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THRESHOLD PHENOMENA ON PRODUCT SPACES: BKKKL REVISITED (ONCE MORE)

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Abstract
We revisit the work of [Bourgain et al., 1992] – referred to as "BKKKL" in the title – about influences on Boolean functions in order to give a precise statement of threshold phenomenon on the product space $\{1,\dots,r\}^{\mathbb{N}}$, generalizing one of the main results of [Talagrand, 1994].

1 Introduction

The theory of threshold phenomena can be traced back to [Russo, 1982], who described it as an "approximate zero-one law" (see also [Margulis, 1974], [Kahn et al., 1988] and [Talagrand, 1994]). These phenomena occur on $\{0,1\}^n$ equipped with the probability measure μ_p which is the product of n Bernoulli measures with the same parameter $p \in [0,1]$. We say that an event $A \subset \{0,1\}^n$ is increasing if the indicator function of A is coordinate-wise nondecreasing. When the influence of each coordinate on an increasing event A is small (see the definition of γ hereafter), and when the parameter p goes from 0 to one, the probability that A occurs, $\mu_p(A)$, grows from near zero to near one on a short interval of values of p: this is the *threshold phenomenon.* The smaller the maximal influence of a coordinate on A is, the smaller is the bound obtained on the length of the interval of values of p. More precisely, for any j in $\{1,\dots,n\}$, define A_j to be the set of configurations in $\{0,1\}^n$ which are in A and such that j is *pivotal* for A in the following sense:

$$A_j = \{x \in \{0,1\}^n \text{ s.t. } x \in A, \text{ and } T_j(x) \notin A\} ,$$

where $T_j(x)$ is the configuration in $\{0,1\}^n$ obtained from x by "flipping" coordinate j to $1-x_j$. It is shown in [Talagrand, 1994], Corollary 1.3, that if you denote by γ the maximum over p and j of the probabilities $\mu_p(A_j)$, then, for every $p_1 < p_2$,

$$\mu_{p_1}(A)(1-\mu_{p_2}(A)) \le \gamma^{K(p_2-p_1)} , \tag{1}$$

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where K is a universal constant. This result was also derived independently by [Friedgut and Kalai, 1996], and both results were built upon an earlier paper by [Kahn et al., 1988], where one can find an important breakthrough with the use of Bonami-Beckner's hypercontractivity estimates. A much simpler proof, giving the best constants up to now, was obtained later by [Falik and Samorodnitsky, 2007], and their result will be one of the main tool that we shall use in this paper. See also [Rossignol, 2006] for a more complete description of threshold phenomena, and [Friedgut, 2004, Hatami, 2006] for related questions when the event A is not monotone.

This kind of phenomenon is interesting in itself, but has also been proved useful as a theoretical tool, notably in percolation (see [Bollobás and Riordan, 2006c, Bollobás and Riordan, 2006b, Bollobás and Riordan, 2006a, van den Berg, 2007]). It seems to be partly folklore that this phenomenon occurs on other product spaces than $\{0,1\}^n$. Notably, Theorem 5 in [Bollobás and Riordan, 2006a] gives a threshold result for symmetric functions on $\{1,2,3\}^n$ with an extremely short proof, mainly pointing to [Friedgut and Kalai, 1996]. A strongly related result is [Bourgain et al., 1992], where it is proved that for any subset A of a product probability space of dimension n, there is one coordinate that has influence of order at least $\log n/n$ on A. Although the result in [Bourgain et al., 1992] is stated in terms of influences and not in terms of threshold phenomena, the proof can be rephrased and slightly adapted to show that threshold phenomena occur on various product spaces.

