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DICHOTOMY IN A SCALING LIMIT UNDER WIENER MEASURE WITH DENSITY

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Abstract
In general, if the large deviation principle holds for a sequence of probability measures and its rate functional admits a unique minimizer, then the measures asymptotically concentrate in its neighborhood so that the law of large numbers follows. This paper discusses the situation that the rate functional has two distinct minimizers, for a simple model described by the pinned Wiener measures with certain densities involving a scaling. We study their asymptotic behavior and determine to which minimizers they converge based on a more precise investigation than the large deviation's level.

1 Introduction and results

This paper deals with a sequence of probability measures $\{\mu_N\}_{N=1,2,\ldots}$ on the space $\mathcal{C} = C(I,\mathbb{R}), I = [0,1]$ defined from the pinned Wiener measures involving a proper scaling with densities determined by a class of potentials W. The large deviation principle (LDP) is easily established for $\{\mu_N\}$ and the unnormalized rate functional is given by Σ^W; see (1.3) below. The aim of the present paper is to prove the law of large numbers (LLN) for $\{\mu_N\}$ under the situation that Σ^W admits two minimizers $\bar{h}$ and $\hat{h}$. We will specify the conditions for the potentials W, under which the limit points under μ_N are either $\bar{h}$ or $\hat{h}$ as $N \to \infty$.

1.1 Model

Let $\nu_{0,0}$ be the law on the space $\mathcal{C}$ of the Brownian bridge such that $x(0) = x(1) = 0$. The canonical coordinate of $x \in \mathcal{C}$ is described by $x = \{x(t); t \in I\}$. For $a, b \in \mathbb{R}$, $x \in \mathcal{C}$ and $N = 1, 2, \ldots$, we set

$$h^N(t) = \frac{1}{\sqrt{N}} x(t) + \bar{h}(t), \quad t \in I, \tag{1.1}$$

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where $\bar{h} = \bar{h}_{a,b}$ is the straight line connecting a and b, i.e. $\bar{h}(t) = (1-t)a + tb,\ t \in I$; see Figure 1 below. The law on $\mathcal{C}$ of h^N with x distributed under $\nu_{0,0}$ is denoted by $\nu_N = \nu_{N,a,b}$. In other words, ν_N is the law of the Brownian bridge connecting a and b with covariance $E^{\nu_N}[x(t_1)x(t_2)] - E^{\nu_N}[x(t_1)]E^{\nu_N}[x(t_2)] = (t_1 \wedge t_2 - t_1 t_2)/N$, $t_1, t_2 \in I$.
Let $W = W(r)$ be a (measurable) function on $\mathbb{R}$ satisfying the condition:

$$\begin{aligned}&\text{There exists } A > 0 \text{ such that } \lim_{r\to\infty} W(r) = 0, \lim_{r\to-\infty} W(r) = -A \text{ and}\\ &-A \le W(r) \le 0 \text{ for every } r \in \mathbb{R}.\end{aligned} \tag{W.1}$$

We consider the distribution $\mu_N = \mu_{N,a,b}$ on $\mathcal{C}$ defined by

$$\mu_N(dh) = \frac{1}{Z_N} \exp\left\{ -N \int_I W(Nh(t))\, dt \right\} \nu_N(dh), \tag{1.2}$$

where Z_N is the normalizing constant. Under $\mu_{N,a,b}$, negative h has an advantage since the density becomes larger if it takes negative values. This causes a competition, especially when $a, b > 0$, between the effect of the potential W pushing h to the negative side and the boundary conditions a, b keeping h at the positive side.
The model introduced here can be regarded as a continuous analog of the so-called $\nabla\varphi$ interface model in one dimension under a macroscopic scaling; see Section 3.

