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WEAK CONVERGENCE OF REFLECTING BROWNIAN MOTIONS

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1. Introduction. We will show that if a sequence of domains D_k increases to a domain D then the reflected Brownian motions in D_k's converge to the reflected Brownian motion in D, under mild technical assumptions. Our theorem follows easily from known results and is perhaps known as a "folk law" among the specialists but it does not seem to be recorded anywhere in an explicit form. The purpose of this note is to fill this gap. As the theorem itself is not hard to prove, we will start with some remarks explaining the significance of the result in the context of a currently active research area.

Very recently some progress has been made on the "hot spots" conjecture (Bañuelos and Burdzy [1], Burdzy and Werner [4]), and more papers are being written on the topic. The conjecture was stated in 1974 by J. Rauch and very little was published on the problem since then (see [1] for a review). The conjecture is concerned with the maximum of the second Neumann eigenfunction for the Laplacian in a Euclidean domain. Reflected Brownian motion was used in [1] and [4] to prove the conjecture for some classes of domains and also to give a counterexample. D. Jerison and N. Nadirashvili (private communication) have an argument proving the conjecture for planar convex domains. The question of whether the conjecture holds in planar simply connected domains seems to be the most interesting open problem in the area.

A possible way of constructing a counterexample to the "hot spot" conjecture for a class of domains might be first to fix a domain D in the class and consider a sequence of domains D_k increasing to D. Then one could consider reflected Brownian motions in D_k's and study their limit, if any, as $k \to \infty$. If one could find domains D_k with rough boundaries such that as $k \to \infty$, the reflecting Brownian motions in D_k would look more and more like a reflecting Brownian motion in D with drift, diffusion, or holding on the boundary of D then this might provide a basis for a class of counterexamples. A counterexample was constructed along these lines in Bass and Burdzy [2] but the crucial difference is that in that paper, a conditioned rather than reflected Brownian motion was used. The moral of this note is that a similar construction is impossible for the reflected Brownian motion. In other words, one cannot use "rough boundary" effects to construct counterexamples to the "hot spots" conjecture.

2. Weak convergence. A domain D is said to have continuous boundary if for every $z \in \partial D$ there is a ball $B(z,r)$ centered at z with radius $r>0$ such that $D \cap B(x,r)$ is the region above the graph of a continuous function. Reflecting Brownian motion can be constructed as a strong Markov process not only on a domain with smooth or Lipschitz boundary with Hölder cusps (see Lions and Sznitman [10], Bass and Hsu [3], Fukushima and Tomisaki [9] and the references therein) but on a bounded domain D with continuous boundary as well. Suppose that $n \geq 1$ and assume that D is a bounded domain in $\mathbf{R^n}$ with continuous boundary. Let $W^{1,2}(D) = \{f \in L^2(D,dx) : \nabla f \in L^2(D,dx)\}$ and let $\mathcal{E}(f,g) = \frac{1}{2}\int_D \nabla f \cdot \nabla g dx$ for $f,g \in W^{1,2}(D)$. It is known (see Theorem 2 on page 14 of Maz'ja [12]) that $(\mathcal{E}, W^{1,2}(D))$ is a regular Dirichlet space on $\overline{D}$ (see [8]); that is, $W^{1,2}(D) \cap C(\overline{D})$ is dense both in $(W^{1,2}(D), \mathcal{E}_1^{1/2})$ and in $(C(\overline{D}), \|\cdot\|_\infty)$, where $\mathcal{E}_1 = \mathcal{E} + (\ ,\ )_{L^2(D)}$. Therefore there is a strong Markov process X associated with $(\mathcal{E}, W^{1,2}(D))$, having continuous sample paths on $\overline{D}$; one can construct a consistent Markovian family of distributions for the process starting from every point in $\overline{D}$ except possibly for a subset of ∂D having zero capacity (see [5]). Thus constructed process X is the reflecting Brownian motion on D in the sense that this definition agrees with all other standard definitions in smooth domains.

Suppose that $\{D_k,\ k \geq 1\}$ is an increasing sequence of domains with continuous boundaries such that $\bigcup_{k=1}^\infty D_k = D$. Let X^k be the reflecting Brownian motion on $\overline{D}_k$ for $k \geq 1$. We use P_x^k (P_x) to denote the law of X^k (X) starting from x, respectively.

