*Elect. Comm. in Probab.* **6** (2001) 67–71

**ELECTRONIC COMMUNICATIONS in PROBABILITY**

# A 2-DIMENSIONAL SDE WHOSE SOLUTIONS ARE NOT UNIQUE

JAN M. SWART
*Mathematical Institute*
*University of Erlangen-Nuremberg*
*Erlangen, Germany*
email: `swart@mi.uni-erlangen.de`

*submitted* June 19, 2001 *Final version accepted* July 12, 2001

AMS 2000 Subject classification: Primary: 60H10, 60H20; Secondary: 60J60.
Stochastic differential equation, pathwise uniqueness / strong uniqueness, diffusion process.

*Abstract*

*In 1971, Yamada and Watanabe showed that pathwise uniqueness holds for the SDE* $dX = \sigma(X)dB$ *when* $\sigma$ *takes values in the* $n\times m$ *matrices and satisfies* $|\sigma(x)-\sigma(y)| \leq |x-y|\log(1/|x-y|)^{1/2}$. *When* $n = m = 2$ *and* $\sigma$ *is of the form* $\sigma_{ij}(x) = \delta_{ij}s(x)$, *they showed that this condition can be relaxed to* $|\sigma(x) - \sigma(y)| \leq |x - y|\log(1/|x - y|)$, *leaving open the question whether this is true for general* $2 \times m$ *matrices. We construct a* $2 \times 1$ *matrix-valued function which negatively answers this question. The construction demonstrates an unexpected effect, namely, that fluctuations in the radial direction may stabilize a particle in the origin.*

## 1 Introduction

In 1971, Yamada and Watanabe [5] showed that pathwise uniqueness holds for the SDE $dX_t = \sigma(X_t)dB_t$ where $B$ is $m$-dimensional Brownian motion and $\sigma$ is an $n\times m$ matrix-valued function satisfying

$$|\sigma(x) - \sigma(y)| \leq \rho(|x - y|) \qquad (|x - y| < \varepsilon), \tag{1}$$

where $\rho$ is a continuous nondecreasing function on some nonzero interval $[0, \varepsilon)$ such that $\rho(0) = 0$ and

$$\begin{array}{ll} \text{(i)} & \int_{0+} \rho^{-2}(r)r\,dr = \infty \\ \text{(ii)} & r \mapsto \rho^2(r)r^{-1} \text{ is concave.} \end{array} \tag{2}$$

For example, we may take $\rho(r) = r\log(1/r)^{1/2}$. (In (1), $|\cdot|$ denotes any norm on the space of real $n \times m$ matrices; all such norms are equivalent.)
Moreover, they show by example that in dimensions $n \geq 3$, the conditions (2) are in some sense as far as one can go. To be precise, if $\int_{0+} \rho^{-2}(r)r\,dr < \infty$ and $\rho$ is subadditive, then there exists an isotropic $n \times n$ coefficient $\sigma$ (i.e., $\sigma$ of the form $\sigma_{ij}(x) = \delta_{ij}s(x)$ with $s$ some real-valued function), such that $\sigma(0) = 0$, (1) holds, and such that the equation $dX_t = \sigma(X_t)dB_t$ with initial condition $X_0 = 0$ has, apart from the zero solution, other non-zero solutions.

For isotropic $2 \times 2$ coefficients, they show that the conditions (2) can be relaxed to

$$\begin{array}{l} \int_{0+} \rho^{-2}(r) r \log(1/r) dr = \infty \\ q \mapsto q^3 e^{2/q} \rho^2(e^{-1/q}) \text{ is concave on some interval } [0, \varepsilon'). \end{array} \tag{3}$$

For example, the function $\rho(r) = r \log(1/r)$ satisfies (3) but not (2). For such isotropic $2 \times 2$ coefficients, they show that the conditions (3) are in some sense optimal.
Thus, they leave unanswered the question whether the conditions (2) can be improved for general $2 \times m$ matrices. (The 1-dimensional case was treated in [4].) We will see that this is not the case, and that for general $2 \times m$ matrices, the conditions (2) are in some sense optimal.

## 2 A SDE whose solutions are not unique

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^2$ and let $B_\varepsilon := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < \varepsilon\}$ $(\varepsilon > 0)$.

