Elect. Comm. in Probab. **1** (1996) 1–6

ELECTRONIC COMMUNICATIONS in PROBABILITY

ON THE NON-CONVEXITY OF THE TIME CONSTANT IN FIRST-PASSAGE PERCOLATION [1]

HARRY KESTEN
Department of Mathematics, Cornell University, Ithaca, NY 14853
e-mail: kesten@math.cornell.edu

AMS 1991 Subject classification: 60K35
Keywords and phrases. First-passage percolation, time constant, convexity.

Abstract
We give a counterexample to a conjecture of Hammersley and Welsh (1965) about the convexity of the time constant in first–passage percolation, as a functional on the space of distribution functions. The present counterexample only works for first–passage percolation on $\mathbb{Z}^d$ *for d large.*

1 Introduction

First–passage percolation was introduced by Hammersley and Welsh (1965). They considered an i.i.d. family of random variables $\{t(e)\}$, where e runs through the edges of $\mathbb{Z}^d$. Here, and throughout this paper, $\mathbb{Z}^d$ is regarded as a graph with edges between any two points $u = (u_1, \dots, u_d) \in \mathbb{Z}^d$ and $v = (v_1, \dots, v_d) \in \mathbb{Z}^d$ if and only if

$$|u - v| := \sum_{i=1}^{d} |u_i - v_i| = 1. \tag{1}$$

It is assumed throughout that the $t(e)$ are positive (i.e., ≥ 0) and that

$$\mathbb{E}\{t(e)\} < \infty. \tag{2}$$

The *passage time* of a path $r = (e_1, e_2, \dots, e_n)$ on $\mathbb{Z}^d$ is defined as

$$T(r) = \sum_{i=1}^{n} t(e_i),$$

and the *travel time* from u to v is defined as

$$T(u, v) = \inf\{T(r) : r \text{ a path on } \mathbb{Z}^d \text{ from } u \text{ to } v\} \tag{3}$$

[1] Research supported in part by the NSF through a grant to Cornell University. The author further thanks the ETH in Zürich for its hospitality and support while this research was being carried out.

(see also Smythe and Wierman (1978) or Kesten (1986) for more details). Let F be the common distribution of the $t(e)$. It was proven by Kingman (1968) that under condition (2), there exists a constant $0 \leq \mu(F) = \mu(F, d) < \infty$, which depends only on F and the dimension d, such that

$$\frac{1}{n} T(\mathbf{0}, n\xi_1) \to \mu(F) \text{ w.p.1 and in } L_1. \tag{4}$$

Here $\xi_1 = (1, 0, \dots, 0)$ is the first coordinate vector; μ is called the *time constant*. Hammersley and Welsh (1965) (see Conjecture 6.5.5) conjectured that μ is a convex functional on the space of distribution functions on $[0, \infty)$ satisfying (2). That is, they conjectured that for any two such distribution functions F_0 and F_1,

$$\mu(\alpha F_1 + (1-\alpha)F_0, d) \leq \alpha\mu(F_1, d) + (1-\alpha)\mu(F_0, d),\ 0 \leq \alpha \leq 1. \tag{5}$$

In Remark 4 at the end of the next section we give some pairs F_0, F_1 with $F_0 \neq F_1$ for which (5) holds, even with strict inequality when $0 < \alpha < 1$. However, our main result here is an example for which (5) fails. The present example relies on the results of Hara (1990) on the correlation function for percolation in high dimensions. Once one has a single counterexample to (5), simpler examples may well be forthcoming. There is no reason to believe that dimension plays a significant role in this problem and we therefore doubt the general validity of (5) in any dimension ≥ 2.