Note that if $x \in D$ then $x \in D_k$ for sufficiently large k.

Theorem 1 *For each $T>0$, there is a subset H of $[0,T]$ with Lebesgue measure T such that for each $x \in D$, the finite dimensional distributions of $\{X_t^k,\ t \in H\}$ under P_x^k converge to those of $\{X_t,\ t \in H\}$ under P_x.*

Proof. The proof is the same as that of Theorem 3.6 in Chen [5]. ∎

Theorem 2 *For each $x \in D$, $\{P_x^k,\ k \geq 1\}$ is tight on $C([0,\infty), \mathbf{R^n})$, the space of continuous $\mathbf{R^n}$-valued functions equipped with the local uniform topology. Therefore as $k \to \infty$, P_x^k converge weakly to P_x on $C([0,\infty), \mathbf{R^n})$.*

Proof. In view of Theorem 1, it suffices to show that the family $\{P_x^k\}$ is tight on $C([0,\infty), \mathbf{R^n})$. Fix some $x \in D$ and choose k_0 such that $x \in D_{k_0}$. We will consider only $k \geq k_0$. Let $r>0$ be such that $r < \operatorname{dist}(x, \partial D_{k_0})$ and $\tau_k = \inf\{t>0 : |X_t^k - x| \geq r\}$. Note that for each $k \geq k_0$, the process $\{X_t^k,\ 0 \leq s < \tau_k\}$ is a Brownian motion killed upon leaving $B(x,r)$. By Lemma II.1.2 in Stroock [13], there is a constant $c>0$ such that for each $k \geq k_0$,

$$P_x^k(t > \tau_k) \leq c \exp\left(-\frac{r}{ct}\right) \quad \text{for } t>0. \tag{1}$$

Let $a>0$. For each $T>0$ and $\varepsilon>0$, by (1) and the strong Markov property of X^k,

$$\begin{aligned} P^k_x\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|X^k_t-X^k_s\right|>\varepsilon\right) &\le P^k_x\left(a\ge\tau_k\right)+P^k_x\left(\sup_{\substack{a\le s,t\le T\\ |t-s|\le\delta}}\left|X^k_t-X^k_s\right|>\varepsilon,\, a<\tau_k\right)\\ &\le c\exp\left(-\frac{r}{c\,a}\right)+P^k_\mu\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|X^k_t-X^k_s\right|>\varepsilon\right), \end{aligned}$$

where μ is the sub-probability distribution at time a of Brownian motion killed upon leaving $B(x,r)$ and P^k_μ is the law of X^k with initial distribution μ. Let m_k denote the Lebesgue measure on D. Note that $\phi=d\mu/dm_k$ is bounded and independent of k for $k\ge k_0$. Using an idea from Takeda (Theorem 3.1 of [14]), it can be shown that

$$\lim_{\delta\to 0}\sup_{k\ge 1}P^k_\mu\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|X^k_t-X^k_s\right|>\varepsilon\right)=0. \tag{2}$$

Indeed, by Lyons–Zheng's forward and backward martingale decomposition (see Theorem 5.7.1 of [8]),

$$X^k_t-X^k_0=\frac{1}{2}W^k_t-\frac{1}{2}\left(W^k_T\circ r_T-W^k_{T-t}\circ r_T\right)\quad\text{for all } 0\le t\le T,\ P^k_{m_k}\text{-a.s.} \tag{3}$$

where W^k is a martingale additive functional of X^k (it is in fact an n-dimensional Brownian motion, see the calcuation on page 302 of [5] or on page 211 of [8]) and r_T is the time reversal operator of X at time T, i.e., $X_t(r_T(\omega))=X_{T-t}(\omega)$ for each $0\le t\le T$. Since X^k is symmetric under $P^k_{m_k}$, for $k\ge k_0$,

