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TWO-SIDED EXIT PROBLEM FOR A SPECTRALLY NEGATIVE α-STABLE ORNSTEIN-UHLENBECK PROCESS AND THE WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract
The Laplace transform of the first exit time from a finite interval by a spectrally negative α-stable Ornstein-Uhlenbeck process ($1 < \alpha \leq 2$) is provided in terms of the Wright's generalized hypergeometric function ${}_2\Psi_1$. The Laplace transform of first passage times is also derived for some related processes such as the process killed when it enters the negative half line and the process conditioned to stay positive. The law of the maximum of the associated bridges is computed in terms of the q-resolvent density. As a byproduct, we deduce some interesting analytical properties for some Wright's generalized hypergeometric functions.

1 Introduction and main result

Let $Z := (Z_t, t \geq 0)$ be a spectrally negative α-stable process, with $\alpha \in (1, 2]$, defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We recall that Z is a càdlàg process with stationary and independent increments which fulfils the scaling property $(Z_{ct}, t \geq 0) \stackrel{(d)}{=} (c^{1/\alpha} Z_t, t \geq 0)$, for any $c > 0$, where $\stackrel{(d)}{=}$ denotes equality in distribution. Due to the absence of positive jumps, it is possible to extend its characteristic exponent on the negative imaginary line to derive its Laplace exponent, ψ, which has the following form

$$\psi(u) = u^{\alpha}, \quad u \geq 0. \tag{1.1}$$

We refer to Bertoin's monograph [3, Chap. VII] for an excellent account on stable processes. Doob [8] introduced the α-stable Ornstein-Uhlenbeck process $X := (X_t, t \geq 0)$, with parameter $\lambda > 0$, defined, for any $t \geq 0$ and $x \in \mathfrak{R}$, by

$$X_t = e^{-\lambda t} \left(x + Z_{\tau(t)} \right) \tag{1.2}$$

where $\tau(t) = \frac{e^{\alpha\lambda t}-1}{\alpha\lambda}$. Note that for $t > 0$, X is the solution to the linear stochastic differential equation

$$dX_t = -\lambda X_t\, dt + dZ_t,$$

with $X_0 = x$. Next, let

$$H_{0,a} = \inf\{s \geq 0;\ X_s \notin (0,a)\}$$

be the first exit time from the interval $(0,a)$ by X. The aim of this paper is to characterize the Laplace transform of the stopping time $H_{0,a}$. This generalizes the result of Bertoin [3] who solved the exit problem, for a completely asymmetric stable process (i.e. the case $\lambda = 0$), in terms of the Mittag-Leffler functions. The main motivation to study this problem is to understand better the fluctuations of one-sided Markov processes beyond the Lévy ones. Beside, the exit time distribution of these processes is a key quantity in many applied fields, see, for instance, the Kyprianou's monograph [16] for a wide range of applications of exit time distributions and Patie [19] for the pricing of lookback options in affine term structure models. We proceed by recalling that the Wright's generalized hypergeometric (for short Wgh) function is defined as, see [9, 1,Section 4.1],

$${}_p\Psi_q\left(\begin{matrix} (A_1,a_1),\dots(A_p,a_p) \\ (B_1,b_1)\dots(B_q,b_q) \end{matrix} \,\middle|\, z \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(A_i n + a_i)}{\prod_{i=1}^{q}\Gamma(B_i n + b_i)} \frac{z^n}{n!}$$

where p,q are nonnegative integers, $a_i \in \mathfrak{C}\ (i = 1\dots p), b_j \in \mathfrak{C}\ (j = 1\dots q)$ and the coefficients $A_i \in \mathfrak{R}^+\ (i = 1\dots p)$ and $B_j \in \mathfrak{R}^+\ (j = 1\dots q)$ are such that $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0$. Under such conditions, it follows from the following asymptotic formula of ratio of gamma functions, see e.g. [17],

$$\frac{\Gamma(z+\gamma)}{\Gamma(z+\gamma+\alpha)} \quad = \quad z^{-\alpha}\left[1 + \frac{(-\alpha)(2\gamma-\alpha-1)}{2z} + O(z^{-2})\right], \quad |\arg z| < \pi - \delta,$$

that ${}_p\Psi_q(z)$ is an entire function with respect to z. Next, we introduce the functions

$$\mathcal{N}_q(x) = \frac{\alpha(\alpha\lambda)^{1-\frac{1}{\alpha}}x^{\alpha-1}}{\Gamma(\frac{q}{\alpha\lambda})}\, {}_2\Psi_1\left(\begin{matrix} (1,1),\big(1, 1+\frac{q}{\alpha\lambda}-\frac{1}{\alpha}\big) \\ (\alpha,\alpha) \end{matrix} \,\middle|\, \alpha\lambda x^{\alpha} \right),\ \Re(x) \geq 0,\ \Re(q) > \lambda(1-\alpha),$$

and

$$\overline{\mathcal{N}}^q(x) = \frac{1}{\Gamma(\frac{q}{\alpha\lambda})}\, {}_2\Psi_1\left(\begin{matrix} (1,1),\big(1,\frac{q}{\alpha\lambda}\big) \\ (\alpha,1) \end{matrix} \,\middle|\, \alpha\lambda x^{\alpha} \right), \quad \Re(x) \geq 0,\ \Re(q) > \lambda(1-\alpha).$$

In particular, we write

$$\begin{aligned} \mathcal{N}(x) &= \lim_{q\to 0} \Gamma\left(\frac{q}{\alpha\lambda}\right) \mathcal{N}_q(x) \\ &= \alpha(\alpha\lambda)^{1-\frac{1}{\alpha}}x^{\alpha-1}{}_2\Psi_1\left(\begin{matrix} (1,1),\big(1,1-\frac{1}{\alpha}\big) \\ (\alpha,\alpha) \end{matrix} \,\middle|\, \alpha\lambda x^{\alpha} \right), \quad \Re(x) \geq 0. \end{aligned}$$

We stress that these specific entire functions are members of the class of generalized Fox-Wright functions which recently have played an increasingly significant role in various types of applications see e.g. [12] and [27]. We also mention that Schneider [23] and later Zolotarev [29] used these functions for some interesting representations of stable distributions. We are now ready to state our main result.