1.2 LDP and LLN

The LDP holds for μ_N on $\mathcal{C}$ as $N \to \infty$ under the uniform topology. The speed is N and its unnormalized rate functional is given by

$$\Sigma^W(h) = \frac{1}{2} \int_I \dot{h}^2(t)\, dt - A\big|\{t \in I; h(t) \le 0\}\big|, \tag{1.3}$$

for $h \in H^1_{a,b}(I)$, i.e., for absolutely continuous h with derivatives $\dot{h}(t) = dh/dt \in L^2(I)$ satisfying $h(0) = a$ and $h(1) = b$, where $|\{\cdots\}|$ stands for the Lebesgue measure. For more precise formulation, see Theorem 6.4 in [2] for a discrete model. Under our continuous setting, the proof is essentially the same or even easier than that. Indeed, when $W = 0$, the LDP follows from Schilder's theorem, while, when $W \ne 0$, $W(Nh(t))$ in (1.2) behaves as $-A1_{\{h(t)\le 0\}}$ from the condition (W.1) and can be regarded as a weak perturbation. We omit the details.
The LDP immediately implies the concentration property for μ_N:

$$\lim_{N\to\infty} \mu_N\left(\mathrm{dist}_\infty(h^N, \mathcal{H}^W) \le \delta\right) = 1 \tag{1.4}$$

for every $\delta > 0$, where $\mathcal{H}^W = \{h^*;\text{ minimizers of } \Sigma^W\}$ and dist_∞ denotes the distance in $\mathcal{C}$ under the uniform norm $\|\cdot\|_\infty$. In particular, if Σ^W has a unique minimizer h^*, then the LLN holds under μ_N:

$$\lim_{N\to\infty} \mu_N(\|h^N - h^*\|_\infty \le \delta) = 1 \tag{1.5}$$

for every $\delta > 0$.

1.3 Structure of $\mathcal{H}^W$

It is easy to see that $\mathcal{H}^W = \{\bar{h}\}$ when $a, b \le 0$, and $\mathcal{H}^W = \{\check{h}\}$ when $a > 0, b < 0$ (or $a < 0, b > 0$), where $\check{h}$ is a certain line connecting a and b with a single corner at the level 0; see Section 6.3, Case 2 in [2] for details. The interesting situation arises when $a > 0, b \ge 0$ (or $a \ge 0, b > 0$).

We now assume that $a, b > 0$. The straight line $\bar{h}$ is always a possible minimizer of Σ^W. If $a+b < \sqrt{2A}$, there is another possible minimizer $\hat{h}$ of Σ^W. Indeed, let $\hat{h}$ be the curve composed of three straight line segments connecting four points $(0, a), P_1(t_1, 0), P_2(1-t_2, 0)$ and $(1, b)$ in this order; see Figure 2. The angles at two corners P_1 and P_2 are both equal to $\theta \in [0, \pi/2]$, which is determined by the Young's relation (free boundary condition): $\tan\theta = \sqrt{2A}$. More precisely saying, we have $t_1 = a/\sqrt{2A}, t_2 = b/\sqrt{2A}$ with $t_1 + t_2 < 1$ (from $a + b < \sqrt{2A}$), and

$$\hat{h}(t) = \begin{cases} a - \sqrt{2A}\,t, & t \in I_1 = [0, t_1], \\ \quad 0, & t \in I_2 = [t_1, 1-t_2], \\ b - \sqrt{2A}(1-t), & t \in I_3 = [1-t_2, 1]. \end{cases}$$

Then, $\{\bar{h}, \hat{h}\}$ is the set of all critical points of Σ^W (see Section 6.3, Case 1 in [2]), and this implies that $\mathcal{H}^W \subset \{\bar{h}, \hat{h}\}$.

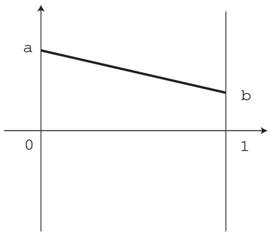

Figure 1: The function $\bar{h}$.

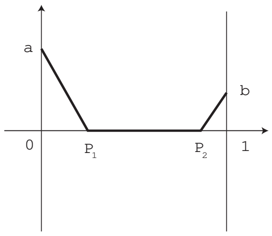

Figure 2: The function $\hat{h}$.