2 The Example

Let $p_c = p_c(d)$ be the critical probability for (Bernoulli) edge percolation on $\mathbb{Z}^d$ (see Grimmett (1989) for a precise definition) and let δ_a be the distribution which puts mass 1 on the point a. Take

$$F_0 = p_c\delta_0 + (1-p_c)\delta_2$$

and

$$F_1 = q\delta_1 + (1-q)\delta_2 \tag{6}$$

for some $0 < q < 1$. Thus F_0 puts masses p_c and $1-p_c$ on the points 0 and 2, respectively, while F_1 puts masses q and $1-q$ on the points 1 and 2, respectively. We shall show that there exist some $d_0 < \infty$ and $\alpha_0(d) > 0$ such that (5) fails for $d \geq d_0$ and $\alpha \leq \alpha_0$.
To prove this claim we first observe that

$$\mu(F_0) = 0 \tag{7}$$

by virtue of Theorem 6.1 in Kesten (1986) and the inequality $p_c = p_T$ which is due to Menshikov (1986) and Aizenman and Barsky (1987) (see Ch.3 in Grimmett (1989)). The right hand side of (5) therefore equals $\alpha\mu(F_1)$, and it suffices to prove that

$$\limsup_{\alpha \downarrow 0} \frac{\mu(\alpha F_1 + (1-\alpha)F_0)}{\alpha} = \infty. \tag{8}$$

In fact, we shall show the stronger property

$$\mu(\alpha F_1 + (1-\alpha)F_0) \geq C_1(d)\alpha^{1/2}\left[\log\frac{1}{\alpha}\right]^{-1} \tag{9}$$

for some constant $C_1 > 0$ and small α. To this end we introduce the distribution function

$$G(p) = p\delta_0 + (1-p)\delta_1$$

and take for $\{s_p(e) : e \in \mathbb{Z}^d\}$ an i.i.d. family of random variables with common distribution function $G(p)$. The joint distribution of all the $s_p(e)$, $e \in \mathbb{Z}^d$, is denoted by $\mathbb{P}_p$. $\mathbb{E}_p$ will be expectation with respect to $\mathbb{P}_p$. Now

$$\alpha F_1(x) + (1-\alpha)F_0(x) \le G((1-\alpha)p_c, x), \; x \in \mathbb{R},$$

so that a random variable with distribution $\alpha F_1 + (1-\alpha)F_0$ is stochastically larger than any $s_{(1-\alpha)p_c}(e)$. Consequently,

$$\mu(\alpha F_1 + (1-\alpha)F_0) \ge \mu\left(G\left((1-\alpha)p_c\right)\right).$$

(9) will therefore follow if we prove

$$\mu(G(p)) \ge C_2(d)(p_c - p)^{1/2}\left[\log \frac{1}{p_c - p}\right]^{-1} \tag{10}$$

for some constant $C_2 > 0$ and small $p_c - p > 0$. Here and in several other places we use the well known fact that $0 < p_c(d) < 1$ (see Broadbent and Hammersley (1957), Hammersley (1957), (1959); also Grimmett (1989), Theorem 1.10). We reduce one step further by an appeal to a lower bound for $\mu(G(p))$ of Chayes, Chayes and Durrett (1986). Their Theorem 3.1 with $\varepsilon = (p_c - p)/2$ yields

$$\mu(G(p)) \ge m\Big(\frac{p+p_c}{2}\Big)\left[\log\Big(\frac{2(1-p)}{p_c - p}\Big)\right]^{-1} \tag{11}$$

for $p < p_c$, where

$$\begin{aligned} m(p) \quad = \quad & \lim_{n\to\infty} \frac{-1}{n} \log P_p\{\exists \text{ path } r = (e_1, \ldots, e_k) \text{ on } \mathbb{Z}^d \text{ from } 0 \text{ to } n\xi_1 \text{ with} \\ & \qquad s_p(e_i) = 0 \text{ for } 1 \le i \le k\}. \end{aligned}$$

Now (10) is immediate from (11) and Theorem 1.1 of Hara (1990) which shows that for $d \ge$ some d_0 and suitable constants $0 < C_i(d) < \infty$,

$$C_4(p_c - p)^{1/2} \le m(p) \le C_5(p_c - p)^{1/2}, \; \frac{1}{2}p_c \le p < p_c. \tag{12}$$

Remark 1 The above construction of a counterexample for (5) works as long as

$$m(p) \ge (p_c - p)^{1-\varepsilon} \text{ for some } \varepsilon > 0 \text{ and small } p_c - p > 0. \tag{13}$$

This is generally believed to be true for all $d \ge 3$, but it is known not to be the case for $d = 2$ (cf. Kesten (1987)).