Theorem 1.1. *Let $0 \leq x \leq a$ and $q \geq 0$. Then,*

$$\mathbb{E}_x[e^{-qH_{0,a}}] = \overline{\mathcal{N}}_q(x) + \frac{\mathcal{N}_q(x)}{\mathcal{N}_q(a)} \left(1 - \overline{\mathcal{N}}_q(a)\right).$$

Remark 1.2. *In the case $\alpha = 2$, that is when X is the classical Ornstein-Uhlenbeck process, a straightforward computation leads to the known result, see e.g. [6, Formula 7.3.0.1],*

$$\mathbb{E}_x[e^{-qH_{0,a}}] = \frac{S\left(\frac{q}{\lambda}; \sqrt{2\lambda}a; \sqrt{2\lambda}x\right) + S\left(\frac{q}{\lambda}; \sqrt{2\lambda}x; 0\right)}{S\left(\frac{q}{\lambda}; \sqrt{2\lambda}a; 0\right)}$$

where

$$S(q; x; y) = \frac{\Gamma(q)}{\pi} e^{(x^2+y^2)/4} \left(\mathcal{D}_{-q}(-x)\mathcal{D}_{-q}(y) - \mathcal{D}_{-q}(x)\mathcal{D}_{-q}(-y)\right),\ q > 0$$

and

$$\mathcal{D}_{-q}(z) = 2^{q/2} e^{-z^2/4} \frac{1}{2\Gamma(q)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+q}{2}\right)}{n!} (-1)^n (\sqrt{2}z)^n,\ z \in \mathfrak{C},$$

stands for the parabolic cylinder function.

2 Proof

Let H_a (resp. κ_a) denotes the upward (resp. downward) first passage time of X at the level a. The proof of the Theorem is based on the following identities, for $0 \leq x \leq a$,

$$\begin{aligned} \mathbb{E}_x[e^{-qH_{0,a}}] &= \mathbb{E}_x[e^{-q\kappa_0}\mathbb{I}_{\{\kappa_0 < H_a\}}] + \mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a < \kappa_0\}}] \\ &= \mathbb{E}_x[e^{-q\kappa_0}] + \mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a < \kappa_0\}}]\left(1 - \mathbb{E}_a[e^{-q\kappa_0}]\right) \end{aligned} \tag{2.1}$$

where the last line follows from the strong Markov property. Thus, we need to compute the Laplace transform of the first passage time at 0 and the first exit time of the interval $(0, a]$ from above of X. That will be done in several steps. First, we provide some general results on X which will be useful for the sequel. Then, we compute the resolvent density of X at the origin. Finally, by combining some martingale and potential theory devices, we derive the above mentioned Laplace transforms.

2.1 Some preliminary results

We start by recalling that the distribution of Z_1 is absolutely continuous with a continuous, everywhere positive density denoted by p, i.e. $\mathbb{P}(Z_1 \in dx) = p(x)dx$. It is plain from (1.2) and the scaling property of Z that

$$\mathbb{E}_x\left[e^{uX_t}\right] = \exp\left(e^{-\lambda t}x + \frac{\psi(u)}{\alpha\lambda}\left(1 - e^{-\alpha\lambda t}\right)\right),\ u \geq 0.$$

By letting $t \to \infty$, we recover the fact that X is ergodic with unique invariant measure denoted by $p^\lambda(x)dx$, see e.g. Sato [22]. For any $x \in \mathfrak{R}$, we write $\mathbb{P}_x$ for the law of X started from x. Its semigroup is specified by the kernel $\mathbb{P}_x(X_t \in dy) = p_t(x,y)p^\lambda(y)dy$, $t > 0$, with

$$p_t(x,y)p^\lambda(y) = \tau(t)^{-1/\alpha} e^{\lambda t} p\left(\tau(t)^{-1/\alpha}(e^{\lambda t}y - x)\right), \quad x, y \in \mathfrak{R}. \tag{2.2}$$

The choice of considering the transition densities with respect to the invariant measure will be justified in Section 2.3. Moreover, it is known that each point of the real line is regular (for itself), that is for any $x \in \mathfrak{R}$, $\mathbb{P}_x(\tilde{H}_x = 0) = 1$, where $\tilde{H}_x = \inf\{s > 0; X_s = x\}$, see Shiga [26]. As a consequence, for each singleton $\{y\} \in \mathfrak{R}$, X admits a local time, denoted by L^y_t, see e.g. [5]. The continuous additive functional L^y is determined by its q-potential, u^q, which is finite for any $q > 0$ and given by

$$u^q(x,y) = \mathbb{E}_x\left[\int_0^\infty e^{-qt}\, dL^y_t\right].$$

From the definition of L^y and the strong Markov property, we obtain the identity, see [5, Chap. V.3],

$$\mathbb{E}_x\left[e^{-qH_y}\right] = \frac{u^q(x,y)}{u^q(y,y)}, \quad x, y \in \mathfrak{R}.$$

For any $q > 0$, let R^q be the q-resolvent of X which is defined, for every positive measurable function f, by

$$\begin{aligned} R^q f(x) &= \int_0^\infty e^{-qt} \int f(y) p_t(x,y) p^\lambda(y) dy\, dt, \quad x \in \mathfrak{R}, \\ &= \int f(y) r^q(x,y) p^\lambda(y) dy \end{aligned}$$

where $r^q(.,.)$ is the resolvent density of X with respect to the invariant measure. We summarize in the following some properties of the characteristics of X.

