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SOME RESULTS FOR POISONING IN A CATALYTIC MODEL

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Abstract

We obtain new results concerning poisoning/nonpoisoning in a catalytic model which has previously been introduced and studied. We show that poisoning can occur even when the arrival rate of one gas is smaller than the sum of the arrival rates of the other gases, and that poisoning does not occur when all gases have equal arrival rates.

1 Introduction

Grannan and Swindle (1991) introduced a collection of interacting particle systems to model catalytic surfaces. In these models n types of gases represented by states $\{1, 2, ..., n\}$ fall on the sites of the integer lattice Z^d according to independent rate 1 Poisson processes. A vacant site is represented by state 0. An arriving gas molecule is of type i with probability p_i. Therefore molecules of gas i fall upon vacant sites at rate p_i, where $\sum p_i = 1$. No two different gases can occupy adjacent sites and so if a molecule of type i falls upon a vacant site adjacent to a molecule of type $j, j \neq i$, then the two gases react and both sites are left vacant. If there are several adjacent sites with molecules different from the arriving molecule, one of them is selected uniformly at random to react with the arriving molecule.

Grannan and Swindle (1991) defined ”poisoning” as the configuration converging a.s. as time goes to infinity. In such a case, the limit is necessary the “all i configuration” for some $i \in \{1, 2, ..., n\}$. Let δ_i be the point mass at the configuration where everything is in state i and define *coexistence* as the existence of a stationary distribution which is not a convex combination of the $\{\delta_i\}$’s. Heuristically, poisoning implies noncoexistence (but this statement is

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not precise). Grannan and Swindle (1991) proved the following result. (They proved somewhat more than what we state.)

Theorem 1 *(Grannan and Swindle) (i). If $n = 2$, d is arbitrary and $p_1 \neq 1/2$, then coexistence does not occur.*
(ii). There exists $\epsilon_0 > 0$ such that for any n and d, if $p_1 > 1 - \epsilon_0$, then for any initial state having infinitely many 0's, a.s. poisoning occurs with the limiting configuration being all 1's; i.e., $\eta_t \to 1$, the configuration consisting of all 1's.
(iii). For any n and d, if $p_1 < \frac{1}{2d\lambda_c(d)+1}$ (where $\lambda_c(d)$ is the critical value for the contact process in d–dimensions), then, for any initial state having infinitely many 0's, a.s. the system does not converge to the all 1 configuration.
(iv). For every d, there exists N_d such that if $n \geq N_d$, there exist $p_1, \ldots, p_n$ so that coexistence occurs. For $d = 1$, one can have coexistence with $n = 5$.

(i) was proved using a type of "energy" argument, (ii) was quite involved and (iii) and (iv) were proved by arguments involving stochastic comparison with a supercritical contact process. It is not known whether coexistence can occur for less than 5 gases in 1 dimension.
Mountford and Sudbury (1992) strengthened part (ii) (using a submartingale argument) of the above result by showing:

Theorem 2 *(Mountford and Sudbury) Let n and d be arbitrary and $p_1 > 1/2$. Then, if the initial state,η_0, has infinitely many 1's or 0's, then poisoning occurs with $\eta_t \to 1$ a.s.*

A question left open is whether it is possible for a gas to poison a surface with its arrival rate being less than 1/2. Our first result proves this to be the case.

Theorem 3 *If $d = 1$ and $n = 4$, if the first gas has rate 0.47 and the other three have rates $0.53/3$, then the first gas will poison the surface with probability 1 if the initial state of the surface has an infinite number of 0's or 1's.*

It is trivial to observe that if poisoning occurs from every initial state, then there cannot be coexistence. However, one should not think of these things as being synonymous as we will see. In addition, the proof of Theorem 2 shows that for any d and n, if $p_1 > 1/2$, then coexistence does not occur. For the interesting case $p_1 = p_2$, Mountford (1992) proved the following result. (What we state is just a special case of what he proved.)

Theorem 4 *(Mountford) For $d = 1$ and $n = 2$, the only translation invariant stationary distributions are $\{p\delta_1 + (1-p)\delta_2\}_{p\in[0,1]}$.*

This *strongly* suggests that coexistence does not occur here. Our second result says that this noncoexistence result is not due to poisoning.