Remark 2 It is known (see Kesten (1986), Theorem 6.9) that μ is continuous, in the sense that if $F_n \Rightarrow F$, then $\mu(F_n) \to \mu(F)$. One can therefore slightly modify the distributions F_0 and F_1 above to obtain a counterexample to (5) with continuous distributions F_0, F_1.

Remark 3 In our construction above we concentrated on the time constant in the first coordinate direction. However, the same example works in any direction. More precisely, define for any vector $\mathbf{0} \neq x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$\mu(F, d, x) = \lim_{n \to \infty} \frac{1}{n} T(\mathbf{0}, \lfloor nx \rfloor), \tag{14}$$

where $\lfloor nx \rfloor =$ stands for $(\lfloor nx_1 \rfloor, \dots, \lfloor nx_d \rfloor)$, and $T(u, v)$ is defined by (3) and $t(e)$ have the common distribution F; the limit in (14) exists with probability 1 and in L^1 by Section 3 of Kesten (1986). If we set $F_\alpha = \alpha F_1 + (1-\alpha)F_0$, then by equations (3.11)–(3.15) there, it holds that for all $x \in \mathbb{R}^d$

$$\mu(F_0, d, x) = 0,$$

$$\mu(F_1, d, x) \leq |x| \mu(F_1, d),$$

and

$$\mu(F_\alpha, d, x) \geq \frac{1}{d} |x| \mu(F_\alpha, d)$$

(see also first line on p.160 of Kesten (1986)); $|x| = \sum_1^d |x_i|$ as in (1)). Therefore, for $d \geq d_0$, there exists an $\alpha_1(d) > 0$ so that

$$\mu(F_\alpha, d, x) > \alpha \mu(F_1, d, x) + (1 - \alpha) \mu(F_0, d, x)$$

for all $0 < \alpha \leq \alpha_1$ and $x \neq \mathbf{0}$.

Remark 4 Here is an example of a pair of distribution functions F_0 and F_1 for which (5) *does* hold. Let $F_0 = \delta_a$ and $F_1 = \delta_b$ for some $a, b > 0$. Since the passage time of any path r of k edges equals ka when the $t(e)$ have distribution F_0, it is clear that $\mu(F_0) = a$. Similarly $\mu(F_1) = b$. Moreover, if the $t(e)$ have common distribution $F_\alpha = \alpha F_1 + (1 - \alpha)F_0$, then

$$\mathbb{E}\{t(e)\} = \alpha b + (1 - \alpha) a.$$

Consequently (see Hammersley and Welsh (1965), Theorem 4.1.9),

$$\mu(F_\alpha) < \alpha b + (1 - \alpha) a = \alpha \mu(F_1) + (1 - \alpha) \mu(F_0), \qquad 0 < \alpha < 1.$$

A more interesting, and less singular class of examples for which (5) holds can be constructed by means of van den Berg and Kesten (1993). Let F_1 be a distribution on $[0, \infty)$ which has a finite mean and satisfies the following condition: If $r := \min(\mathrm{supp}(F_1))$, then

$$F_1(r) < p_c \qquad \text{in case } r = 0,$$

$$F_1(r) < \vec{p}_c \qquad \text{in case } r > 0,$$

where $\vec{p}_c = \vec{p}_c(d)$ is the critical probability of oriented bond percolation on $(\mathbb{Z}_+)^d$. In the terminology of van den Berg and Kesten (1993) this means that F_1 is *useful*. Let $F_0(x) = F_1(\beta x)$ for some $\beta > 0$, $\beta \neq 1$. Then (5) holds for this pair F_0, F_1.
To prove our last claim, let s and $I^{(\alpha)}$ be two independent random variables such that s has distribution F_1 and

$$P\{I^{(\alpha)} = 1\} = \alpha, \; P\Big\{I^{(\alpha)} = \frac{1}{\beta}\Big\} = 1 - \alpha.$$