1.4 Results

This paper is concerned with the critical case where both $\bar{h}$ and $\hat{h}$ are minimizers of Σ^W, i.e. $\Sigma^W(\bar{h}) = \Sigma^W(\hat{h})$; note that $\Sigma^W(\bar{h}) = (a-b)^2/2$ and $\Sigma^W(\hat{h}) = \sqrt{2A}(a+b) - A$. In fact, in the following, we always assume the conditions (W.1) and

$$a, b > 0 \quad \text{and} \quad \Sigma^W(\bar{h}) = \Sigma^W(\hat{h}), \tag{W.2}$$

which is actually equivalent to the condition: $a, b > 0$ and $\sqrt{a} + \sqrt{b} = (2A)^{1/4}$; see Appendix B of [1].

Theorem 1.1. (*Concentration on $\bar{h}$*) *In addition to the conditions* (W.1) *and* (W.2), *if*

$$W(r) = 0 \text{ for all } r \ge K \tag{W.3}$$

is fulfilled for some $K \in \mathbb{R}$, then (1.5) *holds with $h^* = \bar{h}$.*

Theorem 1.2. *(Concentration on $\hat{h}$) In addition to* (W.1) *and* (W.2)*, if the following three conditions*

$$^\exists\lambda_1, \alpha_1 > 0 \text{ such that } W(r) \sim -\lambda_1 r^{-\alpha_1} \text{ (i.e. the ratio tends to 1) as } r \to \infty \tag{W.4}$$

$$^\exists\lambda_2, \alpha_2 > 0 \text{ such that } W(r) \le -A + \lambda_2 |r|^{-\alpha_2} \text{ as } r \to -\infty \tag{W.5}$$

$$0 < \alpha_1 < \min\{\alpha_2/(\alpha_2+1), \alpha_2/2\} \text{ and } \int_{I_1 \cup I_3} \hat{h}(t)^{-\alpha_1} dt > \int_I \bar{h}(t)^{-\alpha_1} dt \tag{W.6}$$

are fulfilled, then (1.5) *holds with* $h^* = \hat{h}$.

The rate functional Σ^W of the LDP is determined only from the limit values $W(\pm\infty)$, but for Theorems 1.1 and 1.2 we need more delicate information on the asymptotic properties of W as $r \to \pm\infty$ to control the next order to the LDP. The roles of the above conditions might be explained in a rather intuitive way as follows: The condition (W.3) (with $K = 0$) means that W is large at least for $r \ge 0$ so that the force pushing the Brownian path downward is weak and not enough to push it down to the level of $\hat{h}$. On the other hand in Theorem 1.2, since the values of $Nh(t)$ in (1.2) are very large for t close to 0 or 1, compared with (W.3), the Brownian path is pushed downward because of the condition (W.4) and, once it reaches near the level 0, the condition (W.5) forces it to stay there. This makes the Brownian path reach the level of $\hat{h}$. In the special case where $a = b = \sqrt{A/8}$ ($t_1 = t_2 = 1/4$), the second condition in (W.6) is fulfilled if $1/2 < \alpha_1 < 1$, and such α_1, which simultaneously satisfies the first condition in (W.6), exists if $\alpha_2 > 1$.
Section 2 gives the proofs of Theorems 1.1 and 1.2. Section 3 explains the relation between the (continuous) model discussed in this paper and the so-called $\nabla\varphi$ interface model (discrete model) in one dimension in a rather informal manner. The analysis is, in general, simpler for continuous models than discrete models. The same kind of problem is discussed for weakly pinned Gaussian random walks, which may involve hard walls, by [1] in which the coexistence of $\bar{h}$ and $\hat{h}$ in the limit is established under a certain situation; see also [3]. In our setting, the pinning effect can be generated from potentials having compact supports and taking negative values near $r = 0$. Our condition (W.1) on W excludes the potentials of pinning type and of hard wall type.

2 Proofs

From (1.4) followed by LDP together with our basic assumption $\mathcal{H}^W = \{\bar{h}, \hat{h}\}$, for the proofs of Theorems 1.1 and 1.2, it is sufficient to show that the ratio of probabilities

$$\frac{\mu_N(\|h^N - \hat{h}\|_\infty \le \delta)}{\mu_N(\|h^N - \bar{h}\|_\infty \le \delta)}$$

converges either to 0 or to $+\infty$, respectively, as $N \to \infty$ for small enough $\delta > 0$. This will be established by (2.2) and (2.3) for Theorem 1.1 and by (2.5)–(2.7) for Theorem 1.2, below.

