Elect. Comm. in Probab. **1** (1996) 7–10

ELECTRONIC COMMUNICATIONS in PROBABILITY

A PROOF OF A CONJECTURE OF BOBKOV AND HOUDRE

S. KWAPIEN and M.PYCIA
Department of Mathematics, Warsaw University, ul.Banacha 2, 02-097 Warsaw, Poland.
e-mail: kwapstan@mimuw.edu.pl, mpycia@mimuw.edu.pl

W. SCHACHERMAYER
Department of Statistics, University of Vienna, Bruennerstrasse 72, A-1210 Wien, Austria.
e-mail: wschach@stat1.bwl.univie.ac.at

AMS 1991 Subject classification: 60E05
Keywords and phrases: Gaussian Distribution, Characterization.

Abstract
S.G. Bobkov and C. Houdré recently posed the following question on the Internet ([1]): Let X, Y be symmetric i.i.d. random variables such that:

$$\mathbb{P}\{\frac{|X+Y|}{\sqrt{2}} \geq t\} \leq \mathbb{P}\{|X| \geq t\},$$

for each $t > 0$. Does it follow that X has finite second moment (which then easily implies that X is Gaussian)? In this note we give an affirmative answer to this problem and present a proof. Using a different method K. Oleszkiewicz has found another proof of this conjecture, as well as further related results.

We prove the following:

Theorem. Let X, Y be symmetric i.i.d random variables. If, for each $t > 0$,

$$\mathbb{P}\{|X+Y| \geq \sqrt{2}t\} \leq \mathbb{P}\{|X| \geq t\}, \tag{1}$$

then X is Gaussian.

Proof. Step 1. $\mathbb{E}\{|X|^p\} < \infty$ for $0 \leq p < 2$.
For this purpose it will suffice to show that, for $p < 2$, X has finite weak p'th moment, i.e., that there are constants C_p such that

$$\mathbb{P}\{|X| \geq t\} \leq C_p t^{-p}.$$

To do so, it is enough to show that, for $\epsilon > 0, \delta > 0$, we can find t_0 such that, for $t \geq t_0$, we have

$$\mathbb{P}\{|X| \geq (\sqrt{2}+\epsilon)t\} \leq \frac{1}{2-\delta}\mathbb{P}\{|X| \geq t\}. \tag{2}$$

Fix $\epsilon > 0$. Then:

$$\begin{aligned}
\mathbb{P}\{|X+Y| \geq \sqrt{2}t\} &= 2\mathbb{P}\{X+Y \geq \sqrt{2}t\} \\
&\geq 2\mathbb{P}\{X \geq (\sqrt{2}+\epsilon)t, Y \geq -\epsilon t, \text{ or } Y \geq (\sqrt{2}+\epsilon)t, X \geq -\epsilon t\} \\
&= 2(2\mathbb{P}\{X \geq (\sqrt{2}+\epsilon)t\}\mathbb{P}\{Y \geq -\epsilon t\} - \mathbb{P}\{X \geq (\sqrt{2}+\epsilon)t\}\mathbb{P}\{Y \geq (\sqrt{2}+\epsilon)t\}) \\
&= 2\mathbb{P}\{|X| \geq (\sqrt{2}+\epsilon)t\}(\mathbb{P}\{Y \geq -\epsilon t\} - \frac{1}{2}\mathbb{P}\{X \geq (\sqrt{2}+\epsilon)t\}) \\
&\geq (2-\delta)\mathbb{P}\{|X| \geq (\sqrt{2}+\epsilon)t\},
\end{aligned}$$

where $\delta > 0$ may be taken arbitrarily small for t large enough. Using (1) we obtain inequality (2).

Step 2. Let $\alpha_1, ..., \alpha_n$ be real numbers such that $\alpha_1^2 + ... + \alpha_n^2 \leq 1$ and let $(X_i)_{i=1}^{\infty}$ be i.i.d. copies of X; then

$$\mathbb{E}\{|\alpha_1 X_1 + ... + \alpha_n X_n|\} \leq \sqrt{2}\mathbb{E}\{|X|\}.$$

We shall repeatedly use the following result:

Fact: Let S and T be symmetric random variables such that $\mathbb{P}\{|S| \geq t) \leq \mathbb{P}\{|T| \geq t)$, for all $t > 0$, and let the random variable X be independent of S and T. Then

$$\mathbb{E}\{|S+X|\} \leq \mathbb{E}\{|T+X|\}.$$

Indeed, for fixed $x \in \mathbb{R}$, the function $h(s) = \frac{|s+x|+|s-x|}{2}$ is symmetric and non-decreasing in $s \in \mathbb{R}_+$ and therefore

$$\mathbb{E}\{|S+x|\} = \mathbb{E}\{\frac{|S+x|+|S-x|}{2}\} \leq \mathbb{E}\{\frac{|T+x|+|T-x|}{2}\} = \mathbb{E}\||T+x|\}.$$