$$\begin{aligned} &P^k_\mu\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|X^k_t-X^k_s\right|>\varepsilon\right)\\ \le\ &\|\phi\|_\infty P^k_{m_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|X^k_t-X^k_s\right|>\varepsilon\right)\\ \le\ &\|\phi\|_\infty P^k_{m_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|W^k_t-W^k_s\right|>\varepsilon\right)+\|\phi\|_\infty P^k_{m_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|W^k_{T-t}\circ r_T-W^k_{T-s}\circ r_T\right|>\varepsilon\right)\\ =\ &2\|\phi\|_\infty P^k_{m_k}\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|W^k_t-W^k_s\right|>\varepsilon\right)\\ =\ &2\|\phi\|_\infty\,|D_k|\,P\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|B_t-B_s\right|>\varepsilon\right), \end{aligned}$$

where B is the standard n-dimensional Brownian motion and $|D_k|$ is the volume of D_k. Claim (2) now follows. Hence

$$\limsup_{\delta\to 0}\sup_{k\ge 1}P^k_x\left(\sup_{\substack{0\le s,t\le T\\ |t-s|\le\delta}}\left|X^k_t-X^k_s\right|>\varepsilon\right)\le c\exp\left(-\frac{r}{c\,a}\right).$$

Letting $a \downarrow 0$, we have that for each $T > 0$ and $\varepsilon > 0$,

$$\lim_{\delta \to 0} \sup_{k \geq 1} P_x^k \left(\sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq \delta}} \left| X_t^k - X_s^k \right| > \varepsilon \right) = 0.$$

Thus the family $\{P_x^k, \, k \geq 1\}$ is tight on $C([0, \infty), \mathbf{R^n})$. Since each of its weak limit distributions must be P_x by Theorem 1, we conclude that P_x^k's converge weakly to P_x on $C([0, \infty), \mathbf{R^n})$ as $k \to \infty$. ∎

3. Extension. Although Theorem 2 provides sufficient information for those interested in the "hot spots" conjecture, it is natural to ask about possible generalizations of the result to larger classes of domains D. It seems that the main technical, or one could even say philosophical problem with any such generalization is the question of existence and uniqueness (or definition) of a reflecting Brownian motion in D rather than proving the convergence. We will indicate in a few words what is known about the existence of reflecting Brownian motion in an arbitrary domain and then we will show how the weak convergence result might be extended.

When D is an arbitrary bounded domain in $\mathbf{R^n}$, reflecting Brownian motion, in general, can not be constructed on the Euclidean closure $\overline{D}$ of D as a strong Markov process. For example, reflecting Brownian motion on a planar disk with a slit removed can not be a strong Markov process on the Euclidean closure of the domain. However for any bounded open set D, one can always find a suitable compact metric space D^* that contains D as a dense open subset such that $(\mathcal{E}, W^{1,2}(D))$ is a regular Dirichlet space on D^*. For example, the Martin-Kuramochi compactification introduced by Fukushima in [7] can play the role of D^*. Therefore a conservative strong Markov process X^* with continuous sample paths can be constructed on D^*. Fukushima [7] was the first person to construct reflecting Brownian motion on D^* for arbitrary bounded domain D, with a consistent Markovian family of distributions for the process starting from every point in D^* except possibly for a subset of ∂D^* having zero capacity. Let $\widetilde{P}_x$ denote the law of X^* starting from x.

More recently there were some efforts to construct a reflecting Brownian motion on the Euclidean closure $\overline{D}$ of D rather than on an abstract compactification D^* of D. In [15], among other things, Williams and Zheng constructed stationary reflecting Brownian motion on the Euclidean closure $\overline{D}$ of a bounded domain D with the initial distribution being the normalized Lebesgue measure in D. It is possible to modify that construction to obtain a reflecting Brownian motion starting from any fixed point in D, by noting that for each starting point, the reflecting Brownian motion in D has the same behavior as the standard Brownian motion before it hits ∂D. In [5], among other things, Chen used quasi-continuous projection from D^* to $\overline{D}$ in order to construct a reflecting Brownian motion on $\overline{D}$ for every starting point in D or even more generally, for every starting point in D^* except possibly for a subset of ∂D^* having zero capacity. Besides the construction, papers [15] and [5] discuss approximations of the reflecting Brownian motion and give sufficient conditions for it to be a semimartingale. A necessary and sufficient condition can be found in Chen, Fitzsimmons and Williams [6] for stationary reflecting Brownian motion on the closure of a bounded Euclidean domain to be a quasimartingale on each compact time interval.