2.1 Proof of Theorem 1.1

In view of the scaling, we may assume $K = 0$ in the condition (W.3) without loss of generality. Introduce the first and the last hitting times $0 \le \tau_1 < \tau_2 \le 1$ of $h^N(t)$ to 0 on the event $\Omega_0 = \{h^N \text{ hits } 0\}$, respectively, by $\tau_1 = \inf\{t \in I; h^N(t) = 0\}$ and $\tau_2 = \sup\{t \in I; h^N(t) = 0\}$.

Then, from the condition (W.3) with $K=0$, the strong Markov property of $h^N(t)$ under ν_N shows that

$$\begin{aligned} &Z_N\mu_N(\|h^N-\hat h\|_\infty\le\delta)\\ &\quad\le\int_{t_1-c\le s_1<s_2\le 1-t_2+c} E^{\nu_{0,0}^{s_1,s_2}}\left[\exp\left\{-N\int_{s_1}^{s_2}W(\sqrt{N}x(s))\,ds\right\}\right]\nu_N(\tau_1\in ds_1,\tau_2\in ds_2)\\ &\qquad+\nu_N(\Omega_0^c,\|h^N-\hat h\|_\infty\le\delta),\end{aligned}$$

where $\nu_{0,0}^{s_1,s_2}$ (more generally $\nu_{\alpha,\beta}^{s_1,s_2}$) is the law on the space $C([s_1,s_2],\mathbb{R})$ of the Brownian bridge such that $x(s_1)=x(s_2)=0$ (or $x(s_1)=\alpha, x(s_2)=\beta$) and $c=\delta/\sqrt{2A}$ arises from the condition $\|h^N-\hat h\|_\infty\le\delta$. However, in the first term, the conditions (W.1) and (W.3) with $K=0$ imply that

$$-N\int_{s_1}^{s_2}W(\sqrt{N}x(s))\,ds\le ANX^{s_1,s_2},$$

where $X^{s_1,s_2}=|\{s\in[s_1,s_2];x(s)<0\}|$ is the occupation time of x on the negative side. Since $X^{s_1,s_2}=(s_2-s_1)X^{0,1}$ in law and $\nu_{0,0}(X^{0,1}\in ds)=ds$ (see (6) in [6] for more general formulas), we obtain that

$$E^{\nu_{0,0}^{s_1,s_2}}\left[\exp\left\{-N\int_{s_1}^{s_2}W(\sqrt{N}x(s))\,ds\right\}\right]\le\int_I e^{AN(s_2-s_1)s}\,ds\le\frac{e^{AN(s_2-s_1)}}{AN(s_2-s_1)}.$$

Lemma 2.1. *The joint distribution of (τ_1,τ_2) under ν_N is given by*

$$\begin{aligned}&\nu_N(\tau_1\in ds_1,\tau_2\in ds_2)\\ &\quad=\frac{abN}{2\pi\sqrt{(s_2-s_1)s_1^3(1-s_2)^3}}\exp\left\{\frac{N}{2}(a-b)^2-\frac{a^2N}{2s_1}-\frac{b^2N}{2(1-s_2)}\right\}ds_1ds_2,\end{aligned}$$

for $0<s_1<s_2<1$.

Proof. Let Q_N be the law on $\mathcal{C}$ of $y(t)=\sqrt{N}h^N(t)$ under ν_N, let P_a be the Wiener measure starting at a and $\mathcal{F}_{[T_1,T_2]}=\sigma\{y(t);t\in[T_1,T_2]\}$ for $0\le T_1<T_2\le 1$. Then, for every $0<\bar s_1<\bar s_2<1$, we have on $\mathcal{F}_{[0,\bar s_1]}\otimes\mathcal{F}_{[\bar s_2,1]}$

$$Q_N(dy)=\frac{p(\bar s_2-\bar s_1,y_{\bar s_1},y_{\bar s_2})}{p(1,a\sqrt{N},b\sqrt{N})}P_{a\sqrt{N}}|_{\mathcal{F}_{[0,\bar s_1]}}\otimes\hat P_{b\sqrt{N}}|_{\mathcal{F}_{[\bar s_2,1]}}(dy),$$