Theorem 5 *Let n and d be arbitrary. If there is $i \neq j$ with $p_i = p_j$, then, starting from all 0's, the probability of poisoning in states i or j is 0. In particular, if all of the p_i's are equal, then, starting from all 0's, the probability of poisoning is 0.*

Theorem 3 is proved in Section 2, Theorem 5 is proved in Section 3, some further comments and questions are given in Section 4 and finally there is an appendix which provides some assistance for one of the proofs.

2 Poisoning in 1-dimension with rate $< 1/2$

In this section, we prove Theorem 3.

Proof of Theorem 3: The method used will be to consider a block of adjacent 1's and find conditions under which they tend to spread. At the right-hand end of such a block there must be a 0. We condition on the types occupying the next three spaces to the right. There are 18 essentially different possibilities as the reader can check; see column 1 of Table 1 for a list of these. Blocks such as 030 and 020 or 203 and 304 are considered equivalent.

Consider the same model but defined only on the nonnegative integers. Equivalently, at time 0, there is a 0 in position -1 and no particles are allowed to arrive at this position. We show that if there is a 1 at the origin, then with a fixed positive probability, independent of the configuration to the right of the origin, the origin remains in state 1 forever and the block of 1's containing the origin approaches ∞. (Of course if the origin remains in state 1 forever, it must be the case that the block of 1's containing the origin approaches ∞.) If this can be done, it follows that in the original model, any 1 will spread in both directions poisoning all of the integers with a fixed positive probability. Since there are initially infinitely many 0's or 1's, it easily follows that poisoning in state 1 occurs a.s.

In order to prove the statement concerning the spread of 1's, for each configuration η on the nonnegative integers which has a 1 or 0 at the origin, we define its weight $W(\eta)$ as follows. Let $B(\eta)$ be the block of 1's starting from the origin in the configuration η. (This may be empty or infinite.) If $|B(\eta)| = \infty$ (equivalently $\eta \equiv 1$), then we let $W(\eta) := \infty$. Otherwise, $W(\eta)$ is defined to be $|B(\eta)|$ plus the score, as defined in column 2 of Table 1, of the block following the first 0 after $B(\eta)$. Note that when the block is empty, then the 0 at the origin is the first 0 after $B(\eta)$ and so $W(\eta)$ is then the score of the block sitting at locations 1,2 and 3. The idea is to define the scores of the blocks (abc) in such a way that the expected change in the weight is always positive, in which case we will obtain a submartingale. (Columns 3 and 4 are not used in this proof and so they can be ignored at this point; they will be used later on for certain explanations.)

Letting $\{\eta_t\}_{t\geq 0}$ denote our process on the nonnegative integers, a very long and tedious calculation, left to the reader, shows that there exists $c > 0$ such that for all η with $B(\eta) \neq \emptyset$,

$$\frac{dE[W(\eta_t)|\eta_0 = \eta]}{dt}|_{t=0} \geq c. \tag{1}$$

[While this long detailed calculation is being left to the reader, the appendix contains a discussion which aids the reader in making this calculation; perhaps it takes 2 hours of work to check the above with the aid of the appendix and 5 hours otherwise.] It follows that with positive probability, uniformly in η, with $B(\eta) \neq \emptyset$, the block of 1's will grow to ∞ before the 1 at the origin changes which is what we wanted to show. To carefully do this, we follow the argument for Theorem 3 in Sudbury (1999) and proceed as follows. Let T be the (possibly infinite) stopping time when the block of 1's at the origin disappears. Equation (1) implies that $\{W(\eta_{t\wedge T})\}_{t\geq 0}$ is a submartingale with respect to the natural filtration of $\{\eta_t\}_{t\geq 0}$. Moreover, it can be shown that Equation (1), together with the fact that the jumps downward are bounded, implies that for ϵ sufficiently small, for any initial configuration η with $B(\eta) \neq \emptyset$, $U_t := 1 - (1-\epsilon)^{W(\eta_{t\wedge T})}$ is a bounded submartingale and thus tends a.s. to a limit U_∞. Let us assume further for the moment that $|B(\eta_0)| \geq 2$. This gives us that $U_0 \geq 1 - (1-\epsilon)^2$. If the block of 1's dies out, then $U_\infty \leq 1 - (1-\epsilon)^{1.997}$. Since $E[U_0] = E[U_\infty]$, it follows that it cannot be the case that the block of 1's dies out a.s. Since however U_t converges a.s., it must be the case that the block of 1's grows to infinity with a uniform positive probability, independent of η_0 with $|B(\eta_0)| \geq 2$. If $|B(\eta_0)| = 1$, it is clear that with a uniform positive probability, independent of η_0, the single 1 at the origin spreads and reaches size at least 2 at time 1. At that point, one can apply the previous argument.