Being asked by Rob van den Berg for a reference on generalizations of (1) to $\{1,\dots,r\}^{\mathbb{N}}$, we could not find a truly satisfying one. The work of [Paroissin and Ycart, 2003] is close in spirit to what we were looking for, but is stated only for symmetric sets in finite dimension. Also, Theorem 3.4 in [Friedgut and Kalai, 1996] is even closer to what we need but is not quite adapted to $\{1,\dots,r\}^{\mathbb{N}}$ since the quantity $\gamma = \max_j \mu_p(A_j)$ is replaced by the maximum of all influences, which is worse than the equivalent of γ in $\{1,\dots,r\}^{\mathbb{N}}$. The purpose of the present note is to provide an explicit statement of the threshold phenomenon on $\{1,\dots,r\}^{\mathbb{N}}$, with a rigorous, detailed proof. We insist strongly on the fact that the spirit of what is written in this note can be seen as already present in [Bourgain et al., 1992], [Friedgut and Kalai, 1996] and [Talagrand, 1994].

Our goal will be accomplished in two steps. The first one, presented in section 2, is a general functional inequality on the countable product $[0,1]^{\mathbb{N}}$ equipped with its Lebesgue measure. Then, in section 3, we present the translation of this result into a threshold phenomenon on $\{1,\dots,r\}^{\mathbb{N}}$. This is the main result of this note, stated in Corollary 3.1.

2 A functional inequality on $[0,1]^{\mathbb{N}}$, following [Bourgain et al., 1992]

In [Talagrand, 1994], inequality (1) is derived from a functional inequality on $(\{0,1\}^n, \mu_p)$ (Theorem 1.5 in [Talagrand, 1994]). Falik and Samorodnitsky's main result is also a functional inequality on $(\{0,1\}^n, \mu_p)$, with a slightly different flavour but the same spirit: it improves upon the classical Poincaré inequality essentially when the discrete partial derivatives of the function at hand have low L^1-norm with respect to their L^2-norm. Such inequalities have been extended to some continuous settings in [Benaim and Rossignol, 2006], where they were called "modified Poincaré inequalities". The discrete partial derivative is then replaced by a semi-group which is required to satisfy a certain hypercontractivity property.

In this section, we take a different road to generalize the modified Poincaré inequality of Talagrand (Theorem 1.5 in [Talagrand, 1994]). This is done by combining the approach of [Bourgain et al., 1992] and [Falik and Samorodnitsky, 2007]. This is also very close in spirit

to what is done in [Friedgut, 2004]. We will obtain a functional inequality on $[0,1]^{\mathbb{N}}$ equipped with the Lebesgue measure, which can be seen as a modified Poincaré inequality. All measures considered in this section are Lebesgue measures on Lebesgue measurable sets.
First, we need some notations. Let $(x_{i,j})_{\substack{i\geq 1\\ j\geq 0}}$ be independent symmetric Bernoulli random variables. For each j, the random variable $\sum_{i\geq 1}\frac{x_{i,j}}{2^i}$ is uniformly distributed on $[0,1]$, whereas $\sum_{i=1}^m \frac{x_{i,j}}{2^i}$ is uniformly distributed on $\{\frac{k}{2^m};\ k=0,\dots,2^m-1\}$. For positive integers m and n, define a random variable $X^{m,n}$ with values in $[0,1]^{\mathbb{N}}$ as follows:

$$(X^{m,n})_j = \begin{cases} \sum_{i=1}^m \frac{x_{i,j}}{2^i} & \text{if } j\leq n \\ \\ \sum_{i\geq 1}\frac{x_{i,j}}{2^i} & \text{if } j>n \end{cases}$$

For any real function f on $[0,1]^{\mathbb{N}}$, we define the following random variables:

$$\Delta_{i,j}^{m,n} f = f(X^{m,n}) - \mathbb{E}_{x_{i,j}}[f(X^{m,n})]\ ,$$

where $\mathbb{E}_{x_{i,j}}$ denotes the expectation with respect to $x_{i,j}$ only. Define λ to be the Lebesgue measure on $[0,1]$, and if f belongs to $L^2([0,1]^{\mathbb{N}},\lambda^{\otimes\mathbb{N}})$, denote by $Var_\lambda(f)$ the variance of f with respect to $\lambda^{\otimes\mathbb{N}}$.
Finally, define, for any positive integer n and any real numbers $y_1,\dots,y_n$:

$$f_n(y_0,\dots,y_n) = \int f(y_0,\dots,y_n,y_{n+1},\dots) \bigotimes_{k\geq n+1} d\lambda(y_k)\ .$$

We shall use the following hypothesis on f:

$$\text{For every integer } n,\, f_n \text{ is Riemann-integrable.} \tag{2}$$

The following result can be seen as a generalization of Theorem 1.5 in [Talagrand, 1994].