where $p(s,a,b)$ is the heat kernel and $\hat P_{b\sqrt{N}}$ is the inversion of $P_{b\sqrt{N}}$ under the map $\hat y(t)=y(1-t)$. This implies that

$$\nu_N(\tau_1\in ds_1,\tau_2\in ds_2)=\frac{1}{\sqrt{s_2-s_1}}e^{\frac{N}{2}(a-b)^2}P_{a\sqrt{N}}(\tau\in ds_1)P_{b\sqrt{N}}(1-\tau\in ds_2),$$

where τ is the hitting time to 0. Therefore the conclusion of the lemma follows from

$$P_a(\tau\in ds)=\frac{a}{\sqrt{2\pi s^3}}e^{-\frac{a^2}{2s}}\,ds,\quad a>0,$$

see, e.g., (6.3) in [5], p.80. □

This lemma, combined with the above computations, shows that

$$\begin{aligned}&Z_N\mu_N(\|h^N-\hat{h}\|_\infty\le\delta)\\&\quad\le\frac{ab}{2\pi A}\int_{t_1-c\le s_1<s_2\le 1-t_2+c}\frac{e^{-Nf(s_1,s_2)}}{(s_2-s_1)^{3/2}s_1^{3/2}(1-s_2)^{3/2}}\,ds_1ds_2\\&\qquad+\nu_N(\|h^N-\hat{h}\|_\infty\le\delta),\end{aligned}\tag{2.1}$$

where

$$f(s_1,s_2)=\frac{a^2}{2s_1}+\frac{b^2}{2(1-s_2)}-\frac{1}{2}(a-b)^2-A(s_2-s_1).$$

Since $f(s_1,s_2)=\Sigma^W(\hat{h}_{s_1,s_2})-\Sigma^W(\hat{h})$ for the curve $\hat{h}_{s_1,s_2}$ defined similarly to $\hat{h}$ with $t_1,1-t_2$ replaced by s_1,s_2, respectively, we see that $f(s_1,s_2)\ge 0$ and f attains its minimal value 0 at $(s_1,s_2)=(t_1,1-t_2)$. Furthermore, it behaves near $(t_1,1-t_2)$ as

$$f(s_1,s_2)=\frac{1}{2}a^2t_1^{-3}(s_1-t_1)^2+\frac{1}{2}b^2t_2^{-3}(1-s_2-t_2)^2+o\left((s_1-t_1)^2+(1-s_2-t_2)^2\right).$$

This proves that the first term in the right hand side of (2.1) behaves as $(A|I_2|^{3/2}N)^{-1}$ as $N\to\infty$. Therefore, for every $0<\delta<\|\bar{h}-\hat{h}\|_\infty$, by noting that $\nu_N(\|h^N-\hat{h}\|_\infty\le\delta)\le e^{-CN}$ for some $C>0$ (since the LDP holds for ν_N with speed N and the unnormalized rate functional $\Sigma^0(h)$), we have that

$$\lim_{N\to\infty}Z_N\mu_N(\|h^N-\hat{h}\|_\infty\le\delta)=0.\tag{2.2}$$

On the other hand, the condition (W.3) implies for every $0<\delta<(a\wedge b)$ that

$$\lim_{N\to\infty}Z_N\mu_N(\|h^N-\bar{h}\|_\infty\le\delta)=\lim_{N\to\infty}\nu_{0,0}(\|x\|_\infty\le\sqrt{N}\delta)=1.\tag{2.3}$$

Thus, the conclusion of Theorem 1.1 follows from (2.2) and (2.3) noting that (1.4) holds with $\mathcal{H}^W=\{\bar{h},\hat{h}\}$.