Table 1
Scores for essentially different blocks

block	Score	Expression	Follower
222	0.000	0.0007	2
220	0.163	0.0012	2
202	0.295	0.0022	2
203	0.339	0.0002	3
022	0.354	0.0034	2
200	0.404	0.0002	2
201	0.493	0.0018	00
020	0.498	0.0031	2
002	0.570	0.0032	2
000	0.664	0.0055	2
001	0.827	0.0058	00
010	0.920	0.0044	2
102	1.008	0.0034	22
100	1.157	0.0036	22
011	1.173	0.0060	00
101	1.456	0.0584	02
110	1.555	0.0040	222
111	1.997	0.0835	0222

Remarks:
(i). The score (abc) represents whether this triple is likely to aid the 1's in spreading or make them more likely to contract.
(ii). The above proof is *not* computer assisted in the sense that one can check the correctness of the proof without a computer; it suffices to use a hand calculator or in fact even hand calculations suffice (the latter requiring a good deal of patience). Nonetheless, a computer was essential in helping us find the proof and in particular helping us find the scores for the various blocks of length 3. More discussion concerning this point follows below.
(iii). One might hope that we could have carried out the above proof using blocks of length 2 instead of blocks of length 3 but it seems that this is not possible if we want to have $p_1 < 1/2$ as also described below.

While it is not needed at all for checking the correctness of the proof of Theorem 3, we now nonetheless explain in some detail how the scores in Table 1 were arrived at which in turn allowed us to obtain the proof. We first tried blocks of size 2. Our method was to try various values of p_1. As in the proof, we considered only the situation at the right-hand end. We wrote down the equations for the instantaneous rates of change of the expected weight for the various blocks of size 2. The basic idea was to find the values of the $(ab), ab \neq 22$ which make all these rates 0. Having done this we wrote down the equation for (22). If the rate was > 0 then we would have found a value of p_1 which allows the block of 1's to spread. If the rate was < 0, then we would need a larger value of p_1. (In fact we found values of the

(ab) which gave a rate very close to 0 and then increased p_1 so that the rate of change of the expected weight was strictly positive for all blocks). When we computed this change in the expected weight, we computed it assuming to the right of the block one had what one would guess is the most disadvantageous scenario for spreading of the 1's. The only case considered was $p_2 = p_3 = p_4$. A little experimentation with other possibilities suggested they would give less favorable results.
Unfortunately it was impossible to make all the necessary expressions positive with $p_1 < 0.5$. This meant we had to go up to the next level, looking at blocks of length 3 to the right of the block of 1's and their immediate 0. There were then 18 fundamentally different expressions that must be made > 0. Again, we assumed that $p_2 = p_3 = p_4$. We assumed that $1 > 0 > 2, 3, 4$ in the sense that the score for every block is increased if the value in the block is increased by replacing it by a "higher" value. For example, we assumed that $(020) < (000) < (010)$. This assumption is easily checked by looking at the 18 scores in column 2 of Table 1. For each 3 block, there is a follower to the right of it which reasonably can be assumed to be the worst case scenario to the right of this 3 block; this is listed in the fourth column in Table 1. For each 3 block, assuming that the follower listed in column 4 comes right after it, we can compute the rate of change of $W(\eta)$. These rates of changes are as follows immediately below. Finally, the Expression in column 3 is the number obtained when plugging in the scores from column 2 for each of the 3 blocks in the expressions below; this number has to of course be positive.
<u>000</u>

$$p_1((002) + 1 + (100) + (010) + (000)) + 3p_2(-1 + (000) + (200) + (020)) + p_2(002)+$$
$$2p_2(000) - 4(000)$$

<u>203</u>

$$p_1((002) + .5((002) + (200))) + p_2(-1 + (020) + (200) + (002))$$
$$+p_2(-1 + (020) + (002) + ((002) + (200))/2) - 2(203)$$

<u>100</u>

$$p_1(2 + (022) + (110) + (100)) + 3p_2((010) - 1 + (000))/2$$
$$+3p_2(000) + 2p_2(100) + p_2(102) - 3(100)$$

<u>111</u>

$$p_1 4 + 3p_2(-1 + (011)2)/2 - (111) + 2p_2((110) - (111))$$

<u>001</u>

$$p_1(1 + (010) + (101) + (011)) + 3p_2(-1 + (000) + (201) + (000)) - 3(001)+$$
$$(1 - p_1)((000) - (001))$$