Theorem 2.1. *Let f be a real measurable function on $[0,1]^{\mathbb{N}}$. Define, for $p\geq 0$:*

$$N_p(f) = \limsup_{n\to\infty}\limsup_{m\to\infty}\sum_{j=0}^{n}\sum_{i=1}^{m}\mathbb{E}(|\Delta_{i,j}^{m,n} f|^p)^{\frac{2}{p}}\ .$$

Suppose that f belongs to $L^2([0,1]^{\mathbb{N}})$, and satisfies hypothesis (2). Then,

$$N_2(f) \geq \frac{1}{2}Var_\lambda(f)\log\frac{Var_\lambda(f)}{N_1(f)}\ .$$

Proof: Denote by $Y^{m,n}$ the first n coordinates of $X^{m,n}$. Theorem 2.2 in [Falik and Samorodnitsky, 2007] implies that:

$$\sum_{j=0}^{n}\sum_{i=1}^{m}\mathbb{E}(\Delta_{i,j}^{m,n} f^2) \geq \frac{1}{2}Var(f_n(Y^{m,n}))\log\frac{Var(f_n(Y^{m,n}))}{\sum_{j=0}^{n}\sum_{i=1}^{m}\mathbb{E}(|\Delta_{i,j}^{m,n} f|)^2}\ .$$

Notice that $\mathbb{E}(f_n(Y^{m,n}))$ is a Riemann-sum of f_n over $[0,1]^n$. Since f_n is Riemann-integrable, $Var(f_n(Y^{m,n}))$ converges to $Var(f_n(U_0,\dots,U_n))$ when m goes to infinity, where $U_0,\dots,U_n$ are independent random variables with uniform distribution on $[0,1]$. Then, this is for instance a consequence of Doob's convergence theorems for martingales bounded in L^2, $f_n(U_1, \dots, U_n)$ converges in L^2 to f as n tends to infinity. Thus,

$$\lim_{n\to\infty}\lim_{m\to\infty} Var(f_n(Y^{m,n})) = Var_\lambda(f) \; .$$

The theorem follows. □

If a function f is coordinate-wise nondecreasing, we shall say it is *increasing*. Now, we can get a simplified version of Theorem 2.1 for increasing functions. To this end, let us define the random variable X^∞ with values in $[0,1]^{\mathbb{N}}$ as follows:

$$\forall j \geq 0, \; (X^\infty)_j = \sum_{i\geq 1} \frac{x_{i,j}}{2^i} \; ,$$

and let:

$$\Delta_{i,j}^\infty f = f(X^\infty) - \mathbb{E}_{x_{i,j}}[f(X^\infty)] \; .$$

Corollary 2.2. *Let f be a real measurable function on $[0,1]^{\mathbb{N}}$, increasing for the coordinate-wise partial order. Define, for $p \geq 0$:*

$$M_p(f) = \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} \mathbb{E}(|\Delta_{i,j}^\infty f|^p)^{\frac{2}{p}} \; .$$

Then,

$$M_2(f) \geq \frac{1}{2} Var_\lambda(f) \log \frac{Var_\lambda(f)}{M_1(f)} \; .$$

