& \times \left(1 - \frac{1}{n} \int_0^n \left[1 - \exp\left(-sx^{-1/\alpha}\right)\right] dx \right)^{n-1} ds.
\end{aligned}
$$

A routine limit argument now shows that this last expression converges to

$$
- \int_0^\infty s\omega^{''}(s) \exp(-\omega(s))\, ds.
$$

Now assume that $ET_n^2 \to 1-\alpha$, with $0<\alpha<1$. From equation (14) we get that

$$
ET_n^2 = n \int_0^\infty t\varphi''(t) (\varphi(t))^{n-1}\, dt \to 1-\alpha,
$$

where $\varphi(t) = E\exp(-tY_1)$, for $t \geq 0$. Arguing as in the proof of Theorem 3 of Breiman (1965) this implies that

$$
s \int_0^\infty t\varphi''(t) \exp(s \log \varphi(t))\, dt \to 1-\alpha, \text{ as } s \to \infty. \tag{15}
$$

For $y \geq 0$, let $q(y)$ denote the inverse of $-\log\varphi(v)$. Changing variables to $t = q(y)$ we get from (15) that

$$
s \int_0^\infty \exp(-sy)\, q(y)\, \varphi''(q(y))\, dq(y) \to 1-\alpha, \text{ as } s \to \infty.
$$

By Karamata's Tauberian theorem, see Theorem 1.7.1 on page 37 of Bingham et al (1987), we conclude that

$$
v^{-1} \int_0^v q(x)\, \varphi''(q(x))\, dq(x) \to 1-\alpha, \text{ as } v \searrow 0,
$$

which, in turn, by the change of variables $y = q(x)$ gives

$$
\frac{\int_0^t y\varphi''(y)\, dy}{-\log\varphi(t)} \to 1-\alpha, \text{ as } t \searrow 0.
$$

Since $-\log(1-s)/s \to 1$ as $s \searrow 0$, this implies that

$$
\frac{\int_0^t y\varphi''(y)\, dy}{1-\varphi(t)} = \frac{t\varphi'(t)}{1-\varphi(t)} + 1 \to 1-\alpha, \text{ as } t \searrow 0,
$$

or in other words

$$
\frac{t\varphi'(t)}{1-\varphi(t)} \to -\alpha, \text{ as } t \searrow 0. \tag{16}
$$

Set $f(x) = -x^{-2}\varphi'(1/x) = (\varphi(1/x))'$, for $x>0$. With this notation we can rewrite (16) as

$$
\frac{xf(x)}{\int_x^\infty f(y)\, dy} \to \alpha, \text{ as } x \to \infty. \tag{17}
$$

By Theorem 1.6.1 on page 30 of Bingham et al (1987) this implies that $f(y)$ is regularly varying at infinity with index $\rho = -\alpha - 1$, which, in turn, by their Theorem 1.5.11 implies that $1 - \varphi(1/x)$ is regularly varying at infinity with index $-\alpha$, which says that $1 - \varphi(s)$ is regularly varying at 0 with index α. Set for $x \geq 0$,

$$U(x) = \int_0^x (1 - G(u))du.$$

We see that for any $s > 0$,

$$\int_0^\infty e^{-sx} dU(x) = s^{-1}(1 - \varphi(s)),$$

which is regularly varying at 0 with index $\alpha - 1$. Now by Theorem 1.7.1 on page 37 of Bingham et al (1987) this implies that $U(x)$ is regularly varying at infinity with index $1 - \alpha$. This, in turn, by Theorem 1.7.2 on page 39 of Bingham et al (1987) implies that $1 - G(x)$ is regularly varying at infinity with index $-\alpha$. Hence $G \in D(\alpha)$.

To finish the proof we must show that

$$ET_n^2 = nE\left(\frac{Y_1}{\sum_{i=1}^n Y_i}\right)^2 \to 1. \tag{18}$$

holds if and only if $Y \in D(0)$. It is well–known going back to Darling (1952), that $Y \in D(0)$ if and only if

$$\max_{1 \leq j \leq n}\left(Y_j / \sum_{i=1}^n Y_i\right) \to_p 1. \tag{19}$$

(Refer to Haeusler and Mason (1991) and the references therein.) Thus clearly whenever $Y \in D(0)$ we have

$$T_n \to_d s_1$$

and therefore we have (18). To go the other way, assume that (18) holds. This implies that

$$\sum_{i=1}^n E\left(D_n^{(i)}\right)^2 = ET_n^2 \to 1,$$

which since $D_n^{(1)} \geq \cdots \geq D_n^{(n)} \geq 0$ and $\sum_{i=1}^n D_n^{(i)} = 1$ forces $ED_n^{(1)} \to 1$. This, in turn, implies (19) and thus $Y \in D(0)$. Hence we have (18) if and only if $Y \in D(0)$. □