Proposition 2.1. *1. For $q > 0$, the mapping $(x,y) \mapsto r^q(x,y)$ is continuous and bounded by $\max(q^{-1}, 0)$ on $\mathfrak{R} \times \mathfrak{R}$.*

2. The q-potential of L^y is related to the resolvent density of X as follows

$$u^q(x,y) = r^q(x,y) p^\lambda(y), \quad x, y \in \mathfrak{R}.$$

3. Finally, the infinitesimal generator of X is given, for $f \in C^2_0(\mathfrak{R})$, the space of twice continuously differentiable functions vanishing at infinity, by

$$\mathbf{A}^{(\lambda)} f(x) = \left(\frac{c_\alpha}{\alpha - 1} - \lambda x\right) f'(x) + \int_{-\infty}^0 \left(f(x+y) - f(x) - y f'(x) \mathbb{I}_{\{|y|<1\}}\right) c_\alpha |y|^{-\alpha-1} dy$$

where $c_\alpha = -\frac{\alpha}{\Gamma(1-\alpha)} > 0$.

Proof. Since each point of the real line is regular (for itself) and X is recurrent, the fine topology coincides with the initial topology of $\mathfrak{R}$, the first claim follows from Bally and Stoica [1, Proposition 3.1]. The second claim is deduced from the following

$$\begin{aligned} R^q f(x) &= \mathbb{E}_x\left[\int_{y \in \mathfrak{R}} dy f(y) \int_0^\infty e^{-qt}\, dL^y_t\right] \\ &= \int_{y \in \mathfrak{R}} dy f(y) u^q(x,y). \end{aligned}$$

Finally, for $f \in C_0^2(\Re)$, we have

$$\begin{aligned}\mathbf{A}^{(\lambda)} f(x) &= \lim_{t\downarrow 0} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t} \\ &= \lim_{t\downarrow 0} \frac{\mathbb{E}\left[f\left(e^{-\lambda t}(x + Z_{\tau(t)})\right)\right] - f(e^{-\lambda t}x)}{\tau(t)} \frac{\tau(t)}{t} + \frac{f(e^{-\lambda t}x) - f(x)}{t}.\end{aligned}$$

As for $t \downarrow 0$, $\tau(t)/t \to 1$, $\frac{\mathbb{E}_x\left[f\left(e^{-\lambda t}(x+Z_{\tau(t)})\right)\right]-f(e^{-\lambda t}x)}{\tau(t)} \to \mathbf{A}^{(0)}f(x)$ and by Taylor expansion $\frac{f(e^{-\lambda t}x)-f(x)}{t} \to -\lambda x f'(x)$ uniformly, we obtain the expression for $\mathbf{A}^{(\lambda)}$. □ □

2.2 The resolvent density at the origin

We carry on by computing the constants $r^q := r^q(0,0)$, the resolvent density at the origin, and $u^q := u^q(0,0)$. These quantities will be useful in the proof of the main result.

Proposition 2.2. *For any $q > 0$, we have*

$$\begin{aligned} r^q &= \frac{1}{\alpha\lambda} B\left(\frac{\alpha-1}{\alpha}, \frac{q}{\alpha\lambda}\right) \\ u^q &= \frac{(\lambda\alpha)^{1/\alpha-1}\,\Gamma(\frac{q}{\alpha\lambda})}{\alpha\Gamma(\frac{q}{\alpha\lambda}+1-\frac{1}{\alpha})}\end{aligned}$$

where B and Γ stand for the beta and the gamma functions.

Proof. From the expression of the density of the semigroup (2.2), we get

$$\begin{aligned} r^q &= \int_0^\infty e^{-qt} p_t(0,0)dt \\ &= \frac{p(0)}{p^\lambda(0)} \int_0^\infty e^{-qt} \tau_\lambda(t)\, dt \\ &= \frac{p(0)}{p^\lambda(0)} (\lambda\alpha)^{1/\alpha-1} B\left(1-\frac{1}{\alpha}, \frac{q}{\alpha\lambda}\right)\end{aligned}$$

where $\tau_\lambda(t) = \tau(t)^{-1/\alpha} e^{\lambda t}$ and the last line follows after performing the change of variable $u = e^{\alpha\lambda t} - 1$ and from the following integral representation of the Beta function, see e.g [13, p. 376],

$$B(z_1, z_2) = \int_0^\infty u^{z_1-1}(1+u)^{-z_1-z_2} du, \quad \Re(z_i) > 0,\ i = 1, 2.$$