Proof: We only need to show that f satisfies the hypotheses of Theorem 2.1, and that $N_p(f) \leq M_p(f)$, at least when p equals 1 and 2. Since f is coordinate-wise increasing, so is f_n for every n, and thus hypothesis (2) is satisfied. The function f is trivially in $L^2([0,1]^{\mathbb{N}}, \lambda^{\otimes\mathbb{N}})$ since it is a real measurable increasing function on $[0,1]^{\mathbb{N}}$, and therefore is bounded. We shall use the following notation: for $\varepsilon \in \{0,1\}$, $f(X^{m,n}|x_{i,j} = \varepsilon)$ denotes the value of f at $X^{m,n}$ where the value of $x_{i,j}$ is forced to be ε, and for $t \in [0,1]$, $f(X^{m,n}|y_j = t)$ denotes the value of f at $X^{m,n}$ where the value of $(X^{m,n})_j$ is replaced by t. We also use the notation $\mathbb{E}_{(x_{i',j})_{i'<i}}(g(X^{m,n}))$ to denote the expectation with respect to the random variables $(x_{i',j})_{i'<i}$. For $j \leq n$, and $p \geq 1$,

$$\begin{aligned}
\mathbb{E}_{(x_{i',j})_{i'\leq i}}(|\Delta_{i,j}^{m,n} f|^p) &= \frac{1}{2^p}\mathbb{E}_{(x_{i',j})_{i'<i}}(|f(X^{m,n}|x_{i,j}=1) - f(X^{m,n}|x_{i,j}=0)|^p) \; , \qquad (3)\\
&= \frac{1}{2^p}\sum_{\varepsilon\in\{0,1\}^{i-1}} \frac{1}{2^{i-1}} \left| f\left(X^{m,n}\middle| y_j = \sum_{i'=1}^{i-1}\frac{\varepsilon_{i'}}{2^{i'}} + \frac{1}{2^i} + \sum_{i'>i}\frac{x_{i',j}}{2^{i'}}\right)\right.\\
&\qquad \left. - f\left(X^{m,n}\middle| y_j = \sum_{i'=1}^{i-1}\frac{\varepsilon_{i'}}{2^{i'}} + \sum_{i'>i}\frac{x_{i',j}}{2^{i'}}\right)\right|^p \; ,\\
&= \frac{1}{2^{p+i-1}}\sum_{k=0}^{2^{i-1}-1} \left| f\left(X^{m,n}\middle| y_j = \frac{k}{2^{i-1}} + \frac{1}{2^i} + \sum_{i'>i}\frac{x_{i',j}}{2^{i'}}\right)\right.\\
&\qquad \left. - f\left(X^{m,n}\middle| y_j = \frac{k}{2^{i-1}} + \sum_{i'>i}\frac{x_{i',j}}{2^{i'}}\right)\right|^p \; ,
\end{aligned}$$

Let us define $t_k = \frac{k}{2^i} + \sum_{i'>i} \frac{x_{i',j}}{2^{i'}}$. Notice that $t_k < t_{k+1}$. Then,

$$\begin{aligned}
\mathbb{E}_{(x_{i',j})_{i' \leq i}}(|\Delta_{i,j}^{m,n} f|^p) &= \frac{1}{2^{p+i-1}} \sum_{k=0}^{2^{i-1}-1} |f(X^{m,n}|y_j = t_{2k+1}) - f(X^{m,n}|y_j = t_{2k})|^p \ , \\
&\leq \frac{(2\|f\|_\infty)^{p-1}}{2^{p+i-1}} \sum_{k=0}^{2^{i-1}-1} |f(X^{m,n}|y_j = t_{2k+1}) - f(X^{m,n}|y_j = t_{2k})| \ , \\
&\leq \frac{\|f\|_\infty^p}{2^{i-1}} \ ,
\end{aligned}$$

since f is increasing. Thus, when n and j are fixed, $(\mathbb{E}(|\Delta_{i,j}^{m,n} f|^p \ \mathbb{1}_{i \leq m})^2)_{i \geq 1}$ is dominated by $(2^{2-2i}\|f\|_\infty^{2p})_{i\geq 1}$, whose sum converges. On the other hand, since f is coordinate-wise increasing, the function $y_j \mapsto f((y_i)_{i\geq 1})$ is Riemann-integrable for any fixed $(y_i)_{i\neq j}$ and any j. Thus,

$$\lim_{m\infty} \mathbb{E}(|\Delta_{i,j}^{m,n} f|^p) = \mathbb{E}(|\Delta_{i,j}^{\infty} f|^p) \ .$$