Proof of Theorem 2. First assume that for some non–degenerate random variable R,

$$R_n \to_d R. \tag{20}$$

By Jensen's inequality for any $r \geq 1$,

$$\left|\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}\right|^r \leq \frac{\sum_{i=1}^n |X_i|^r Y_i}{\sum_{i=1}^n Y_i},$$

Thus for any $p > 2$

$$E\left|\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}\right|^p \leq E|X|^p.$$

This implies that whenever $R_n \rightarrow_d R$, where R is non–degenerate, then

$$ER_n^2 = EX^2 nE\left(\frac{Y_1}{\sum_{i=1}^n Y_i}\right)^2 \rightarrow EX^2\left(1-\alpha\right),$$

where necessarily $0 \leq \alpha < 1$. Thus by Proposition 1, $Y \in D\left(\alpha\right)$, with $0 \leq \alpha < 1$.
Breiman (1965) shows that whenever $Y \in D\left(\alpha\right)$, with $0 \leq \alpha < 1$, then (20) holds for some non–degenerate random variable R. To be specific, when $\alpha = 0$, $R =_d X$ and when $0 < \alpha < 1$, it can be shown by using the methods of Le Page et al (1981) that

$$R =_d \frac{\sum_{i=1}^\infty X_i\left(\Gamma_i\right)^{-1/\alpha}}{\sum_{i=1}^\infty \left(\Gamma_i\right)^{-1/\alpha}}. \tag{21}$$

This completes the proof of Theorem 2. □

The proof just given is highly dependent on the assumption that $E|X|^p < \infty$ for some $p > 2$. To replace it by the weaker assumption $E|X| < \infty$ would require an entirely different approach. Therefore the complete Breiman conjecture remains open. In the next section we provide some applications of our results to the study of the asymptotic distribution of relative ratio and self–normalized sums.

1 Applications

1.1 Application to relative ratio sums

Let $\{Y_i\}_{i\geq 1}$ be a sequence of i.i.d. Y non–negative random variables and for any $n \geq 0$ let $S_n = \sum_{i=1}^n Y_i$, where $S_0 := 0$. For any $n \geq 1$ and $0 \leq t \leq 1$, consider the *relative ratio sum*

$$V_n\left(t\right) := \frac{S_{[nt]}}{S_n}. \tag{22}$$

Our first corollary characterizes when such relative ratio sums converge in distribution to a non–degenerate law.

Corollary 4 *For any* $0 < t < 1$

$$V_n\left(t\right) \rightarrow_d V\left(t\right), \tag{23}$$

where $V\left(t\right)$ *is non–degenerate if and only if* $Y \in D\left(\alpha\right)$, *with* $0 \leq \alpha < 1$.

The proof Corollary 1 will be an easy consequence of the following proposition. Independent of $\{Y_i\}_{i\geq 1}$ let $\{\epsilon_i\left(t\right)\}_{i\geq 1}$ be a sequence of i.i.d. $\epsilon\left(t\right)$ random variables, where $P\left\{\epsilon\left(t\right) = 1\right\} = t = 1 - P\left\{\epsilon\left(t\right) = 0\right\}$, with $0 < t < 1$. For any $n \geq 1$ and $0 < t < 1$ let $[nt]$ denote the integer part of nt and set

$$N_n\left(t\right) = \sum_{i=1}^n \epsilon_i\left(t\right).$$

Proposition 5 *For all* $0 < t < 1$,

$$\frac{\sum_{i=1}^{N_n(t)} Y_i}{S_n} = V_n\left(t\right) + o_P\left(1\right). \tag{24}$$

Proof of Proposition 2. We have

$$\frac{\sum_{i=1}^{N_n(t)} Y_i}{S_n} = V_n(t) + \frac{\sum_{i=1}^{N_n(t)} Y_i - S_{[nt]}}{S_n}$$

and, clearly,

$$\left| \frac{\sum_{i=1}^{N_n(t)} Y_i - S_{[nt]}}{S_n} \right| =_d \frac{\sum_{i=1}^{M_m(t)} Y_i}{S_n},$$

where $M_m(t) = |N_n(t) - [nt]|$. Now (recalling that we define $0/0 = 0$), we have

$$E\left(E\left[\frac{\sum_{i=1}^{M_m(t)} Y_i}{S_n} | N_n(t) \right] \right) \leq \frac{E\,|N_n(t) - [nt]|}{n}.$$