Finally, by means of the continuity and the scaling property of the stable densities, we deduce that $\lim_{x\to 0} \frac{p(x)}{p^\lambda(x)} = (\alpha\lambda)^{-\frac{1}{\alpha}}$, which gives the first claim. Finally, recalling that $u^q = r^q p^\lambda(0)$, with $p^\lambda(0) = (\alpha\lambda)^{\frac{1}{\alpha}} \frac{\sin(\frac{\pi}{\alpha})}{\pi} \Gamma\left(1+\frac{1}{\alpha}\right)$, see [28, Formula 4.9.1], the proof is completed by using the recurrence and reflection formulae for the Gamma function

$$\Gamma(\nu+1) = \nu\Gamma(\nu) \text{ and } \Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin(\pi\nu)}. \quad \square$$

□

Next, we leave aside the proof of the Theorem and start a short digression regarding the description of the length of the excursions away from 0 by X. Let us denote by $\sigma := (\sigma_l, l \geq 0)$ the right continuous inverse of the continuous and increasing functional $(L_t^0, t \geq 0)$. It is plain that σ is a subordinator. Let us now introduce the recurrent δ-dimensional radial Ornstein-Uhlenbeck process with drift parameter $\mu > 0$, which is defined, for $0 < \delta < 1$, as the non-negative solution to

$$dR_t = \left(\frac{\delta - 1}{2R_t} - \mu R_t\right) dt + dB_t$$

where B is a Brownian motion. It is well-known that, under the condition $1 \leq \delta < 2$, the point $\{0\}$ is a recurrent state for R and we denote by $\sigma^{(\delta,\mu)}$ the inverse local time at 0 of R. We are now ready to state the following.

Corrolary 2.3. *The Lévy measure of the subordinator σ is absolutely continuous with density g given, for any $s > 0$, by*

$$g(s) \quad = \quad \frac{(\alpha-1)}{\Gamma(\frac{1}{\alpha})} e^{\frac{\lambda}{2}s} \left(\frac{\alpha\lambda}{\sinh\left(\frac{\alpha\lambda}{2}s\right)}\right)^{2-1/\alpha}.$$

Moreover, we have the following identity

$$(\sigma_l, l \geq 0) \stackrel{(d)}{=} (\sigma_l^{\left(\frac{2}{\alpha}, \frac{\alpha\lambda}{2}\right)}, l \geq 0)$$

where the normalization of the local time of R is $C = \frac{(\alpha-1)}{\Gamma(\frac{1}{\alpha})}$.

Proof. It is well known that the Laplace exponent of σ_1 is expressed in terms of the q-potential of the local time as follows, see e.g. [5, Chap. V.3],

$$-\log\left(\mathbb{E}\left[e^{-q\sigma_1}\right]\right) = \frac{1}{u^q}.$$

Moreover, since the transition probabilities of X are diffuse, σ is a driftless subordinator. Thus, its Laplace exponent has the following form

$$\frac{1}{u^q} = \int_0^\infty (e^{-qs} - 1) G(ds)$$

for a measure G on $]0, \infty]$ satisfying $\int_0^\infty (1 \wedge s) G(ds) < \infty$. Denoting $C_\alpha = \left(\frac{1}{\alpha\lambda}\right)^{1/\alpha - 1}$, we have, with $\overline{G}'(s) = G(s)$,

$$\frac{C_\alpha \Gamma(\frac{q}{\alpha\lambda} + 1 - \frac{1}{\alpha})}{\Gamma(\frac{q}{\alpha\lambda})} \quad = \quad q \int_0^\infty e^{-qs} \overline{G}(s)\, ds.$$

We deduce the expression for the density g from the following integral representation of the ratio of gamma functions

$$\frac{\Gamma(\frac{q}{\alpha\lambda} + 1 - \frac{1}{\alpha})}{\Gamma(\frac{q}{\alpha\lambda} + 1)} = \frac{\alpha\lambda}{\Gamma(\frac{1}{\alpha})} \int_0^\infty e^{-qs} (e^{\alpha\lambda s} - 1)^{1/\alpha - 1}\, ds.$$

Finally, comparing with formula (16) in Pitman and Yor [21], we deduce that g is also the density of the Lévy measure of the inverse local time of R with dimension $\delta = \frac{2}{\alpha}$ and parameter $\mu = \frac{\alpha\lambda}{2}$. The identity follows from the injectivity of the Laplace transform and the fact that σ is a Lévy process. □ □

2.3 The dual process

We introduce the dual process of X, denoted by $\hat{X}$, relative to the invariant measure $p^\lambda(y)dy$. That is, denoting by $\hat{R}^q$, the resolvent of $\hat{X}$, we have for every positive measurable function f,

$$\hat{R}^q f(x) = \int f(y) r^q(y,x) p^\lambda(y)dy,\ x \in \Re.$$

Moreover, we shall need the expression of the Laplace transform of H_a, for $x \leq a$, which has been evaluated by Hadjiev [14], see also Novikov [18] for a similar result, as follows

$$\mathbb{E}_x[e^{-qH_a}] \quad = \quad \frac{\mathcal{H}_q(x)}{\mathcal{H}_q(a)}, \quad q \geq 0, \tag{2.3}$$

where

$$\mathcal{H}_q(x) = \frac{1}{\Gamma(\frac{q}{\alpha\lambda})} \int_0^\infty e^{(u\alpha\lambda)^{\frac{1}{\alpha}}x - u} u^{\frac{q}{\alpha\lambda}-1} du.$$

$\mathcal{H}$ admits also a representation in terms of the Wgh functions. Indeed, from the integral representation above, one gets

$$\begin{aligned}
\mathcal{H}_q(x) \quad &= \quad \frac{1}{\Gamma(\frac{q}{\alpha\lambda})} \sum_{n=0}^\infty \frac{\Gamma\left(\frac{n}{\alpha} + \frac{q}{\alpha\lambda}\right)}{n!} (\alpha\lambda)^{\frac{1}{\alpha}} x^n \\
&= \quad \frac{1}{\Gamma(\frac{q}{\alpha\lambda})} \, {}_1\Psi_0\Big(\ \left(\tfrac{1}{\alpha}, \tfrac{q}{\alpha\lambda}\right) \ \big| \, (\alpha\lambda)^{\frac{1}{\alpha}} x \Big).
\end{aligned}$$

We deduce that $\mathcal{H}$ admits an analytical continuation on the whole complex plane with respect to x and is analytic on the domain $\{q \in \mathfrak{C};\ \Re(q) > \frac{1}{\alpha}\}$.