Here is the idea of the construction in [5]. The coordinate maps $D \ni x = (x_1, \cdots, x_n) \to x_i$, $i = 1, \cdots, n$, are elements of $W^{1,2}(D)$. Let φ_i denote the quasi-continuous extension of $x \to x_i$ to all of D^* (see [8]) and set $\varphi = (\varphi_1, \cdots, \varphi_n)$. Define $X = \varphi(X^*)$ as the reflecting Brownian motion on D, which has continuous sample paths on $\overline{D}$. For each x, let P_x denote the law of X under $\widetilde{P}_x$ and call X the reflecting Brownian motion on $\overline{D}$. It is not hard to see that this definition agrees with all other standard definitions in smooth domains. The forward and backward martingale decomposition (3) still holds for thus constructed reflecting Brownian motion X under P_m, where m is the Lebesgue measure on D. Note that $P_m(X_t \in \partial D) = \widetilde{P}_m(X_t^* \in \partial D^*) = 0$ for each fixed $t > 0$. For any domain D_k in an increasing sequence $\{D_k,\, k \geq 1\}$ such that $\bigcup_{k=1}^{\infty} D_k = D$, let $(X^k,\, P_x^k)$ be the reflecting Brownian motion on $\overline{D_k}$ constructed as above. Then the proof of Theorem 2 applies to this sequence and shows that both $\{|D_k|^{-1}\, P_{m_k}^k,\, k \geq 1\}$ and $\{P_x^k,\, k \geq k_0\}$ are tight on $C([0,\infty), \mathbf{R^n})$ for any $x \in D_{k_0}$ for some $k_0 \geq 1$. Theorem 1 and its proof adapted from the proof of Theorem 3.6 in Chen [5] extend to the present case. All these remarks taken together show that Theorem 2 can be extended to arbitrary domains, with the suitable definition of reflected Brownian motion mentioned above.

References

[1] R. Bañuelos and K. Burdzy, On the "hot spots" conjecture of J. Rauch. Preprint.

[2] R. Bass and K. Burdzy, A boundary Harnack principle in twisted Hölder domains. *Ann. Math.* **134** (1991), 253–276.

[3] R. Bass and P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains. *Ann. Probab.*, **19** (1991), 486-508.

[4] K. Burdzy and W. Werner, A counterexample to the "hot spots" conjecture. Preprint.

[5] Z.-Q. Chen, On reflecting diffusion processes and Skorokhod decompositions. *Probab. Theory Rel. Fields*, **94** (1993), 281-351.

[6] Z.-Q. Chen, P. J. Fitzsimmons and R. J. Williams, Reflecting Brownian motions: quasimartingales and strong Caccioppoli sets. *Potential Analysis*, **2** (1993), 219-243.

[7] M. Fukushima, A construction of reflecting barrier Brownian motions for bounded domains. *Osaka J. Math.*, **4** (1967), 183-215.

[8] M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet forms and symmetric Markov processes.* Walter de Gruyter, Berlin, 1994

[9] M. Fukushima and M. Tomisaki, Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps. *Probab. Theory Rel. Fields*, **106** (1996), 521-557.

[10] P. L. Lions and A. S. Sznitman, Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, **37** (1984), 511-537.

[11] T. J. Lyons and W. Zheng, A crossing estimate for the canonical process on a Dirichlet space and a tightness result. *Asterisque*, **157-158** (1988), 249-271.

[12] V. G. Maz'ja, *Sobolev Spaces.* Springer–Verlag, Berlin Heidelberg 1985.

[13] D.W. Stroock, Diffusion semigroups corresponding to uniformly elliptic divergence form operator. *Lect. Notes Math.* **1321**, 316-347, Springer-Verlag (1988).

[14] M. Takeda, On a martingale method for symmetric diffusion processes and its applications. *Osaka J. Math.* **26** (1989), 605-623.

[15] R. J. Williams and W. Zheng, On reflecting Brownian motion—a weak convergence approach. *Ann. Inst. Henri Poincaré*, **26** (1990), 461-488.