2.2 Proof of Theorem 1.2

From the definition (1.2) of μ_N and by recalling (1.1), we have

$$\begin{aligned}&Z_N\mu_N(\|h^N-\hat{h}\|_\infty\le\delta)\\&\quad=E^{\nu_{0,0}}\left[\exp\left\{-N\int_I W(\sqrt{N}x(t)+N\bar{h}(t))\,dt\right\},\|x+\sqrt{N}(\bar{h}-\hat{h})\|_\infty\le\sqrt{N}\delta\right]\\&\quad=E^{\nu_{0,0}}\left[\exp\left\{\hat{F}_N(x)\right\},\|x\|_\infty\le\sqrt{N}\delta\right],\end{aligned}$$

where

$$\begin{aligned}\hat{F}_N(x)=&-N\int_I W(\sqrt{N}x(t)+N\hat{h}(t))\,dt\\&+\sqrt{N}\int_I(\dot{\bar{h}}-\dot{\hat{h}})(t)\,dx(t)-\frac{N}{2}\int_I(\dot{\bar{h}}-\dot{\hat{h}})^2(t)\,dt.\end{aligned}$$

The third line follows by means of the Cameron-Martin formula for $\nu_{0,0}$ transforming $x + \sqrt{N}(\bar h - \hat h)$ into x. However, since $\dot{\bar h}(t) \equiv b-a$ and $\int_I \dot{\hat h}(t)\,dt = \hat h(1) - \hat h(0) = b-a$, we have

$$\frac{1}{2}\int_I (\dot{\bar h} - \dot{\hat h})^2(t)\,dt = -\Sigma^W(\bar h) + \Sigma^W(\hat h) + A(1-t_1-t_2) = A|I_2|,$$

by the condition (W.2). Moreover, since $\dot{\hat h} = -\sqrt{2A}$ on I_1°, 0 on I_2° and $\sqrt{2A}$ on I_3°,

$$\begin{aligned}
&\int_I (\dot{\bar h} - \dot{\hat h})(t)\,dx(t) \\
&\quad = (b-a)(x(1)-x(0)) + \sqrt{2A}(x(t_1)-x(0)) - \sqrt{2A}(x(1)-x(1-t_2)) \\
&\quad = \sqrt{2A}(x(t_1)+x(1-t_2)),
\end{aligned}$$

recall that $x(0)=x(1)=0$ under $\nu_{0,0}$. Therefore, we can rewrite $\hat F_N(x)$ as

$$\begin{aligned}
\hat F_N(x) = -N &\int_{I_1\cup I_3} W\big(\sqrt{N}x(t) + N\hat h(t)\big)\,dt \\
&+ \sqrt{2AN}\big(x(t_1)+x(1-t_2)\big) - N\int_{I_2}\big\{W\big(\sqrt{N}x(t)\big) + A\big\}\,dt \\
=:&\, F_N^{(1)}(x) + F_N^{(2)}(x) + F_N^{(3)}(x).
\end{aligned}$$

To give a lower bound on $F_N^{(1)}$, we consider subintervals $\tilde I_1 = [0, t_1 - \sqrt{2/A}\,\delta]$ and $\tilde I_3 = [1-t_2+\sqrt{2/A}\,\delta, 1]$ of I_1 and I_3, respectively. Then, on the event $\mathcal{A}_1 = \{\|x\|_\infty \le \sqrt{N}\delta\}$, we have for $t \in \tilde I_1 \cup \tilde I_3$,

$$\sqrt{N}x(t) + N\hat h(t) \ge -N\delta + N\hat h(t) \ge N\delta \longrightarrow \infty \quad (\text{as } N\to\infty),$$

and also $\sqrt{N}x(t) + N\hat h(t) \le N(\hat h(t)+\delta)$. Accordingly, by the condition (W.4), for every sufficiently small $\epsilon > 0$, the integrand of $F_N^{(1)}$ times $-N$ is bounded from below as

$$-NW\big(\sqrt{N}x(t) + N\hat h(t)\big) \ge (\lambda_1-\epsilon)N^{1-\alpha_1}(\hat h(t)+\delta)^{-\alpha_1},$$

which implies, by recalling $-W \ge 0$, that

$$F_N^{(1)} \ge (\lambda_1-\epsilon)N^{1-\alpha_1}\int_{\tilde I_1\cup\tilde I_3}(\hat h(t)+\delta)^{-\alpha_1}\,dt =: (\lambda_1-\epsilon)C_1(\delta)N^{1-\alpha_1},$$

on $\mathcal{A}_1$ for sufficiently large N.