**Proposition** *Let $\varepsilon > 0$ and let $\sigma : B_\varepsilon \to \mathbb{R}^2$ be given by $\sigma(0) = 0$ and*

$$\sigma(x) := \rho(|x|)|x|^{-1}(x_2, -x_1) \qquad (x \neq 0), \tag{4}$$

*where $\rho$ is a continuous function on $[0, \varepsilon)$ such that $\rho(0) = 0$, $\rho > 0$ on $(0, \varepsilon)$, and*

$$\int_{0+} \rho^{-2}(r) r \, dr < \infty. \tag{5}$$

*Then, on some probability space equipped with a one-dimensional Brownian motion $B$ and for some $T > 0$, there exists a nonzero solution $(X_t)_{t \in [0,T)}$ of the SDE*

$$dX_t = \sigma(X_t) dB_t \qquad \textit{with initial condition} \quad X(0) = 0. \tag{6}$$

**Remark** In the example above $|\sigma(x)| = \rho(|x|)$. If $\rho$ is additionally nondecreasing and concave, then

$$|\sigma(x) - \sigma(y)| \leq 4\rho(\tfrac{1}{2}|x - y|) \qquad (|x - y| \leq 2\varepsilon). \tag{7}$$

To see this, without loss of generality assume that $|x| \leq |y|$ and $|y| > 0$. Define $z := \frac{|x|}{|y|} y$ and estimate

$$|\sigma(x) - \sigma(y)| \leq |\sigma(x) - \sigma(z)| + |\sigma(z) - \sigma(y)|. \tag{8}$$

Set $h := \frac{1}{2}|x - z|$ and $t := \frac{1}{2}|x + z|$, and note that $|x| = \sqrt{t^2 + h^2}$. Then the first term in (8) can be estimated as

$$|\sigma(x) - \sigma(z)| = \rho(|x|)|x|^{-1}|x - z| = \frac{\rho(\sqrt{t^2 + h^2})}{\sqrt{t^2 + h^2}} 2h \leq 2\rho(\tfrac{1}{2}|x - z|), \tag{9}$$

where we used that, since $\rho$ is concave and $\rho(0) = 0$, the function $r \mapsto \rho(r) r^{-1}$ is nonincreasing. The second term in (8) can be estimated as

$$|\sigma(z) - \sigma(y)| = \rho(|y|) - \rho(|x|) \leq \rho(|x - y|) \leq 2\rho(\tfrac{1}{2}|x - y|). \tag{10}$$

Combining (9) and (10) and using that $\rho$ is nondecreasing, we arrive at (7).

**Proof of the Proposition** Let $(B'_t)_{t\in\mathbb{R}}$ a stationary Brownian motion on the unit circle $[0,2\pi)$ (in particular, $B'_t$ is uniformly distributed on $[0,2\pi)$ for every $t\in\mathbb{R}$). Then the $B_\varepsilon$-valued process

$$X'_t := (\sin(B'_t)e^{t/2}, \cos(B'_t)e^{t/2}) \qquad (t < 2\log\varepsilon) \tag{11}$$

solves the SDE

$$dX'_t = \sigma'(X'_t)dB'_t \qquad \text{with} \quad \sigma'(x) := (x_2, -x_1). \tag{12}$$

We will use a time transformation to transform $X'$ into a nonzero solution of (6). Let $a'$ denote the $2\times 2$ matrix $a'(x) := \begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix}$ and let $A', A$ be the differential operators

$$A'f(x) := \sum_{i,j=1}^{2} a'_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}f(x) \quad \text{and} \quad Af(x) := \rho^2(|x|)|x|^{-2}A'f(x). \tag{13}$$

Then $X'$ solves the martingale problem for $A'$, i.e., for every $u\in(-\infty,2\log\varepsilon)$ and $f\in\mathcal{C}^2(\mathbb{R}^2)$ the process $({M'}_t^{u,f})_{t\in[u,2\log\varepsilon)}$ given by

$${M'}_t^{u,f} := f(X'_t) - \int_u^t A'f(X'_s)ds \qquad (t\in[u,2\log\varepsilon)) \tag{14}$$

is a martingale with respect to the filtration generated by $X'$. To complete the proof, it suffices to show that there exists a nonzero process $(X_t)_{t\in[0,T)}$ with continuous sample paths, starting in $X_0 = 0$, such that for each $f\in\mathcal{C}^2(\mathbb{R}^2)$, the process $(M_t^{0,f})_{t\in[0,T)}$ is a martingale with respect to the filtration generated by $X$, where we define, for $u\in[0,T)$,

$$M_t^{u,f} := f(X_t) - \int_u^t Af(X_s)ds \qquad (t\in[u,T)). \tag{15}$$

Standard results (see [2], Chapter 5), then show that there exists a weak solution of (6) that is equal in distribution to $X$.
Define a strictly increasing function $\psi : (-\infty, 2\log\varepsilon)\to(0,\infty)$ by

$$\psi(s) := \int_0^{e^{s/2}} 2\rho^{-2}(r)r\,dr, \tag{16}$$

put $T := \lim_{s\to 2\log\varepsilon}\psi(s)$, and let $\psi^{-1} : (0,T)\to(-\infty,2\log\varepsilon)$ denote the inverse of $\psi$. Set $X_0 := 0$ and