Lemma 2.4. *$\hat{X}$ is an α-stable Ornstein-Uhlenbeck of parameter λ driven by $\hat{Z} = -Z$, the dual of Z with respect to the Lebesgue measure. Consequently, for $q \geq 0$, $x \leq a$, we have*

$$\mathbb{E}_a[e^{-q\hat{H}_x}] \quad = \quad \frac{\mathcal{H}_q(-x)}{\mathcal{H}_q(-a)}$$

where $\hat{H}_x$ stands for the upward first passage time of $\hat{X}$ at the level x. Moreover, for $q > 0$,

$$\begin{aligned}
r^q(a,a) \quad &= \quad r^q \mathcal{H}_q(a)\mathcal{H}_q(-a) \\
r^q(x,a) \quad &= \quad r^q \mathcal{H}_q(x)\mathcal{H}_q(-a).
\end{aligned}$$

Proof. We write $\mathfrak{L}(p)(u) := \int_\Re e^{ux} p(x)dx$. We have $\mathfrak{p}^\lambda(u) = e^{\frac{\psi(u)}{\alpha\lambda}}$ where $\mathfrak{p}^\lambda(u) := \mathfrak{L}(p^\lambda)(u)$, $\Re(u) \geq 0$. Then, denoting by $\mathbf{A}^{(-\lambda)}$ the infinitesimal generator of $X^{(-\lambda)}$, the stable Ornstein-Uhlenbeck process with parameter $-\lambda$, we derive easily

$$\begin{aligned}
\mathfrak{L}\left(\mathbf{A}^{(-\lambda)} p^\lambda\right)(u) \quad &= \quad \lambda\left(-u\frac{\partial}{\partial u}\mathfrak{p}^\lambda(u) - \mathfrak{p}^\lambda(u)\right) + \psi(u)\mathfrak{p}^\lambda(u) \\
&= \quad -\lambda \mathfrak{p}^\lambda(u).
\end{aligned}$$

Hence, $p^\lambda(.)$ is $-\lambda$-invariant for $X^{(-\lambda)}$. We deduce, from the expression of the semigroup (2.2) and the stationarity of X, the following absolute continuity relationship, with obvious notation,

$$d\mathbb{P}_x^{(\lambda)}{}_{|\mathcal{F}_t} = e^{\lambda t} \frac{p^\lambda(X_t)}{p^\lambda(x)} d\mathbb{P}_x^{(-\lambda)}{}_{|\mathcal{F}_t}, \quad t > 0.$$

Next, we write $\mathbb{P}_x(\hat{X}_t \in dy) = \hat{p}_t(x,y)p^{\lambda}(y)dy$ and $\mathbb{P}_x(e^{\lambda t}Z_{-\tau(-t)} \in dy) = q_t^{(-\lambda)}(x,y)dy$. Using the duality, with respect to the Lebesgue measure, between Z and $-Z$ and the above absolute continuity relationship, we get that

$$\begin{aligned}
\hat{p}_t(x,y) &= (p^{\lambda}(y))^{-1}\tau(t)^{-1/\alpha}e^{\lambda t}\hat{p}\left(\tau(t)^{-1/\alpha}(e^{\lambda t}y - x)\right) \\
&= (p^{\lambda}(y))^{-1}(-\tau(-t))^{-1/\alpha}p\left((-\tau(-t))^{-1/\alpha}(e^{-\lambda t}x - y)\right) \\
&= (p^{\lambda}(y))^{-1}e^{\lambda t}q_t^{(-\lambda)}(y,x) \\
&= p_t(y,x),
\end{aligned}$$

which gives the first claim. The expression of the Laplace transform of $\hat{H}_x$ is obtained by following a line of reasoning similar to Shiga [26, Theorem 3.1]. Finally, by means of the duality relationship between the resolvent densities

$$\hat{r}^q(x,y) = r^q(y,x), \quad x,y \in \Re,$$

we complete the proof from the identities

$$\begin{aligned}
r^q(a,a) &= r^q\frac{\mathbb{E}_a[e^{-q\hat{H}_0}]}{\mathbb{E}_0[e^{-qH_a}]} \\
&= r^q\mathcal{H}_q(a)\mathcal{H}_q(-a)
\end{aligned}$$

and

$$r^q(x,a) = \mathbb{E}_x[e^{-qH_a}]r^q(a,a). \quad \square$$

$\square$

2.4 The first hitting and passage time at 0

We recall that the θ-scale function of Z, denoted by $\mathcal{W}^{\theta}$, has been computed by Bertoin [4] as follows

$$\mathcal{W}^{\theta}(x) = x^{\alpha-1}\mathcal{E}_{\alpha,\alpha}((\theta^{1/\alpha}x)^{\alpha}), \quad \Re(x) \geq 0,$$

where $\mathcal{E}_{\alpha,\beta}$ stands for the Mittag-Leffler function which is expressed in term of the Wgh function by

$$\mathcal{E}_{\alpha,\beta}(x) = {}_1\Psi_1\left(\begin{array}{c}(1,1)\\(\beta,\alpha)\end{array}\middle|\, x\right).$$

We write simply $\mathcal{V}(\theta^{1/\alpha}x) = \alpha\theta^{1-1/\alpha}\mathcal{W}^{\theta}(x)$.