To give lower bounds on $F_N^{(2)}$ and $F_N^{(3)}$, we introduce two more events

$$\begin{aligned}
&\mathcal{A}_2 = \{x(t_1)\ge 0,\ x(1-t_2)\ge 0\}, \\
&\mathcal{A}_3 = \{x(t) \le -N^{-\kappa} \text{ for all } t \in \tilde I_2 := [t_1 + N^{-\frac12-\kappa}, 1-t_2-N^{-\frac12-\kappa}]\},
\end{aligned}$$

where $0<\kappa<1/2$ will be chosen later. Then, obviously $F_N^{(2)} \ge 0$ on $\mathcal{A}_2$. If $x \in \mathcal{A}_3$, noting that $-W(r) - A \ge -A$ for all $r\in\mathbb{R}$, we have from (W.5)

$$\begin{aligned}
F_N^{(3)} &\ge -2AN^{\frac12-\kappa} + N\int_{\tilde I_2}\big\{-W\big(\sqrt{N}x(t)\big) - A\big\}\,dt \\
&\ge -2AN^{\frac12-\kappa} - \lambda_2 N^{1-\alpha_2(\frac12-\kappa)}|\tilde I_2|,
\end{aligned}$$

for sufficiently large N. These estimates on $F_N^{(1)}, F_N^{(2)}$ and $F_N^{(3)}$ are summarized into

$$\hat{F}_N \geq (\lambda_1 - \epsilon) C_1(\delta) N^{1-\alpha_1} - 2AN^{\frac{1}{2}-\kappa} - \lambda_2 N^{1-\alpha_2(\frac{1}{2}-\kappa)} |\tilde{I}_2| \tag{2.4}$$

on $\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$ for sufficiently large N.
The next lemma gives a lower bound on the probability $\nu_{0,0}(\mathcal{A}_2 \cap \mathcal{A}_3)$.

Lemma 2.2. *There exists $C > 0$ such that*

$$\nu_{0,0}(\mathcal{A}_2 \cap \mathcal{A}_3) \geq CN^{-\frac{1}{2}-2\kappa} \exp\{-36N^{\frac{1}{2}-\kappa}\}.$$

Proof. Consider an auxiliary event

$$\mathcal{A}_4 = \{x(t_1 + N^{-\frac{1}{2}-\kappa}),\, x(1 - t_2 - N^{-\frac{1}{2}-\kappa}) \in [-3N^{-\kappa}, -2N^{-\kappa}]\}.$$

Then, by the Markov property, we have

$$\begin{aligned}
&\nu_{0,0}(\mathcal{A}_2 \cap \mathcal{A}_3) \geq \nu_{0,0}(\mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4) \\
&\quad = E^{\nu_{0,0}}\Big[\nu_{0,\alpha}\big(x(t_1) \geq 0\big) \cdot \nu_{\alpha,\beta}\big(x(t) \leq -N^{-\kappa}, {}^\forall t \in \tilde{I}_2\big) \cdot \nu_{\beta,0}\big(x(1-t_2) \geq 0\big), \mathcal{A}_4\Big],
\end{aligned}$$

where $\alpha = x(t_1 + N^{-\frac{1}{2}-\kappa})$, $\beta = x(1 - t_2 - N^{-\frac{1}{2}-\kappa})$ and $\nu_{0,\alpha} = \nu_{0,\alpha}^{0,t_1+N^{-\frac{1}{2}-\kappa}}$, $\nu_{\alpha,\beta} = \nu_{\alpha,\beta}^{t_1+N^{-\frac{1}{2}-\kappa}, 1-t_2-N^{-\frac{1}{2}-\kappa}}$, $\nu_{\beta,0} = \nu_{\beta,0}^{1-t_2-N^{-\frac{1}{2}-\kappa},1}$. However,