$$X_t := X'_{\psi^{-1}(t)} \qquad (t\in(0,T)). \tag{17}$$

The substitution of variables

$$s = \psi(s') \qquad ds = 2\rho^{-2}(e^{s'/2})e^{s'/2}\cdot\tfrac{1}{2}e^{s'/2}ds' \tag{18}$$

shows that for every $u\in(0,T)$, $t\in[u,T)$ and $f\in\mathcal{C}^2(\mathbb{R}^2)$

$$\begin{aligned} M_t^{u,f} &= f(X_t) - \int_{\psi^{-1}(u)}^{\psi^{-1}(t)} Af(X_{\psi(s')})\rho^{-2}(e^{s'/2})e^{s'}ds' \\ &= f(X'_{\psi^{-1}(t)}) - \int_{\psi^{-1}(u)}^{\psi^{-1}(t)} Af(X'_{s'})\rho^{-2}(|X'_{s'}|)|X'_{s'}|^2ds' \\ &= f(X'_{\psi^{-1}(t)}) - \int_{\psi^{-1}(u)}^{\psi^{-1}(t)} A'f(X'_{s'})ds' = {M'}_{\psi^{-1}(t)}^{\psi^{-1}(u),f}, \end{aligned} \tag{19}$$

where we have used that $|X'_{s'}| = e^{s'/2}$. The filtration generated by $X'$ is just a time transform of the filtration generated by $X$, and thus we see that for every $u \in (0,T)$, the process $(M_t^{u,f})_{t\in[u,T)}$ is a martingale with respect to the filtration generated by $X$. A simple approximation argument shows that $(M_t^{0,f})_{t\in[0,T)}$ is also a martingale. $\Box$

**Remark** Also for SDE's with a drift term, both in dimensions $n = 1$ and $n \geq 2$, Yamada and Watanabe [4, 5] give conditions on the smoothness of the coefficients that are sufficient to guarantee pathwise uniqueness. Their examples and the one above show that their conditions are, in some sense, the best possible in every dimension $n \geq 1$. Of course, this does not exclude the possibility that distribution uniqueness or even pathwise uniqueness may hold for *some* SDE's with less smooth coefficients; see for example [1, 3].

## 3 Stabilization by radial fluctuations

Define $\sigma^1, \sigma^2 : B_{e^{-1}} \to \mathbb{R}^2$ by

$$\sigma^1(x) := \log(1/|x|)(x_2, -x_1) \quad \text{and} \quad \sigma^2(x) := \log(1/|x|)(x_1, x_2). \tag{20}$$

Then, according to the Proposition above, there exists a nonzero solution of the SDE

$$dX_t = \sigma^1(X_t)dB_t^1 \qquad \text{with initial condition} \quad X(0) = 0. \tag{21}$$

On the other hand, as we will see shortly, there exist no nonzero solutions of the SDE

$$dX_t = \sigma^1(X_t)dB_t^1 + \sigma^2(X_t)dB_t^2 \qquad \text{with initial condition} \quad X(0) = 0, \tag{22}$$

where $B^1$ and $B^2$ are two independent Brownian motions. Indeed, this follows from the already mentioned result of Yamada and Watanabe for isotropic $2\times 2$ coefficients. Recall that the function $\rho(r) = r\log(1/r)$ satisfies (3) but not (2). Since $\sum_{k=1}^2 \sigma_i^k(x)\sigma_j^k(x) = \delta_{ij}|x|^2\log(1/|x|)^2$, solutions to (22) are equal in distribution to solutions of the SDE

$$dX_t^i = \sum_{j=1}^2 \delta_{ij}|X_t|\log(1/|X_t|)dB_t^j \qquad \text{with initial condition} \quad X(0) = 0, \tag{23}$$

to which the result of Yamada and Watanabe is applicable. Thus, we see that adding fluctuations in the radial direction to the SDE (21) may prevent solutions from escaping from the origin in finite time.

**Acknowledgments** This work was sponsored by the 'Graduiertenkolleg Stochastische Prozesse und Probabilistische Analysis' and the DFG.

## References

[1] M. T. Barlow, One dimensional stochastic differential equations with no strong solution. J. London Math. Soc. 26 (1982) 335–347.

[2] S. N. Ethier and T. G. Kurtz, *Markov processes; Characterization and Convergence*, John Wiley & Sons, New York, 1986.

[3] J. M. Swart, Pathwise uniqueness for a SDE with non-Lipschitz coefficients. preprint 2000.

[4] T. Yamada and S. Watanabe, On the uniqueness of solutions of stochastic differential equations. J. Math. Kyoto Univ. 11 (1971) 155–167.

[5] T. Yamada and S. Watanabe, On the uniqueness of solutions of stochastic differential equations II. J. Math. Kyoto Univ. 11 (1971) 553–563.