<u>102</u>

$$p_1(2 + (100)) + 3p_2(-1 + (010) + (002))/2 + p_2((002) + (100) + (002)) - 2(102)$$

<u>220</u>

$$p_1((020) + ((200) + (220))/2) + p_2(-1 + (022))$$
$$+p_2(-1 + (022) + (020) + (220) + (200)) - 2(220)$$

<u>022</u>

$$p_1(1 + (002)) + 3p_2(-1 + (002)) + 2p_2(002) - 2(022)$$

<u>202</u>

$$p_1((002) + ((200) + (002))/2) + p_2(-1 + (020))+$$
$$p_2(-1 + (020) + (002) + (002) + (200)) - 2(202)$$

<u>200</u>

$$p_1((000) + (000) + (200)) + p_2(-1 + (020) + (220) + (202))$$
$$+p_2(-1 + (020) + (000)3 + 2(200)) - 3(200)$$

<u>020</u>

$$p_1(1 + (202) + 1.5(000) + .5(020)) + p_2(-1 + (002) + (220) + (022))$$
$$+2p_2(-1 + (002) + 1.5(000) + .5(020)) - 3(020)$$

<u>201</u>

$$p_1((001)2) + p_2(-1 + (020) + (200) - 1 + (020) + 2(001) + (200)) - 2(201)+$$
$$(1 - p_1)((200) - (201))$$

<u>010</u>

$$p_1((1 + (102) + (110) + (010)) + 3p_2(-1 + (001) + (000)) + p_2((000)2 + (010)) - 3(010)$$

<u>002</u>

$$p_1(1 + (022) + (102) + (000)) + p_2(-1 + (000) + (202) + (022))+$$
$$2p_2(-1 + (000) + (302) + (000)) - 3(002)$$

<u>011</u>

$$p_1(1 + (110) + (111)) + 3p_2(-1 + (001) + (001)) - 2(011) + (1 - p_1)((010) - (011))$$

<u>110</u>

$$p_1(3 + (110)) + 3p_2(-1 + (011) + (010))/2 + p_2(2(100) + (110)) - 2(110)$$

<u>101</u>

$$p_1(2 + (102) + (111)) + 3p_2(-1 + (010) + (001) + (001) + (100))/2 - 2(101)+$$
$$2p_2((100) - (101))$$

<u>222</u>

$$p_1(022) + p_2(-2 + 3(022))$$

A trial value of p_1 is chosen and the first 17 equations are then solved. All 18 equations are not used, because, since the total rates in and out of the blocks must balance, the set of equations would be singular. The last expression, which is for (222) was then tested. If it was positive, a suitable set of values would have been found. If it was not, p_1 was replaced by a larger value and the procedure was started again. A suitable set of scores was found for $p_1 = 0.4699$. The expressions above were then independently checked (with the package Minitab) using $p_1 = 0.47$ to ensure all the expressions came out strictly positive. The first few decimal places of these expressions are given in column 3 of Table 1. (It should be noted that the computer program tests the transitions for all possible scenarios to the right of the blocks of length 3 given above since these scenarios can influence the outcome. The expressions given above use simply the best guess as to which of the values $0, 1, 2, 3, 4$ will give the least advantageous result for the spread of the block of 1's. The guesses are in fact correct as can be checked.) The checks

made by Minitab can then also be carried out by hand and/or pocket calculator obtaining the rigorous noncomputer assisted proof presented above.
One can next attempt to prove that even lower values of p_1 can spread with 4 types or alternatively study this problem with a different number of types. In either case, one has to deal with blocks of length greater than 3 and then the equations become too complicated to exhibit or to do calculator or hand computations with. The computer program used can only continue up until blocks of length 5 as the particular version of Fortran used only allows arrays of size up to 4096. Further, one can consider the case when there are infinitely many gases. In this limiting case it is assumed that any new arrival will react with an adjacent gas unless it is of type 1 in which case there will be no reaction if the arriving gas is also type 1. Since this situation is more favorable to gas 1, it is possible to use only blocks of length 2 to show that in this case, gas 1 can spread with a rate of 0.46.
Theorem 3 may then be improved as follows using the computer program mentioned above. Belief in the following theorem thus relies on belief that the program is correct. The program gives the same results as hand calculation for blocks of length 2 and 3.