Now take a sequence $\beta_1, ..., \beta_n \in \{2^{-k/2} : k \in \mathbb{N}_0\}$, such that $\alpha_i \leq \beta_i < \sqrt{2}\alpha_i$. Then $\beta_1^2 + ... + \beta_n^2 \leq 2$ and

$$\mathbb{E}\{|\alpha_1 X_1 + ... + \alpha_n X_n|\} \leq \mathbb{E}\{|\beta_1 X_1 + ... + \beta_n X_n|\}.$$

If there is $i \neq j$ with $\beta_i = \beta_j$ we may replace $\beta_1, \ldots, \beta_n$ by $\gamma_1, \ldots, \gamma_{n-1}$ with $\sum_{i=1}^n \beta_i^2 = \sum_{j=1}^{n-1} \gamma_j^2$ and

$$\mathbb{E}\{|\sum_{i=1}^n \beta_i X_i|\} \leq \mathbb{E}\{|\sum_{j=1}^{n-1} \gamma_j X_j|\}. \tag{3}$$

Indeed, supposing without loss of generality that $i = n-1$ and $j = n$ we let $\gamma_i = \beta_i$, for $i = 1, \ldots, n-2$ and $\gamma_{n-1} = \sqrt{2}\beta_{n-1} = \sqrt{2}\beta_n$. With this definition we obtain (3) from (1) and the above mentioned fact.

Applying the above argument a finite number of times we end up with $1 \leq m \leq n$ and numbers $(\gamma_j)_{j=1}^m$ in $\{2^{-k/2} : k \in \mathbb{N}_0\}$, $\gamma_i \neq \gamma_j$, for $i \neq j$, satisfying $\sum_{j=1}^m \gamma_j^2 \leq 2$ and

$$\mathbb{E}\{|\sum_{i=1}^n \alpha_i X_i|\} \leq \mathbb{E}\{|\sum_{j=1}^m \gamma_j X_j|\}.$$

To estimate this last expression it suffices to consider the extreme case $\gamma_j = 2^{-(j-1)/2}$, for $j = 1, \ldots, m$. In this case — applying again repeatedly the argument used to obtain (3):

$$\begin{aligned}
\mathbb{E}\{|\sum_{j=1}^{m} 2^{-(j-1)/2} X_j|\} &\leq \mathbb{E}\{|\sum_{j=1}^{m-1} 2^{-(j-1)/2} X_j + 2^{-(m-1)/2} X_m|\} \\
&\leq \mathbb{E}\{|\sum_{j=1}^{m-2} 2^{-(j-1)/2} X_j + 2^{-(m-2)/2} X_m|\} \\
&\leq \mathbb{E}\{|X_1 + X_2|\} \leq \mathbb{E}\{|\sqrt{2} X_1|\} = \sqrt{2}\mathbb{E}\{|X_1|\}.
\end{aligned}$$

Step 3. $\mathbb{E}\{X^2\} < \infty$.
We deduce from Step 2 that for a sequence $(\alpha_i)_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$ the series

$$\sum_{i=1}^{\infty} \alpha_i X_i$$

converges in mean and therefore almost surely. Using the notation

$$[S] = \begin{cases} S \text{ if } |S| \leq 1, \\ \operatorname{sign}(S) \text{ if } |S| \geq 1. \end{cases}$$

for a random variable S, we deduce from Kolmogorov's three series theorem that

$$\sum_{i=1}^{\infty} \mathbb{E}\{[\alpha_i X_i]^2\} < \infty.$$

Suppose now that $\mathbb{E}\{X^2\} = \infty$; this implies that for every $C > 0$, we can find $\alpha > 0$ such that

$$\mathbb{E}\{[\alpha X]^2\} \geq C\alpha^2.$$

¿From this inequality it is straightforward to construct a sequence $(\alpha_i)_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} \mathbb{E}\{[\alpha_i X_i]^2\} = \infty, \text{ while } \sum_{i=1}^{\infty} \alpha_i^2 < \infty,$$

a contradiction proving Step 3.

Step 4. Finally, we show how $\mathbb{E}\{X^2\} < \infty$ implies that X is normal. We follow the argument of Bobkov and Houdré [2].
The finiteness of the second moment implies that we must have equality in the assumption of the theorem, i.e.,

$$\mathbb{P}\{|X + Y| \geq \sqrt{2}t\} = \mathbb{P}\{|X| \geq t\}.$$

Indeed, assuming that there is strict inequality in (1) for some $t > 0$, we would obtain that the second moment of $X + Y$ is strictly smaller than the second moment of $\sqrt{2}X$, which leads to a contradiction:

$$2\mathbb{E}\{X^2\} > \mathbb{E}\{(X + Y)^2\} = \mathbb{E}\{X^2\} + \mathbb{E}\{Y^2\} = 2\mathbb{E}\{X^2\}.$$

Hence, $2^{-n/2}(X_1 + \ldots + X_{2^n})$ has the same distribution as X and we deduce from the Central Limit Theorem that X is Gaussian.

References

[1] S.G. Bobkov, C.Houdré (1995): Open Problem, *Stochastic Analysis Digest* **15**

[2] S.G. Bobkov, C. Houdré (1995): A characterization of Gaussian measures via the isoperimetric property of half-spaces, *(preprint)*.