Lemma 2.5. *For any $x, q \geq 0$, we have*

$$\mathbb{E}_x[e^{-qH_0^-}] = \mathcal{H}_q(x) - \mathcal{N}_q(x) \tag{2.4}$$

and

$$\mathbb{E}_x[e^{-q\kappa_0}] = \overline{\mathcal{N}}_q(x) - \alpha^{-1}\mathcal{N}_q(x). \tag{2.5}$$

Proof. We deduce from (1.2) that $T_0^- = \tau(H_0^-)$ a.s., where T_0^- stands for the downward hitting time of Z at 0. Thus,

$$\mathbb{E}_x\left[e^{-qH_0^-}\right] = \mathbb{E}_x\left[(\alpha\lambda T_0^- + 1)^{-\frac{q}{\alpha\lambda}}\right]. \tag{2.6}$$

Moreover, Doney [7] showed that the law of T_0^- is characterized by

$$\int_0^\infty e^{-\beta x}\mathbb{E}_x[e^{-\theta T_0^-}]\,dx = \frac{1}{\beta - \theta^{1/\alpha}} - \frac{\alpha\theta^{1-1/\alpha}}{\beta^\alpha - \theta}$$

where the right-hand side is defined by continuity for $\beta, \theta > 0$. Noting that for $\beta > \theta^{1/\alpha}$, $\frac{1}{\beta^\alpha - \theta} = \sum_{n=1}^\infty \beta^{-\alpha n}\theta^{n-1}$, so inverting the Laplace transform yields, for $x \geq 0$,

$$\mathbb{E}_x[e^{-\theta T_0^-}] = e^{\theta^{1/\alpha}x} - \mathcal{V}(\theta^{1/\alpha}x). \tag{2.7}$$

Next, setting $\bar{\theta} = \alpha\lambda\theta$ and using a device introduced by Shepp [25] in the case of the Brownian motion, we integrate both sides of the previous identity by the measure $\frac{1}{\Gamma(\frac{q}{\alpha\lambda})}e^{-\theta}\theta^{\frac{q}{\alpha\lambda}-1}d\theta$ to get

$$\frac{1}{\Gamma(\frac{q}{\alpha\lambda})}\int_0^\infty \mathbb{E}_x[e^{-\bar{\theta}T_0^-}]e^{-\theta}\theta^{\frac{q}{\alpha\lambda}-1}d\theta \quad = \quad \frac{1}{\Gamma(\frac{q}{\alpha\lambda})}\int_0^\infty \left(e^{\bar{\theta}^{1/\alpha}x} - \mathcal{V}(\bar{\theta}^{1/\alpha}x)\right)e^{-\theta}\theta^{\frac{q}{\alpha\lambda}-1}d\theta.$$

By using the change of variable $u = \theta(\alpha\lambda T_0^- + 1)$ on the left hand side and observing that

$$\frac{1}{\Gamma(\frac{q}{\alpha\lambda})}\int_0^\infty \mathcal{V}\left(\bar{\theta}^{1/\alpha}x\right)e^{-\theta}\theta^{\frac{q}{\alpha\lambda}-1}d\theta \quad = \quad \mathcal{N}_q(x),$$

we obtain from (2.6) that

$$\mathbb{E}_x\left[e^{-qH_0^-}\right] \quad = \quad \mathcal{H}_q(x) - \mathcal{N}_q(x).$$

The expression of the Laplace transform of κ_0 is obtained in a similar way. Indeed, it is plain that $\kappa_0 = \tau(\eta_0)$ a.s. with $\eta_0 = \inf\{s \geq 0; Z_s < 0\}$. Thus,

$$\mathbb{E}_x\left[e^{-q\kappa_0}\right] = \mathbb{E}_x\left[(\alpha\lambda\eta_0 + 1)^{-\frac{q}{\alpha\lambda}}\right].$$

Moreover, we know, see e.g. [4], that

$$\mathbb{E}_x\left[e^{-\bar{\theta}\eta_0}\right] = \mathcal{E}_{\alpha,1}(\bar{\theta}x^\alpha) - \alpha^{-1}\mathcal{V}\left(\bar{\theta}x^\alpha\right).$$

As above, integrating both sides of the latter equation by the measure $\frac{1}{\Gamma(\frac{q}{\alpha\lambda})}e^{-\theta}\theta^{\frac{q}{\alpha\lambda}-1}d\theta$ we obtain (2.5). □ □

2.5 First exit time from above

We denote by r_0^q (resp. $\hat{r}_0^q$), the resolvent density of the process X (resp. $\hat{X}$) killed when entering the negative line.

