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A PERCOLATION FORMULA

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Abstract

Let A be an arc on the boundary of the unit disk $\mathbb{U}$. We prove an asymptotic formula for the probability that there is a percolation cluster K for critical site percolation on the triangular grid in $\mathbb{U}$ which intersects A and such that 0 is surrounded by $K \cup A$.

Motivated by questions of Langlands et al [LPSA94] and M. Aizenman, J. Cardy [Car92, Car] derived a formula for the asymptotic probability for the existence of a crossing of a rectangle by a critical percolation cluster. Recently, S. Smirnov [Smi1] proved Cardy's formula and established the conformal invariance of critical site percolation on the triangular grid. The paper [LSW] has a generalization of Cardy's formula. Another percolation formula, which is still unproven, was derived by G. M. T. Watts [Wat96]. The current paper will state and prove yet another such formula. A short discussion elaborating on the general context of these results appears at the end of the paper.

Consider site percolation on the triangular lattice in $\mathbb{C}$ with small mesh $\delta > 0$, where each site is declared open with probability $1/2$, independently. (See [Gri89, Kes82] for background on percolation.) It is convenient to represent a percolation configuration by coloring the corresponding hexagonal faces of the dual grid; black for an open site, white for a closed site. (The faces are taken to be topologically closed. Some edges are colored by both colors, but that has no significance.) Let $\mathfrak{B}$ denote the union of the black hexagons, intersected with the closed unit disk $\overline{\mathbb{U}}$, and for $\theta \in (0, 2\pi)$ let $\mathcal{A} = \mathcal{A}(\theta)$ be the event that there is a connected component K of $\mathfrak{B}$ which intersects the arc

$$A_\theta := \{e^{is} : s \in [0, \theta]\} \subset \partial\mathbb{U}$$

and such that 0 is surrounded by $K \cup A_\theta$. The latter means that 0 is in a bounded component of $\mathbb{C} \setminus (A_\theta \cup K)$ or $0 \in K$. Figure 1 shows the two distinct topological ways in which this could happen.

Theorem 1.

$$\lim_{\delta \downarrow 0} \mathbf{P}[\mathcal{A}] = \frac{1}{2} - \frac{\Gamma(2/3)}{\sqrt{\pi}\,\Gamma(1/6)}\, F_{2,1}\Big(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, -\cot^2\frac{\theta}{2}\Big) \cot\frac{\theta}{2}\,.$$

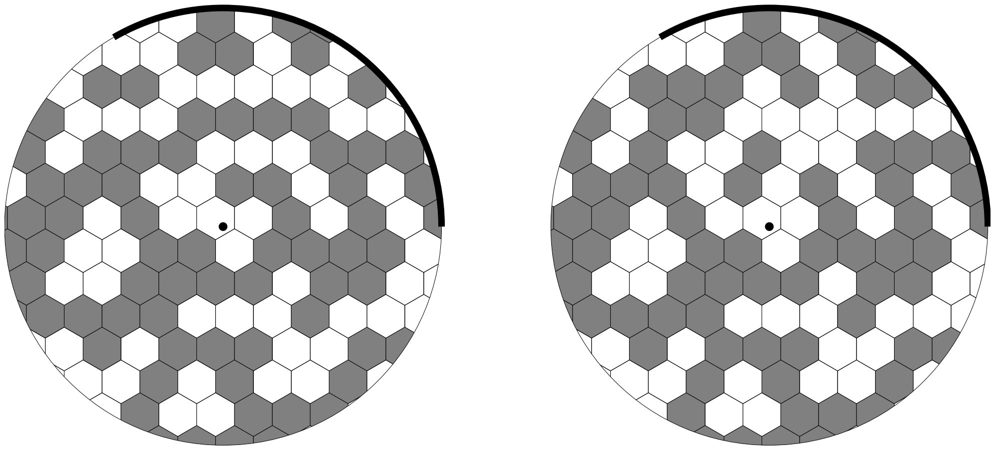

Figure 1: The two topologically distinct ways to surround 0.

Here, $F_{2,1}$ is the hypergeometric function. See [EMOT53, Chap. 2] for background on hypergeometric functions.

There is a second interpretation of the Theorem. Let C_1 be the cluster of either black or white hexagons which contains 0. (If 0 is on the boundary of two clusters of different colors, let C_1 be the black cluster containing 0, say.) Let C_2 be the (unique) cluster which surrounds C_1 and is adjacent to it. Inductively, let C_{n+1} be the cluster surrounding and adjacent to C_n, and of the opposite color. Let m be the least integer such that C_m is not contained in $\mathbb{U}$, and let C'_m be the component of $\overline{\mathbb{U}} \cap C_m$ which surrounds 0. Let $X := 1$ if $C'_m \cap \partial\mathbb{U} \subset A_\theta$, let $X := 0$ if $C'_m \cap A_\theta = \emptyset$, and otherwise set $X := 1/2$. Then $\lim_{\delta\downarrow 0} \mathbf{E}[X] = \lim_{\delta\downarrow 0} \mathbf{P}[\mathcal{A}]$. This is so because

$$\mathcal{A} = \{X = 1\} \cup \{X = 1/2 \text{ and } C_m \text{ is black}\} \cup \{m = 1,\, X > 0 \text{ and } 0 \in \partial C_m\}\,,$$

and the probability that $m = 1$ goes to zero as $\delta \downarrow 0$ (since a.s. there is no infinite cluster).