Thus since $E\,|N_n(t) - [nt]|\,/n \to 0$, we get (24). □

Proof of Corollary 1. Note that

$$\frac{\sum_{i=1}^{N_n(t)} Y_i}{S_n} =_d \frac{\sum_{i=1}^{n} \epsilon_i(t)\, Y_i}{S_n}. \tag{25}$$

Therefore by Proposition 2 and (25), we readily conclude that (23) holds with a non–degenerate $V(t)$ if and only if

$$\frac{\sum_{i=1}^{n} \epsilon_i(t)\, Y_i}{S_n} - t = \frac{\sum_{i=1}^{n} (\epsilon_i(t) - t)\, Y_i}{S_n}$$

converges in distribution to a non-degenerate random variable. Thus Corollary 1 follows from Theorem 2. □

When $Y \in D(0)$, it is easy to apply Proposition 2, (25) and (19) to get that $V_n(t) \to_d \epsilon_1(t)$, and when $Y \in D(\alpha)$, with $0 < \alpha < 1$, one gets from Proposition 2, (25) and by arguing as in Le Page et al (1981), that

$$V_n(t) \to_d \frac{\sum_{i=1}^{\infty} \epsilon_i(t)\, \Gamma_i^{-1/\alpha}}{\sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha}}.$$

Also, one can show using Theorem 1, Theorem 2 and Proposition 1 of Breiman (1965) that

$$\frac{\sum_{i=1}^{n} (\epsilon_i(t) - t)\, Y_i}{S_n} \to_p 0,$$

if and only if there exists a sequence of positive constants $B_n \nearrow$ such that (9) holds. Furthermore, by Proposition 2 and (25), we see that this happens if and only if $V_n(t) \to_p t$.
An easy variation of Corollary 1, says that if $S_n' =_d S_n$, with S_n' and S_n independent, then

$$\frac{S_n'}{S_n} \to_d K,$$

where K is non–degenerate if and only if $Y \in D(\alpha)$, with $0 < \alpha < 1$. Again by using the techniques of Le Page et al (1981) one can show that

$$K =_d \frac{W_\alpha'}{W_\alpha},$$

where $W'_\alpha =_d W_\alpha$, W'_α and W_α are independent and

$$W_\alpha =_d \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha}.$$

Curiously, it can be shown that

$$L_\alpha := \log W_\alpha \text{ and } \log K$$

provide examples of random variables that have finite positive moments of any order, yet have distributions that are not uniquely determined by their moments. To see this, let h_α denote the density of L_α. Using known results about densities of stable laws that can be found in Ibragimov and Linnik (1971) and Zolotarev (1986) it can be proved that L_α has all positive moments and its density h_α is in C^∞. Moreover, it is readily checked that

$$-\int_{-\infty}^{\infty} \frac{\log h_\alpha(x)}{1+x^2} dx < \infty.$$

This implies that the distribution of L_α is not uniquely determined by its moments. Refer to Lin (1997). Furthermore, by a result of Devinatz (1959), this in turn implies that the distribution of $\log K =_d L'_\alpha - L_\alpha$, where L'_α is an independent copy of L'_α, is also not uniquely determined by its moments.

1.2 Application to self–normalized sums

Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. X random variables and consider the *self–normalized sums*

$$S_n(2) = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}}.$$

Logan, Mallows, Rice and Shepp (1973) conjectured that $S_n(2)$ converges in distribution to a standard normal random variable if and only if $EX = 0$ and $X \in D(2)$, and more generally that $S_n(2)$ converges in distribution to a non–degenerate random variable not concentrated on two points if and only if $X \in D(\alpha)$, with $0 < \alpha \leq 2$, where $EX = 0$ if $0 < \alpha < 1$ and X is in the domain of attraction of a Cauchy law in the case $\alpha = 1$. The first conjecture was proved by Giné, Götze and Mason (1997) and the more general conjecture has been recently established by Chistyakov and Götze (2004). Griffin and Mason (1991) attribute to Roy Erickson an elegant proof of the first conjecture of Logan et al (1973) in the case when X is symmetric about 0. We shall use Proposition 1 to extend Erickson's method to provide a short proof of the second conjecture of Logan et al (1973), for the symmetric about 0 case. In the following corollary s and Y are independent random variables, where $P\{s = 1\} = P\{s = -1\} = 1/2$. Since for a random variable X symmetric about 0, $X =_d sY$, where $Y =_d |X|$, it establishes the second Logan et al (1973) conjecture in the symmetric case. It is also Corollary 1 of Chistyakov and Götze (2004). The proof of the second Logan et al (1973) conjecture without the simplifying assumption of symmetry is much more lengthy and requires a lot of serious analysis.