Lemma 2.6. *For any $a \geq x > 0$ and $q \geq 0$, we have*

$$\mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a<\kappa_0\}}] = \frac{\mathcal{N}_q(x)}{\mathcal{N}_q(a)} \tag{2.8}$$

$$\mathbb{P}_x[H_a < \kappa_0] = \frac{\mathcal{N}(x)}{\mathcal{N}(a)}. \tag{2.9}$$

Proof. First, for $0 < x \leq a$, note that

$$\mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a<\kappa_0\}}] = \frac{r_0^q(x,a)}{r_0^q(a,a)}.$$

Next, from the strong Markov property and the absence of negative jumps for $\hat{X}$, we get, for $x, a > 0$,

$$\hat{r}_0^q(a,x) = \hat{r}^q(a,x) - \mathbb{E}_a[e^{-q\hat{H}_0}]\hat{r}^q(0,x).$$

Moreover, the switch identity for Markov processes, see [5, Chap.VI], tells us that $\hat{r}_0^q(x,a) = r_0^q(a,x)$. Thus,

$$r_0^q(x,a) = r^q(x,a) - \mathcal{H}_q(-a)r^q(x,0).$$

However, from (2.4), we deduce that, for $x \geq 0$,

$$\begin{aligned} r^q(x,0) &= \mathbb{E}_x[e^{-qH_0^-}]r^q \\ &= r^q\left(\mathcal{H}_q(x) - \mathcal{N}_q(x)\right). \end{aligned}$$

Hence, from Lemma 2.4, we get

$$r_0^q(x,a) = r^q\mathcal{N}_q(x)\mathcal{H}_q(-a), \quad 0 < x \leq a. \tag{2.10}$$

The continuity of the resolvent density yields

$$r_0^q(a,a) = r^q\mathcal{N}_q(a)\mathcal{H}_q(-a), \quad 0 < a,$$

which proves the first claim. The second assertion (2.9) is obtained by passage to the limit as q tends to 0, see (1.3). □ □

Finally, the proof of the Theorem is completed by combining (2.1), (2.5) and (2.8).

3 Some concluding remarks

We proceed by studying the law of the first passage time of some (Markov) processes, the laws of which are constructed from the one of X. In order to simplify the notation we shall work in the canonical setting. That is, we denote by D the space of càdlàg paths $\omega : [0,\zeta) \longrightarrow \mathfrak{R}$ where $\zeta = \zeta(w)$ is the lifetime. D will be equipped with the Skohorod topology, with its Borel σ-algebra $\mathcal{F}$, and the natural filtration $(\mathcal{F}_t)_{t\geq 0}$. We keep the notation X for the coordinate process. Let $\mathbb{P}_x$ (resp. $\mathbb{E}_x$) be the law (resp. the expectation operator) of the stable Ornstein-Uhlenbeck process starting at $x \in \mathfrak{R}$.

3.1 The α-stable Ornstein-Uhlenbeck Process Conditioned to Stay Positive

It is plain from (2.9) that the function $\mathcal{N}$ is a positive invariant function for the α-stable Ornstein-Uhlenbeck process killed at κ_0. Thus, one can define a new probability measure $\mathbb{P}^{\uparrow}$ on $D\left((0,\infty)\right)$ by setting, for $x>0$,

$$\mathbb{P}^{\uparrow}_x(A)=\frac{1}{\mathcal{N}(x)}\mathbb{E}_x\left[\mathcal{N}(X_t),A,t<\kappa_0\right],\quad A\in\mathcal{F}_t. \tag{3.1}$$

$\mathbb{P}^{\uparrow}$ is a Doob's h-transform of the killed process. It is plain that the lifetime ζ is infinite and that X is a $(\mathbb{P}^{\uparrow}_x,\mathcal{F},\mathcal{F}_t)$ strong Markov process. Moreover, $\mathbb{P}^{\uparrow}_x$ drifts to $+\infty$. It turns out that $\mathbb{P}^{\uparrow}_x$ can be identified as the law of $\mathbb{P}_x$ conditioned by the event "X stays positive". Indeed following [2], we introduce $\mathbb{P}^{(0,y)}_x$, for $0<x<y$, the law of the initial Ornstein-Uhlenbeck process starting from x, conditioned to make its first exit from $(0,y)$ through y and killed when it hits y. Then, from the strong Markov property, we have, for any Borel set $A\in\mathcal{F}_t$,

$$\mathbb{P}^{(0,y)}_x(A,t<\zeta)\quad=\quad\frac{1}{\mathcal{N}(x)}\mathbb{E}_x\left[\mathcal{N}(X_t),A,t<\kappa_0,t<H_y\right].$$

By monotone convergence, as y tends to ∞, we obtain (3.1). Let us still denote by H_a the first passage time of the canonical process X at the level $a>x$. Finally, let $r^q_{\uparrow}(x,y)$ denotes the resolvent density of $(X,\mathbb{P}^{\uparrow})$. Observing that

$$\lim_{x\downarrow 0}\frac{\mathcal{N}_q(x)}{\mathcal{N}(x)}\quad=\quad\frac{1}{\alpha\lambda r^q},$$

we deduce, from the definition of $\mathbb{P}^{\uparrow}_x$, (2.8) and (2.10), the following.