Theorem 1 will be proved by utilizing the relation between the scaling limit of percolation and Stochastic Loewner evolution with parameter $\kappa = 6$ (a.k.a. SLE_6), which was conjectured in [Sch00] and proven by S. Smirnov [Smi1].

We now very briefly review the definition and the relevant properties of chordal SLE. For a thorough treatment, see [RS]. Let $\kappa \geq 0$, let $B(t)$ be Brownian motion on $\mathbb{R}$ starting from $B(0) = 0$, and set $W(t) = \sqrt{\kappa}\, B(t)$. For z in the upper half plane $\mathbb{H}$ consider the time flow $g_t(z)$ given by

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}\,, \qquad g_0(z) = z\,. \tag{1}$$

Then $g_t(z)$ is well defined up to the first time $\tau = \tau(z)$ such that $\lim_{t\uparrow\tau} g_t(z) - W(t) = 0$. For all $t > 0$, the map g_t is a conformal map from the domain $H_t := \{z \in \mathbb{H} : \tau(z) > t\}$ onto $\mathbb{H}$. The process $t \mapsto g_t$ is called Stochastic Loewner evolution with parameter κ, or SLE_κ.

In [RS] it was proven that at least for $\kappa \neq 8$ a.s. there is a uniquely defined continuous path $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$, called the **trace** of the SLE, such that for every $t \geq 0$ the set H_t is equal to the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. In fact, a.s.

$$\forall t \geq 0, \qquad \gamma(t) = \lim_{z \to W(t)} g_t^{-1}(z)\,,$$

where z tends to $W(t)$ from within $\mathbb{H}$. Additionally, it was shown that γ is a.s. transient, namely $\lim_{t\to\infty}|\gamma(t)| = \infty$, and that when $\kappa \in (0,8)$ we have for every $z_0 \in \mathbb{H}$ that $\mathbf{P}\big[z_0 \in \gamma[0,\infty)\big] = 0$.

It was also shown [RS] that γ is a simple path a.s. iff $\kappa \le 4$. When $\kappa > 4$, $\kappa \ne 8$, although not a simple path, γ does not cross itself; that is, a.s. for every $t_0 > 0$ there is a continuous homotopy $H : [0,1]\times[t_0,\infty) \to \overline{\mathbb{H}}$ such that $H(0,t) = \gamma(t)$ and $H\big((0,1]\times(t_0,\infty)\big)\cap\gamma[0,t_0] = \emptyset$. This property easily follows from the fact that $\gamma[t_0,\infty)$ is the image of a continuous path in $\overline{\mathbb{H}}$ (which, by the way, has essentially the same law as γ) under the continuous extension of $g_{t_0}^{-1} : \mathbb{H} \to \mathbb{H}\setminus\gamma[0,t_0]$, to $\overline{\mathbb{H}}$. (See, for example, [RS, Proposition 2.1.ii, Theorem 5.2].)

Fix some $z_0 = x_0 + i\,y_0 \in \mathbb{H}$. Then we may ask if γ passes to the right or to the left of z_0, topologically. (Formally, this should be defined in terms of winding numbers, as follows. Let β_t be the path from $\gamma(t)$ to 0 which follows the arc $|\gamma(t)|\partial\mathbb{U}$ clockwise from $\gamma(t)$ to $|\gamma(t)|$ and then takes the straight line segment in $\mathbb{R}$ to 0. Then γ passes to the left of z_0 if the winding number of $\gamma[0,t]\cup\beta_t$ around z_0 is 1 for all large t. As noted above, γ is a.s. transient, and therefore there is some random time t_0 such that the winding number is constant for $t \in (t_0,\infty)$. This constant is either 0 or 1, since γ does not cross itself, as discussed above.) Theorem 1 will be established by applying the following with $\kappa = 6$:

Theorem 2. *Let $\kappa \in [0,8)$, and let $z_0 = x_0 + i\,y_0 \in \mathbb{H}$. Then the trace γ of chordal* SLE_κ *satisfies*

$$\mathbf{P}\big[\gamma \textit{ passes to the left of } z_0\big] = \frac{1}{2} + \frac{\Gamma(4/\kappa)}{\sqrt{\pi}\,\Gamma\big(\frac{8-\kappa}{2\kappa}\big)}\,\frac{x_0}{y_0}\,F_{2,1}\Big(\frac{1}{2},\frac{4}{\kappa},\frac{3}{2},-\frac{x_0^2}{y_0^2}\Big)\,.$$

When $\kappa = 2, 8/3, 4$ and 8 the right hand side simplifies to $1 + \frac{x_0\,y_0}{\pi\,|z_0|^2} - \frac{\arg z_0}{\pi}$, $\frac{1}{2} + \frac{x_0}{2\,|z_0|}$, $1 - \frac{\arg z_0}{\pi}$ and $\frac{1}{2}$, respectively.