Then F_0 is the distribution of s/β and $F_\alpha = \alpha F_1 + (1-\alpha)F_0$ is the distribution of $I^{(\alpha)}s$. Finally, we write H_α for the distribution of $(\alpha + (1-\alpha)/\beta)s$, i.e.,

$$H_\alpha(x) = F_1\Big(\frac{\beta x}{\beta\alpha + (1-\alpha)}\Big). \tag{15}$$

Then F_α is *more variable than* H_α in the terminology of van den Berg and Kesten (1993). This follows from Theorem 2.6 there with $t = (\alpha + (1-\alpha)/\beta)s$ and $\widetilde{t} = I^{(\alpha)}s$. Therefore, by Theorem (2.9b) in van den Berg and Kesten (1993)

$$\mu(F_\alpha) < \mu(H_\alpha), \qquad 0 < \alpha < 1. \tag{16}$$

But since H_α and F_0 are the distributions of $(\alpha + (1-\alpha)/\beta)s$ and s/β, respectively,

$$\mu(H_\alpha) = \Big(\alpha + \frac{1-\alpha}{\beta}\Big)\mu(F_1), \quad \mu(F_0) = \frac{1}{\beta}\mu(F_1).$$

Thus (16) shows

$$\mu(F_\alpha) < \alpha\mu(F_1) + (1-\alpha)\mu(F_0), \qquad 0 < \alpha < 1.$$

which gives (5) (of course equality holds in (5) when $\alpha = 0$ or $\alpha = 1$).

References

[1] Aizenman, M. and Barsky, D. J. (1987): Sharpness of the phase transition in percolation models *Comm. Math. Phys.* **108**, 489–526.

[2] van den Berg, J. and Kesten, H. (1993): Inequalities for the time constant in first–passage percolation *Ann. Appl. Probab.* **3**, 56–80.

[3] Broadbent, S. R. and Hammersley, J. M. (1957): Percolation processes I. Crystals and mazes *Proc. Cambr. Phil. Soc.* **53**, 629–641.

[4] Chayes, J. T., Chayes, L. and Durrett, R. (1986): Critical behavior of the two–dimensional first passage time *J. Stat. Phys.* **45**, 933–951.

[5] Grimmett, G. (1989): Percolation, Springer–Verlag, New York, N.Y.

[6] Hammersley, J. M. (1957): Percolation processes: Lower bounds for the critical probability *Ann. Math. Statist.* **28**, 790–795.

[7] Hammersley, J. M. (1959): Bornes supérieures de la probabilité critique dans un processus de filtration, in *Le Calcul des Probabilités et ses Applications*, CNRS, Paris. pp.17–37.

[8] Hammersley, J. M. and Welsh, D. J. A. (1965): First–passage percolation, subadditive processes, stochastic networks, and generalized renewal theory, *Bernoulli–Bayes–Laplace Anniversary Volume*, J. Neyman and L. M. LeCam, eds. pp. 61–110, Springer–Verlag New York, N.Y.

[9] Hara, T. (1990): Mean–field critical behavior for correlation length for percolation in high dimensions *Probab. Th. Rel. Fields* **86**, 337–385.

[10] Kesten, H. (1986): Aspects of first passage percolation, in *Lecture Notes in Math.* **1180** P. L. Hennequin, ed. pp.125–264, Springer–Verlag, New York, N.Y.

[11] Kesten, H. (1987): Scaling relations for $2D$–percolation, *Comm. Math. Phys.* **109**, 109–156.

[12] Kingman, J. F. C. (1968): The ergodic theory of subadditive stochastic processes, *J. Roy. Stat. Soc. Ser. B* **30**, 499–510.

[13] Menshikov, M. V. (1986): Coincidence of critical points in percolation problems, *Soviet Math. Doklady* **33**, 856–859.

[14] Smythe, R. T. and Wierman, J. C. (1978): First–Passage Percolation on the Square Lattice, *Lecture Notes in Math.* **671**, Springer–Verlag, New York, N.Y.