Therefore, by Lebesgue's dominated convergence theorem,

$$\lim_{m\infty} \sum_{i=1}^{m} \mathbb{E}(|\Delta_{i,j}^{m,n} f|^p)^{\frac{2}{p}} = \sum_{i=1}^{\infty} \mathbb{E}(|\Delta_{i,j}^{\infty} f|^p)^{\frac{2}{p}} \ ,$$

which implies $N_p(f) = M_p(f)$. The result follows from Theorem 2.1. $\square$

3 Threshold phenomenon on $\{1, \ldots, r\}^{\mathbb{N}}$

Let r be a positive integer. Let $I =]a, b[$ be a connected open subset of $\mathbb{R}$ with $a < b$, and for every t in I, let μ_t be a probability measure on $\{1, \ldots, r\}$, $\nu_{t,n}$ be the product measure $\mu_t^{\otimes n}$ on $H_n = \{1, \ldots, r\}^n$ and $\nu_{t,\mathbb{N}}$ be the product measure $\mu_t^{\otimes \mathbb{N}}$ on $H_{\mathbb{N}} = \{1, \ldots, r\}^{\mathbb{N}}$. We suppose that for every k in $\{1, \ldots, r\}$, the function $t \mapsto \mu_t(\{k\})$ is differentiable on I, and that for every k in $\{2, \ldots, r\}$, $t \mapsto \mu_t(\{k, k+1, \ldots, r\})$ is strictly increasing. Then, we suppose that:

$$\lim_{t\to a} \mu_t(\{1\}) = 1, \text{ and } \lim_{t \to b} \mu_t(\{r\}) = 1 \ .$$

The following result is a generalization of Corollary 1.3 in [Talagrand, 1994].

Corollary 3.1. *Let A be an increasing measurable subset of $\{1, \ldots, r\}^{\mathbb{N}}$. Let $t_1 \leq t_2$ be two real numbers of I. Define:*

$$\gamma_t := \sup_j \nu_{t,\mathbb{N}}(A_j) \ ,$$

$$\gamma_* = \sup_{t\in[t_1,t_2]} \left\{ \max\{\gamma_t, \gamma_t \log \frac{1}{\gamma_t}\} \right\} \ ,$$

and

$$S^* = \inf_{t\in[t_1,t_2]} \inf_{k=2,\ldots,r} \frac{d}{dt}\mu_t(\{k, k+1, \ldots, r\}) \ .$$

Then,

$$\nu_{t_1,\mathbb{N}}(A)(1 - \nu_{t_2,\mathbb{N}}(A)) \leq \gamma_*^{S^*(t_2-t_1)} \ .$$

Proof : Let $f = \mathbb{1}_A$. Suppose first that A depends only on a finite number of coordinates. Then,

$$\frac{d}{dt}\nu_{t,\mathbb{N}}(A) = \sum_{j\geq 0}\int\sum_{k=1}^{r}\mu_t'(k)f(x|x_j=k)\ d\nu_{t,\mathbb{N}}(x)\ ,$$

where $\mu_t'(k) = \frac{d}{dt}\mu_t(\{k\})$. Define, for any $k \in \{1,\ldots,r\}$,

$$S_{t,k} := \sum_{l=k}^{r}\mu_t'(l) = \frac{d}{dt}\mu_t(\{k,k+1,\ldots,r\})\ .$$

By hypothesis, $S_{t,k} \geq 0$ for any k in $\{2,\ldots,r\}$. Notice also that $S_{t,1} = 0$. Letting $S_{t,r+1} := 0$, we have:

$$\begin{aligned}\sum_{k=1}^{r}\mu_t'(k)f(x|x_j=k) &= \sum_{k=1}^{r}(S_{t,k}-S_{t,k+1})f(x|x_j=k)\ ,\\ &= \sum_{k=2}^{r}S_{t,k}(f(x|x_j=k)-f(x|x_j=k-1))\ .\end{aligned}$$