Theorem 6 *(Computer assisted proof) In one dimension, poisoning will occur for rates of gas 1 greater than those appearing in Table 2 (with all other gases having the same rate) if the initial state of the surface has an infinite number of 0's or 1's.*

Table 2
Upper bounds to critical values for poisoning

Number of Gases	Rate of gas 1	Rate of other gases
2	> 0.5	< 0.5
3	0.445	0.277
4	0.432	0.189
∞	0.46	0.000

In the above table, blocks of length 6 were used for 3 gases, blocks of length 5 for 4 gases and blocks of length 2 for infinitely many gases. Random simulation of the process suggested the following critical values: 0.40 for 3 gases, 0.38 for 4 gases and 0.37 for infinitely many gases. (The referee also carried out simulations which agreed with ours in the first two cases but gave a number in $[.361, .363]$ in the third case.)

3 Nonpoisoning

In this section, we prove Theorem 5. The argument here is very similar to that used in Bramson and Griffeath (1989) and Cox and Klenke (2000).
Recall that in this result, we are starting with all 0's. For each $i = 1, \ldots, n$, let A_i be the event that poisoning occurs and that the final state of all sites is i. Clearly, the A_i's are disjoint and their union is the event in question.

Lemma 7 *For each i, $P(A_i) \in \{0, 1\}$. (This lemma does not require any assumption on the rates.)*

Given this lemma, we are done as follows. $P(A_i) = P(A_j)$ by symmetry and by by Lemma 7, these are each 0 or 1. Since they are disjoint, they each must be 0. The last statement of the result follows immediately.

Proof of Lemma 7. Of course, we just need to consider A_1. The idea is that although we have stochastic dynamics, we can encode all the randomness we will need to drive the dynamics into random variables associated to each lattice point. To drive the dynamics, for each location, we need n independent Poisson processes (for the arrivals of the n different types of particles) and we will also need some more random variables at each location to be used to decide which particle will be reacted with if the arriving particle lands next to more than 2 particles of a different type.

The details of how to do this are as follows. Let $\{U_i\}_{i\in Z^d}$ be i.i.d. random variables each uniformly distributed on $[0,1]$. The entire evolution of our process will simply be a function of the $\{U_i\}$'s (i.e., it will be determined by the $\{U_i\}$'s); no further randomness will be needed. On the unit interval $[0,1]$ with the Borel sets and with Lebesgue measure, define random variables $\{X_t\}_{t\geq 0}$, $\{Y_k\}_{k\geq 1}$ and $\{Z_k\}_{k\geq 1}$ such that these three processes are independent of each other, the first process is a rate 1 Poisson process, the Y_k's are i.i.d. with $P(Y_k=j)=p_j$ for $j=1,\dots,n$, and the Z_k's are i.i.d. with each Z_k being a uniform random ordering of the $2d$ neighbors of the origin in Z^d. Note, crucially, that the X_t's, Y_k's and Z_k's are *functions* defined on $[0,1]$.

Note that we know that on *some* probability space, we can define random variables with the above distribution but the point is that we want to define them on the unit interval $[0,1]$ with the Borel sets. It is well known that this probability space is *rich* enough to be able to define these random variables on it. We of course, as usually done in probability theory, do not need to explicitly give what these functions are; it is only required that they have the right distribution.

Now, for each $i\in Z^d$, we can consider the random variables $\{X_t(U_i)\}_{t\geq 0}$, $\{Y_k(U_i)\}_{k\geq 1}$ and $\{Z_k(U_i)\}_{k\geq 1}$. These are independent for different i (since the U_i's are) and have the same distribution as the $\{X_t\}_{t\geq 0}$, $\{Y_k\}_{k\geq 1}$ and $\{Z_k\}_{k\geq 1}$ defined above (because U_i is uniform on $[0,1]$). Note that $X_t(U_i)$ (as well as these others) are random variables since they are a composition of random variables. Note that for this to hold, it was important that the underlying probability space for $\{X_t\}_{t\geq 0}$, $\{Y_k\}_{k\geq 1}$ and $\{Z_k\}_{k\geq 1}$ was the unit interval with the Borel sets and not the unit interval with the collection of Lebesgue measurable sets.