Let $x_t := \operatorname{Re} g_t(z_0) - W(t)$, $y_t := \operatorname{Im} g_t(z_0)$, and $w_t := x_t/y_t$.

Lemma 3. *Almost surely, γ is to the left of z_0 iff $\lim_{t\uparrow\tau(z_0)} w_t = \infty$ and a.s. γ is to the right of z_0 iff $\lim_{t\uparrow\tau(z_0)} w_t = -\infty$.*

Proof. Suppose first that $\kappa \in [0,4]$. In that case, a.s. γ is a simple path and $\tau(z_0) = \infty$, by [RS]. Given γ, we start a planar Brownian motion B from z_0. Suppose that γ is to the left of z_0. This implies that B will first hit $\mathbb{R}\cup\gamma[0,\infty)$ in $[0,\infty)$ or from the right hand side of γ. Since γ is transient, as $t\uparrow\infty$ the probability that B first hits $\gamma[0,t]\cup\mathbb{R}$ from the right hand side of γ or in $[0,\infty)$ tends to 1. By conformal invariance of harmonic measure, this means that the harmonic measure in $\mathbb{H}$ of $[W(t),\infty)$ from $g_t(z_0)$ tends to 1. Therefore, $\lim_{t\uparrow\infty} w_t = \infty$. The argument in the case where γ is to the right of z_0 is entirely similar. Since γ must be either to the left of to the right of z_0, this proves the lemma in the case $\kappa \in [0,4]$.

For $\kappa \in (4,8)$, the analysis is similar. The difference is that a.s. γ is not a simple path, $\tau(z_0) < \infty$, and z_0 is in a bounded component of $\mathbb{H}\setminus\gamma\big[0,\tau(z_0)\big]$ (see [RS]). Clearly, z_0 is not in a bounded component of $\mathbb{H}\setminus\gamma[0,t]$ when $t < \tau(z_0)$. Hence, at time $\tau(z_0)$ the path γ closes a loop around z_0. Since γ does not cross itself, the issue then is whether this is a clockwise or counter-clockwise loop. As above, if the loop is clockwise, then as $t\uparrow\tau(z_0)$ the harmonic measure from z_0 in $\mathbb{R}\cup\gamma[0,t]$ is predominantly on $[0,\infty)$ and the right side of $\gamma[0,t]$. This implies that $w_t\uparrow\infty$. If the loop is counter-clockwise, we get $w_t\downarrow-\infty$, by the same reasoning. This completes the proof. $\square$

Proof of Theorem 2. Writing (1) in terms of the real and imaginary parts gives,

$$dx_t = \frac{2\,x_t\,dt}{x_t^2+y_t^2} - dW(t)\,, \qquad dy_t = -\frac{2\,y_t\,dt}{x_t^2+y_t^2}\,.$$

Itô's formula then gives,

$$dw_t = -\frac{dW(t)}{y_t} + \frac{4\,w_t\,dt}{x_t^2+y_t^2}\,. \tag{2}$$

Make the time change

$$u(t) = \int_0^t \frac{dt}{y_t^2}\,,$$

and set

$$\tilde W(t) = \int_0^t \frac{dW(t)}{y_t}\,.$$

Note that $\tilde W/\sqrt{\kappa}$ is Brownian motion as a function of u. From (2), we now get

$$dw = -d\tilde W + \frac{4\,w\,du}{w^2+1}\,. \tag{3}$$

We got rid of x_t and y_t, and are left with a single variable diffusion process $w(u)$. (This is no mystery, but a simple consequence of scale invariance.) An immediate consequence of this and the lemma is that a.s. $\lim_{t\uparrow\tau(z_0)} u = \infty$, because the diffusion (3) a.s. does not hit $\pm\infty$ in finite time. It is clear that $u(t) < \infty$ when $t < \tau(z_0)$, because y_t is monotone decreasing and positive for $t \in [0, \tau(z_0))$.
Given a starting point $\hat w \in \mathbb{R}$ for the diffusion (3), and given $a, b \in \mathbb{R}$ with $a < \hat w < b$, we are interested in the probability $h(\hat w) = h_{a,b}(\hat w)$ that w will hit b before hitting a. Note that $h\big(w(u)\big)$ is a local martingale. Therefore, assuming for the moment that h is smooth, by Itô's formula, h satisfies

$$\frac{\kappa}{2}\,h''(w) + \frac{4\,w}{w^2+1}\,h'(w) = 0\,, \qquad h(a) = 0\,, \qquad h(b) = 1\,.$$