Define:

$$S_t^* = \inf_{k=2,\ldots,r}S_{t,k} > 0\ .$$

Since f is the indicator function of an increasing event A in $H_{\mathbb{N}}$,

$$\sum_{k=1}^{r}\mu_t'(k)f(x|x_j=k) \geq S_t^*(f(x|x_j=r)-f(x|x_j=1))\ .$$

Thus,

$$\frac{d}{dt}\nu_{t,\mathbb{N}}(A) \geq S_t^*\sum_{j\geq 0}\int f(x|x_j=r)-f(x|x_j=1)\ d\nu_{t,\mathbb{N}}(x)\ . \tag{4}$$

Now, we do not suppose anymore that A depends on finitely many coordinates. Define, for any real function g on $[0,1]$,

$$\frac{d^+}{dt}g(t) = \liminf_{t'\downarrow t}\frac{g(t')-g(t)}{t'-t}\ .$$

It is a straightforward generalization of Russo's formula for general increasing events (see (2.28) in [Grimmett, 1999], and the proof p. 44) to obtain from inequality (4) that when A is measurable, and $f = \mathbb{1}_A$,:

$$\frac{d^+}{dt}\nu_{t,\mathbb{N}}(A) \geq S_t^*\sum_{j\geq 0}\int f(x|x_j=r)-f(x|x_j=1)\ d\nu_{t,\mathbb{N}}(x)\ . \tag{5}$$

Define $I(f)$ the total sum of influences for the event A:

$$I(f) = \sum_{j\geq 0}\int f(x|x_j=r)-f(x|x_j=1)\ d\nu_{t,\mathbb{N}}(x)\ .$$

Let $(u_j)_{j\geq 0}$ be a sequence in $[0,1]^{\mathbb{N}}$. Define a function F_t from $[0,1]^{\mathbb{N}}$ to $\{1,\dots,r\}^{\mathbb{N}}$ as follows:

$$\forall j\in\mathbb{N},\ \forall i\in\{1,\dots,r\},\ (F_t(u))_j = i \text{ if } \mu_t(\{1,\dots,i-1\})\leq u_j < \mu_t(\{1,\dots,i\})\ .$$

Of course, under $\lambda^{\mathbb{N}}$, $F_t(u)$ has distribution $\nu_{t,\mathbb{N}}$. Define g_t to be the increasing, measurable function $f\circ F_t$ on $[0,1]^{\mathbb{N}}$. Using Corollary 2.2,

$$M_2(g_t)\geq \frac{1}{2}Var_\lambda(g_t)\log\frac{Var_\lambda(g_t)}{M_1(g_t)}\ . \tag{6}$$

First, notice that:

$$Var_\lambda(g_t)=Var(f)=\nu_{t,\mathbb{N}}(A)(1-\nu_{t,\mathbb{N}}(A))\ . \tag{7}$$

Then, according to equation (3), and since g_t is increasing and non-negative,

$$\begin{aligned}\sum_{i=1}^{\infty}\mathbb{E}(|\Delta_{i,j}^{m,n}g_t|^2) &\leq \frac{1}{4}\sum_{i=1}^{\infty}\frac{1}{2^{i-1}}\mathbb{E}(g_t(X^{m,n}|y_j=1)-g_t(X^{m,n}|y_j=0))\ ,\\ &= \frac{1}{2}\int f(x|x_j=r)-f(x|x_j=1)\ d\nu_{t,\mathbb{N}}(x)\ .\end{aligned}$$

Thus,

$$M_2(g_t)\leq\frac{1}{2}I(f)\ , \tag{8}$$

Similarly,

$$\sum_{j\geq 0}\sum_{i=1}^{\infty}\mathbb{E}(|\Delta_{i,j}^{\infty}g_t|)\leq I(f)\ . \tag{9}$$

