Elect. Comm. in Probab. **11** (2006), 217–219

ELECTRONIC COMMUNICATIONS in PROBABILITY

A SHORT PROOF OF THE DIMENSION FORMULA FOR LÉVY PROCESSES

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Submitted June 19 2006, *accepted in final form* August 28 2006

AMS 2000 Subject classification: 60G51, 60J25, 60G17
Keywords: Lévy processes, Hausdorff dimension, range

Abstract
We provide a simple proof of the result on the Hausdorff dimension of the range of a Lévy process in a recent paper by Khoshnevisan, Xiao, and Zhong [1].

Let X_t be a Lévy process in $\mathbb{R}^d$ with the Lévy exponent Ψ. There was an "open question" which did not garnered lots of attention until a recent paper on multiparameter Lévy processes by Khoshnevisan et al. [1] showed the simplification of Pruitt's formula in [2] as one of the main applications of their long-proof theorems. Khoshnevisan et al. [1] obtained:

$$\dim_H X([0,1]) = \sup\{\alpha < d : \int |y|^{\alpha-d}\mathrm{Re}\left(\frac{1}{1+\Psi(y)}\right) dy < \infty\} \quad a.s. \tag{1.1}$$

where $X([0,1]) = \{X_s : s \in [0,1]\}$ with the notation $\dim_H$ for the Hausdorff dimension. The present author is still puzzled by why Pruitt himself did not reach the same conclusion whereas he made some interesting remarks about the difficulty of this issue. We show that Pruitt's elegant proof in [2] also yields (1.1).

Proof of (1.1). Let ζ be an exponential random variable with mean 1 independent of X_t, suggesting that we are dealing with a killed process at rate 1. Observe that $E[\int_0^\zeta 1(|X_t| \le r)dt] = \int_0^\infty e^{-t}E[\int_0^\zeta 1(|X_s| \le r)ds|\zeta = t]dt = \int_0^\infty \int_0^t e^{-t}P(|X_s| \le r)dsdt = \int_0^\infty e^{-t}P(|X_t| \le r)dt$. Define

$$\gamma := \sup\{\alpha \ge 0 : \limsup_{r\to 0} r^{-\alpha} \int_0^\infty e^{-t}P(|X_t| \le r)dt < \infty\}. \tag{1.2}$$

Conditioned on ζ, Theorem 1 of Pruitt [2] implies that

$$\dim_H X([0,\zeta]) = \gamma \quad a.s. \tag{1.3}$$

Clearly, $\dim_H X([0,1]) = \dim_H X([0,\zeta])$ $a.s.$ It remains to show that

$$\gamma = \sup\{\alpha < d : \int |y|^{\alpha-d}\mathrm{Re}\left(\frac{1}{1+\Psi(y)}\right) dy < \infty\}. \tag{1.4}$$

Let $g(r) = \int_0^\infty e^{-t}P(|X_t| \le r)dt,\ r > 0$. Clearly $g(r)$ is nondecreasing bounded by 1. For $\alpha > 0$,

$$E|X_t|^{-\alpha} = \alpha \int_0^\infty x^{-\alpha-1}P(|X_t| \le x)dx,$$

$$\int_0^\infty e^{-t}E|X_t|^{-\alpha}dt = \alpha \int_0^1 x^{-\alpha-1}g(x)dx + \alpha \int_1^\infty x^{-\alpha-1}g(x)dx,$$

which shows that $\int_0^\infty e^{-t}E|X_t|^{-\alpha}dt < \infty$ if and only if $\int_0^1 x^{-\alpha-1}g(x)dx < \infty$. This fact and a standard argument imply that

$$\gamma = \sup\{\alpha \ge 0 : \int_0^\infty e^{-t}E|X_t|^{-\alpha}dt < \infty\}.$$

Clearly, $\gamma \le d$. Also note that if $\int_0^1 x^{-d-1}g(x)dx < \infty$ then $\int_0^1 x^{-\alpha-1}g(x)dx < \infty$ for all $\alpha < d$ as well. Thus, we can write

$$\gamma = \sup\{\alpha < d : \int_0^\infty e^{-t}E|X_t|^{-\alpha}dt < \infty\}.$$

There are quite a few ways to compute the integral $\int_0^\infty e^{-t}E|X_t|^{-\alpha}dt$ in terms of Ψ. Since $\gamma \le 2$, we decide to stick to Pruitt's original idea although it works only for $\alpha \le 2$. So, let $\alpha < d$ and $\alpha \in (0, 2]$. Choose a symmetric α-stable process ξ_t in $\mathbb{R}^d$ with Lévy exponent $|x|^\alpha$. Let $p_{\alpha d}(t, \cdot)$ be the density of ξ_t. Note that both $\int_0^\infty p_{\alpha d}(t, x)dt = c|x|^{\alpha-d}$ for some constant $c \in (0, \infty)$ and $\mathrm{Re}[(1 + \Psi(x))^{-1}] > 0$. Following the calculations given by Pruitt [2], p. 375, we find that

$$\begin{aligned}
&\int_0^\infty e^{-t}E|X_t|^{-\alpha}dt \\
&= \int_0^\infty \int_0^\infty \int p_{\alpha d}(t, x)e^{-s(1+\Psi(x))}dxdsdt \\
&= \int_0^\infty \int p_{\alpha d}(t, x) \int_0^\infty e^{-s(1+\Psi(x))}dsdxdt \\
&= \int_0^\infty \int p_{\alpha d}(t, x)\mathrm{Re}\left(\int_0^\infty e^{-s(1+\Psi(x))}ds\right) dxdt \\
&= \int_0^\infty \int p_{\alpha d}(t, x)\mathrm{Re}\left(\frac{1}{1 + \Psi(x)}\right) dxdt \\
&= c \int |x|^{\alpha-d}\mathrm{Re}\left(\frac{1}{1 + \Psi(x)}\right) dx
\end{aligned}$$

where the second equality holds because for fixed $t > 0$,

$$\int_0^\infty \int p_{\alpha d}(t, x)|e^{-s(1+\Psi(x))}|dxds \le \int p_{\alpha d}(t, x)dx \int_0^\infty e^{-s}ds = 1,$$

recalling that $p_{\alpha d}(t, \cdot)$ is a density function. (1.4) has been proved. $\square$

References

[1] Khoshnevisan, D., Xiao, Y. and Zhong, Y. (2003). Measuring the range of an additive Lévy process. *Ann. Probab.* **31**, 1097-1141.

[2] Pruitt, W. E. (1969). The Hausdorff dimension of the range of a process with stationary independent increments. *J. Math. Mech.* **19**, 371-378.