By the maximum principle, these equations have a unique solution, and therefore we find that

$$h(w) = \frac{f(w) - f(a)}{f(b) - f(a)}\,, \tag{4}$$

where

$$f(w) := F_{2,1}(1/2, 4/\kappa, 3/2, -w^2)\,w\,.$$

We may now dispose of the assumption that h is smooth, because Itô's formula implies that the right hand side in (4) is a martingale, and it easily follows that it must equal h. By [EMOT53, 2.10.(3)] and our assumption $\kappa < 8$ it follows that

$$\lim_{w\to\pm\infty} f(w) = \pm\frac{\sqrt{\pi}\,\Gamma\big((8-\kappa)/(2\kappa)\big)}{2\,\Gamma(4/\kappa)}\,. \tag{5}$$

In particular, the limit is finite, which shows that $\lim_{b\to\infty} h_{a,b}(w) > 0$ for all $w > a$. Hence, the diffusion process (3) is transient. Moreover,

$$\mathbf{P}\Big[\lim_{u\to\infty} w(u) = +\infty\Big] = \frac{f(\hat w) - f(-\infty)}{f(\infty) - f(-\infty)}\,.$$

An appeal to the lemma now completes the proof. □

Proof of Theorem 1. As above, let $\mathfrak{B}$ be the intersection of the union of the black hexagons with $\overline{\mathbb{U}}$, and let $\hat{\mathfrak{B}}$ be the union of $\mathfrak{B}$ and the set $S := \{r\, e^{is} : r \geq 1,\, s \in [0, \theta]\}$. Let β be the intersection of $\overline{\mathbb{U}}$ with the outer boundary of the connected component of $\hat{\mathfrak{B}}$ containing S. Then β is a path in $\overline{\mathbb{U}}$ from 1 to $e^{i\theta}$. It is immediate that the event $\mathcal{A}$ is equivalent to the event that 0 appears to the right of the path β; that is, that the winding number of the concatenation of β with the arc A_θ with the clockwise orientation around 0 is 1.
S. Smirnov [Smi1] has shown that as $\delta \downarrow 0$ the law of β tends weakly to the law of the image of the chordal SLE_6 trace γ under any fixed conformal map $\phi : \mathbb{H} \to \mathbb{U}$ satisfying $\phi(0) = 1$ and $\phi(\infty) = e^{i\theta}$. (See also [Smi2].) We may take

$$\phi(z) = e^{i\theta}\, \frac{z + \cot\frac{\theta}{2} - i}{z + \cot\frac{\theta}{2} + i}\,.$$

The theorem now follows by setting $\kappa = 6$ in Theorem 2. $\square$

Discussion. According to J. Cardy (private communication, 2001), presently, the conformal field theory methods used by him to derive his formula do not seem to supply even a heuristic derivation of Theorem 1. On the other hand, it seems that, in principle, probabilities for "reasonable" events involving critical percolation can be expressed as solutions of boundary-value PDE problems, via SLE_6. But this is not always easy. In particular, it would be nice to obtain a proof of Watts' formula [Wat96]. The event $\mathcal{A}$ studied here was chosen because the corresponding proof is particularly simple, and because the PDE can be solved explicitly.

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References

[Car92] John L. Cardy. Critical percolation in finite geometries. *J. Phys. A*, 25(4):L201–L206, 1992.

[Car] John L. Cardy. Conformal Invariance and Percolation, arXiv:math-ph/0103018.

[EMOT53] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vol. I.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. Based, in part, on notes left by Harry Bateman.

[Gri89] Geoffrey Grimmett. *Percolation.* Springer-Verlag, New York, 1989.

[Kes82] Harry Kesten. *Percolation theory for mathematicians.* Birkhäuser Boston, Mass., 1982.

[LPSA94] Robert Langlands, Philippe Pouliot, and Yvan Saint-Aubin. Conformal invariance in two-dimensional percolation. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):1–61, 1994.

[LSW] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents I: Half-plane exponents. *Acta Math.*, to appear. arXiv:math.PR/9911084.

[RS] Steffen Rohde and Oded Schramm. Basic properties of SLE, arXiv:math.PR/0106036.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[Smi1] Stanislav Smirnov. Critical percolation in the plane. I. Conformal invariance and Cardy's formula. II. Continuum scaling limit. Preprint.

[Smi2] Stanislav Smirnov. In preparation.

[Wat96] G. M. T. Watts. A crossing probability for critical percolation in two dimensions. *J. Phys. A*, 29(14):L363–L368, 1996.