For any j in $\mathbb{N}$, define A_j to be the set of configurations in $\{1,\dots,r\}^{\mathbb{N}}$ which are in A and such that j is pivotal for A:

$$A_j=\{x:\ x\in A,\ \text{and}\ f(x|x_j=1)=0\}\ .$$

Since g_t is increasing,

$$\begin{aligned}\mathbb{E}(|\Delta_{i,j}^{\infty}g_t|) &= \mathbb{E}(g_t(X^\infty)-g_t(X^\infty|x_{i,j}=0))\ ,\\ &\leq \mathbb{E}(g_t(X^\infty)-g_t(X^\infty|y_j=0))\ ,\\ &= \int f(x)-f(x|x_j=1)\ d\nu_{t,\mathbb{N}}(x)\ .\end{aligned}$$

and thus, for any $i\geq 1$,

$$\mathbb{E}(|\Delta_{i,j}^{\infty}g_t|)\leq \nu_{t,\mathbb{N}}(A_j)\ . \tag{10}$$

Define $\gamma_t:=\sup_j\nu_{t,\mathbb{N}}(A_j)$. From (9) and (10), we get:

$$M_1(g_t)\leq\gamma_t I(f)\ .$$

This inequality together with (6), (7) and (8) leads to:

$$I(f)\geq Var(f)\log\frac{Var(f)}{\gamma_t I(f)}\ . \tag{11}$$

Therefore,
• either $I(f) > Var(f)\log\frac{1}{\gamma_t}$,
• or $I(f) \leq Var(f)\log\frac{1}{\gamma_t}$, and in this case, plugging this inequality into the right-hand side of (11),

$$I(f) \geq Var(f)\log\frac{1}{\gamma_t\log\frac{1}{\gamma_t}} \ .$$

In any case, defining $\gamma_t^* = \sup\{\gamma_t, \gamma_t\log\frac{1}{\gamma_t}\}$, it follows from (5) that:

$$\frac{d^+}{dt}\nu_{t,\mathbb{N}}(A) \geq S_t^*\nu_{t,\mathbb{N}}(A)(1-\nu_{t,\mathbb{N}}(A))\log\frac{1}{\gamma_t^*} \ .$$

Now, let $\gamma_* = \sup_{t\in[t_1,t_2]}\gamma_t^*$ and $S^* = \inf_{t\in[t_1,t_2]S_t^*}$. We get:

$$\frac{d^+}{dt}\left[\log\frac{\nu_{t,\mathbb{N}}(A)}{1-\nu_{t,\mathbb{N}}(A)} - tS^*\log\frac{1}{\gamma_*}\right] \geq 0 \ ,$$

for any t in $[t_1,t_2[$. It follows from Proposition 2, p. 19 in [Bourbaki, 1949] (it is important to notice that the proof of this Proposition works without modification if the function f equals $g+h$ where g is increasing and h continuous, and if the right-derivative is replaced by d^+/dt) that:

$$\begin{aligned}
\log\frac{\nu_{t_2,\mathbb{N}}(A)(1-\nu_{t_1,\mathbb{N}}(A))}{\nu_{t_1,\mathbb{N}}(A)(1-\nu_{t_2,\mathbb{N}}(A))} &\geq (t_2-t_1)S^*\log\frac{1}{\gamma_*} \ , \\
\frac{\nu_{t_1,\mathbb{N}}(A)(1-\nu_{t_2,\mathbb{N}}(A))}{\nu_{t_2,\mathbb{N}}(A)(1-\nu_{t_1,\mathbb{N}}(A))} &\leq \gamma_*^{S^*(t_2-t_1)} \ ,
\end{aligned}$$

and the result follows. □

Remark: If one wants a cleaner version of the upperbound of Corollary 3.1 in terms of $\eta_* := \sup_{t\in[t_1,t_2]}\sup_j \nu_{t,\mathbb{N}}(A_j)$, simple calculus shows that $\gamma_* \leq \eta_*^{1-1/e} \leq \eta_*^{1/2}$, which leads to:

$$\nu_{t_1,\mathbb{N}}(A)(1-\nu_{t_2,\mathbb{N}}(A)) \leq \eta_*^{S^*(t_2-t_1)/2} \ .$$

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