Corollary 6 *Let $\{Y_i\}_{i\geq 1}$ be a sequence of non–negative i.i.d. Y random variables and independent of them let $\{s_i\}_{i\geq 1}$ be a sequence of independent Rademacher random variables. We* *have*

$$S_n(2) := \frac{\sum_{i=1}^n s_i Y_i}{\sqrt{\sum_{i=1}^n Y_i^2}} =: \frac{\sum_{i=1}^n s_i Y_i}{V_n} \to_d S(2), \tag{26}$$

where $S(2)$ *is a non-degenerate if and only if* $sY \in D(\alpha)$ *, where* $0 \le \alpha \le 2$.

In the proof of Corollary 2 we describe the possible limit laws and when they occur.

Proof of Corollary 2. When $sY \in D(0)$, then by using (19) one readily gets that

$$S_n(2) \to_d s.$$

Whenever $sY \in D(\alpha)$, with $0 < \alpha < 2$, we apply Corollary 1 of Le Page et al (1981) to get that

$$S_n(2) \to_d \frac{\sum_{i=1}^\infty s_i (\Gamma_i)^{-1/\alpha}}{\sqrt{\sum_{i=1}^\infty (\Gamma_i)^{-2/\alpha}}},$$

and when $sY \in D(2)$, Raikov's theorem (see Lemma 3.2 in Giné et al (1997)), implies that for any non–decreasing positive sequence $\{a_i\}_{i\ge 1}$ such that $\sum_{i=1}^n s_i Y_i / a_n \to_d Z$, where Z is a standard normal random variable, one has $\sum_{i=1}^n Y_i^2 / a_n^2 \to_p 1$, which gives

$$S_n(2) \to_d Z.$$

Next assume that $S_n(2) \to_d S(2)$, where $S(2)$ is non–degenerate. By Khintchine's inequality for any $k \ge 1$ we have $E|S_n(2)|^{2k} \le C_k$, for some constant C_k. Hence we can conclude that (26) implies that

$$3 - 2nE\left(\frac{Y_1^4}{\left(\sum_{i=1}^n Y_i^2\right)^2}\right) = ES_n^4(2) \to ES^4(2),$$

which since

$$0 \le nE\left(\frac{Y_1^4}{\left(\sum_{i=1}^n Y_i^2\right)^2}\right) = E\left(\frac{\sum_{i=1}^n s_i Y_i^2}{\sum_{i=1}^n Y_i^2}\right)^2 \le 1,$$

forces

$$nE\left(\frac{Y_1^4}{\left(\sum_{i=1}^n Y_i^2\right)^2}\right) \to 1 - \beta,$$

where $0 \le 1-\beta \le 1$. In the case $0 < 1-\beta \le 1$ Proposition 1 implies that $Y^2 \in D(\beta)$, which says that $Y \in D(\alpha)$, where $\alpha = 2\beta$. When $1-\beta = 0$, it is easy to argue using Markov's inequality that

$$\max_{1\le j\le n} Y_j^2 \Big/ \sum_{i=1}^n Y_i^2 \to_p 0,$$

which by Theorem 1 of Breiman (1965) implies that $Y \in D(2)$. □

1.3 A conjecture

As in Breiman (1965) we shall end our paper with a conjecture. For a sequence of i.i.d. positive random variables $\{Y_i\}_{i\geq 1}$, a sequence of independent Rademacher random variables $\{s_i\}_{i\geq 1}$ independent of $\{Y_i\}_{i\geq 1}$ and $1 \leq p < 2$, we conjecture that

$$S_n(p) := \frac{\sum_{i=1}^n s_i Y_i}{\left(\sum_{i=1}^n Y_i^p\right)^{1/p}} \to_d S(p), \tag{27}$$

where $S(p)$ is a non-degenerate random variable if and only if $Y \in D(\alpha)$, where $0 \leq \alpha < p$. At present we can only verify it for case $p = 1$ and the limit case $p = 2$.

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