We now use these random variables to drive our dynamics as follows. Fix $i\in Z^d$. The arrival times of particles at site i will be taken to be the Poisson process $\{X_t(U_i)\}_{t\geq 0}$. The type of the kth arriving particle at site i will be taken to be $Y_k(U_i)$. If the kth particle arrives and there is at most 1 neighboring particle of a different type, we of course know what to do. However, if there is more than 1 such neighboring particle, then we look at the random ordering $Z_k(U_i)$ of the neighbors of the origin and have our arriving particle choose, among those neighbors which have a particle of a different type, the neighbor $i+u$ with the highest u value according to the ordering $Z_k(U_i)$, and then i and this chosen site react and become 0. It is clear that this generates our interacting particle system.

Now the event A_1 is of course measurable with respect to $\{U_i\}_{i\in Z^d}$ and is also trivially translation invariant. Since any i.i.d. process is ergodic (see for example, Walters (1975)), it follows that $P(A_1)\in\{0,1\}$, as desired.

4 Further remarks and conjectures

We believe the following strengthening of Theorem 5 should be true.

Conjecture 8 *Let n and d be arbitrary. If there are $i\neq j$ such that $p_i=p_j$ and all the other p_k's are no larger, then, starting from all 0's, the probability of poisoning is 0.*

By Theorem 5, the probability that we get poisoned in states i or j is 0. It seems natural that it should be even harder to get poisoned in one of the other states with a smaller p_k. The following monotonicity conjecture seems reasonable and would imply Conjecture 8. Of course, this monotonicity conjecture is not necessary for Conjecture 8 and it cannot be ruled out that there are no other types of monotonicity results which could be used instead to obtain Conjecture 8.

Conjecture 9 *Consider two systems with the same d and n with rates $(p_1, p_2, \ldots, p_n)$ and $(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)$ such that $p_1 \le \tilde{p}_1$ and $p_i \ge \tilde{p}_i$ for $i \ge 2$. Then the process of 1's in the first process is stochastically dominated by the process of 1's in the second process.*

We point out however that the simplest naive way of trying to prove Conjecture 9 doesn't work. Consider the case where $d = 1$, $n = 3$ and our two systems have rates $(1/3, 1/3, 1/3)$ and $(1/2, 1/4, 1/4)$. The obvious way to couple these systems is to let particles fall at the same location at the same time with the probability of the pair (i, j) falling being given by

$$p_{(2,2)} = p_{(3,3)} = 1/4, p_{(1,1)} = 1/3, p_{(2,1)} = p_{(3,1)} = 1/12$$

and if both particles have a choice with whom to react with, they choose the same one. Now, under this coupling, it can in fact happen that there is some location having a 1 in the first system but not having a 1 in the second system. In the following realization, we first have the first process getting a 2 and the second process getting a 1. After this, in the next 5 steps, we have the same type particle arrive in the two systems and the types, in order of arrival, are 2,3,3,3, and 1. The coupled system is then given in Table 3.

Table 3
Coupling of the two systems

system 1 system 2
0000000000 0000000000
0000200000 0000100000
0000220000 0000000000
0000220000 0000300000
0000020000 0003300000
0000000000 0003303000
After this 2 possibilities
0000010000 0003003000
0000010000 0003300000

5 Appendix

In this section, we aid the reader in verifying Equation (1) by suggesting how this should be done.
(1). The reader should first check that the 18 expressions in Section 2 correspond to the rate of change of the expected weight in what the reader would think is the most disadvantageous scenario to the right of the block. In column 4 of Table 1, these supposed most disadvantageous scenarios to the right of the block are listed.
(2). Next, one should plug in the scores for the blocks given in Table 1 into all the expressions for the blocks and check that all the numbers one obtains are positive. The first few decimals for these numbers are listed in column 3 of Table 1.
(3). Last, one has to consider all the other possible scenarios which can in fact sit to the right of the block, compute how each one affects the rate of change of the expected weight and check that all these other cases yield a larger value.
One should observe that our assumed worst case scenario sometimes holds uniformly and sometimes not in the following sense. A 00 to the right of the block 001 is always worse no matter what arrives next. Similarly, a 2 to the right of the block 022 is always worse than a 0. However, for the block 000, it is not uniformly worse to have a 2 to the right rather than a 0. Should a 3 arrive at the right most 0, it would have been worse to have a 0 to the right since then the 3 would have remained but with a 2 to the right, the 3 would not stay. However, if a 1 would have arrived at that position, it would be worse to have a 2 since then the 1 would not stay. However, in all cases, when one averages over all the possibilities, these assumed worst cases are in fact worst case. However, this needs to be checked and this is precisely step (3) above.

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