Corrolary 3.1. *Let $q\geq 0$ and $0<x\leq a$, then*

$$\mathbb{E}^{\uparrow}_x\left[e^{-qH_a}\right]=\frac{\mathcal{N}(a)\mathcal{N}_q(x)}{\mathcal{N}(x)\mathcal{N}_q(a)}.$$

Moreover, the family of probability measures $(\mathbb{P}^{\uparrow}_x)_{x>0}$ converges as $x\to 0^+$ in the sense of finite-dimensional distributions to a law which is characterized by

$$r^q_{\uparrow}(0,y)=\frac{1}{\alpha\lambda}\mathcal{N}(y)\mathcal{H}_q(-y),\ 0<y.$$

3.2 The Law of the Maximum of Bridges

In what follows, we readily extend a result of Pitman and Yor [20] regarding the law of maximum of diffusion bridges to our context. Recall that the α-stable Ornstein-Uhlenbeck process is a strong Markov process with right continuous paths. From (2.2), it is plain that for any $t>0$, its transition densities $p_t(x,y)$ are absolutely continuous with respect to the invariant measure and everywhere positive. Moreover, there exists a second right process $\hat{X}$ in duality with X relative to the invariant measure. Under these conditions Fitzsimmons et al. [11] construct the bridges of X by using Doob's method of h-transform. More precisely, let us denote by $\mathbb{P}^l_{x,y}$ the law of X started at x and conditioned to be at y at time t. We have the following absolute continuity relationship, for $l<t$,

$$d\mathbb{P}^t_{x,y|\mathcal{F}_l}=\frac{p_{t-l}(X_l,y)}{p_t(x,y)}\,d\mathbb{P}_{x|\mathcal{F}_l}. \tag{3.2}$$

Let us denote by M_t the maximum of the canonical process X up to time $t > 0$. The law of the maximum of the bridge of $(X, \mathbb{P})$ is characterized in the following.

Corrolary 3.2. *For $q > 0$, $x, y, a \in \Re$ with $x, y \leq a$, we have*

$$\int_0^\infty e^{-qt}\mathbb{P}^t_{x,y}(M_t \geq a)p_t(x,y)\,dt = r^q(a,y)\frac{\mathcal{H}_q(x)}{\mathcal{H}_q(a)}.$$

In particular, for the standard bridges, with $a \geq 0$, the expression simplifies to

$$\int_0^\infty e^{-qt}\mathbb{P}^t_{0,0}(M_t \geq a)\,p_t(0,0)dt = r^q\left(1 - \frac{\mathcal{N}_q(a)}{\mathcal{H}_q(a)}\right).$$

Proof. Thanks to the absolute continuity relationship (3.2) and Doob's optional stopping theorem, we have

$$\mathbb{P}^t_{x,y}(H_a \in dl)p_t(x,y) = p_{l-t}(a,y)\mathbb{P}_x(H_a \in dl).$$

Then, integrating both sides, we get

$$\mathbb{P}^t_{x,y}(H_a \geq t)p_t(x,y) = \int_0^t p_{l-t}(a,y)\mathbb{P}_x(H_a \in dl). \tag{3.3}$$

Next, we use the fact that $\mathbb{P}^t_{x,y}(H_a \leq t) = \mathbb{P}^t_{x,y}(M_t \geq a)$. Finally by taking the Laplace transform with respect to t, and by noticing the convolution on the right hand side of (3.3), we complete the proof. □ □

Remark 3.3. *It is plain that such a result could be derived for spectrally negative Lévy processes for which one can construct bridges by h-transform. It is, for instance, the case when the density of the Lévy process is absolutely continuous which implies that this density is positive all over the interior of its support, see Sharpe [24].*

3.3 Analytical properties of some Wgh functions

We end up by providing some interesting properties of the Wgh functions.

Corrolary 3.4. *For $1 < \alpha \leq 2$ and $x > 0$, the functions*

$$q \longmapsto \frac{\Gamma\left(q+1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}\right)}\frac{{}_2\Psi_1\left(\begin{matrix}(1,1),\left(1,1-\frac{1}{\alpha}\right)\\(\alpha,\alpha)\end{matrix}\middle|\, x^\alpha\right)}{{}_2\Psi_1\left(\begin{matrix}(1,1),\left(1,q+1-\frac{1}{\alpha}\right)\\(\alpha,\alpha)\end{matrix}\middle|\, x^\alpha\right)},$$

$$q \longmapsto \frac{\Gamma(q)}{{}_1\Psi_0\left(\ \left(\frac{1}{\alpha},q\right)\ \middle|\, x\right)}$$

are Laplace transforms of infinitely divisible distributions concentrated on the positive line. Finally, the mapping

$$q \longmapsto \frac{\Gamma(\alpha)}{{}_1\Psi_1\left(\begin{matrix}(1,1)\\(\alpha,\alpha)\end{matrix}\middle|\, q\right)}$$

is the Laplace transform of a self-decomposable distribution concentrated on the positive line.

Proof. From the absence of positive jumps, it is plain that for any $0 < c < a$, we have, $(H_a, \mathbb{P}^{\uparrow}_{0+}) \overset{(d)}{=} (H_c, \mathbb{P}^{\uparrow}_{0+}) + (H_a, \mathbb{P}^{\uparrow}_{c})$ and from the strong Markov property the two random variables on the right hand side are independent. Then, following Kent [15], by splitting the interval $[0, a]$ in subintervals, we may express $(H_a, \mathbb{P}^{\uparrow}_{0+})$ as the limit of a null triangular array. Thus, $(H_a, \mathbb{P}^{\uparrow}_{0+})$ is infinitely divisible, see Feller [10]. From Corollary 3.1 and by developing the same reasoning with the stopping time $(H_a, \mathbb{P}_0)$, we deduce the first assertion. Finally, in the limit case $\lambda \to 0$, the upward first passage time process $(T_a)_{a>0}$ of the spectrally negative α-stable process conditioned to stay positive is a $\frac{1}{\alpha}$-self-similar additive process. Hence, for any fixed a, the law of T_a is self-decomposable, see e.g. Sato [22]. The proof is completed. □ □

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