$$\begin{aligned}
\nu_{0,\alpha}(x(t_1) \geq 0) &\geq P_0\big(B(N^{-\frac{1}{2}-\kappa}) + \alpha \geq -\alpha\big) - P_0(X \geq -\alpha) \\
&\geq P_0\big(B(1) \geq 6N^{\frac{1}{4}-\frac{\kappa}{2}}\big) - P_0\big(B(1) \geq 2N^{\frac{1}{2}}(t_1 + N^{-\frac{1}{2}-\kappa})^{\frac{1}{2}}\big) \\
&\geq C_1 N^{\frac{\kappa}{2}-\frac{1}{4}} \exp\{-18N^{\frac{1}{2}-\kappa}\} - C_2 N^{-\frac{1}{2}} \exp\{-2t_1 N\},
\end{aligned}$$

for sufficiently large N with $C_1, C_2 > 0$. Indeed, the first line is a consequence of

$$x(t_1) = \alpha + B(N^{-\frac{1}{2}-\kappa}) - X, \quad X := \frac{N^{-\frac{1}{2}-\kappa}}{t_1 + N^{-\frac{1}{2}-\kappa}}\big(B(t_1 + N^{-\frac{1}{2}-\kappa}) + \alpha\big)$$

in law where $B(t)$ is the standard Brownian motion, the second line is seen by the scaling law of B and $6N^{-\kappa} \geq -2\alpha$, $-\alpha \geq 2N^{-\kappa}$ on $\mathcal{A}_4$ and, finally, the third line is shown from

$$\frac{y}{\sqrt{2\pi}(1+y^2)} e^{-y^2/2} \leq P(Y \geq y) \leq \frac{1}{\sqrt{2\pi}y} e^{-y^2/2}, \quad y > 0,$$

for $Y = N(0,1)$; see e.g. [5], p. 112. The probability $\nu_{\beta,0}(x(1-t_2) \geq 0)$ has a similar bound. Finally, on $\mathcal{A}_4$, we have

$$\begin{aligned}
\nu_{\alpha,\beta}\big(x(t) \leq -N^{-\kappa}, {}^\forall t \in \tilde{I}_2\big) &\geq \nu_{0,0}^{0,\bar{t}}\big(x(t) \leq N^{-\kappa}, {}^\forall t \leq \bar{t}\big) \\
&= \nu_{0,0}^{0,1}\big(x(t) \leq \bar{t}^{-1/2} N^{-\kappa}, {}^\forall t \in I\big) \\
&\geq P_0\big(\max_{t \in I} |B(t)| \leq \bar{t}^{-1/2} N^{-\kappa}/2\big) \\
&\geq C_3 N^{-\kappa},
\end{aligned}$$

where $\bar{t} = |\tilde{I}_2| \big(= 1 - t_1 - t_2 - 2N^{-\frac{1}{2}-\kappa} \big)$ and $C_3 > 0$. The first inequality is because the straight line connecting α and β stays below $-2N^{-\kappa}$ on $\mathcal{A}_4$. The second line follows from the scaling law of the Brownian bridge, while the third line is shown by noting that $x(t) = B(t) - tB(1)$ in law. The last inequality is simple because the distribution of $\max_{t\in I} |B(t)|$ admits a positive and continuous density. Therefore, we obtain

$$\nu_{0,0}(\mathcal{A}_2 \cap \mathcal{A}_3) \geq C_4 N^{\kappa-\frac{1}{2}} \cdot N^{-\kappa} \cdot \exp\{-36N^{\frac{1}{2}-\kappa}\} \cdot \nu_{0,0}(\mathcal{A}_4),$$

for sufficiently large N with $C_4 > 0$. However, since we have on $\mathcal{F}_{[0,s_1]} \otimes \mathcal{F}_{[s_2,1]}$

$$\nu_{0,0}(dy) = \frac{p(s_2 - s_1, y_{s_1}, y_{s_2})}{p(1,0,0)} P_0|_{\mathcal{F}_{[0,s_1]}} \otimes \hat{P}_0|_{\mathcal{F}_{[s_2,1]}}(dy), \quad 0 < s_1 < s_2 < 1,$$

choosing s_1, s_2 such that $t_1 + N^{-\frac{1}{2}-\kappa} < s_1 < s_2 < 1 - t_2 - N^{-\frac{1}{2}-\kappa}$ and restricting on the event $\{y; |y_{s_1}|, |y_{s_2}| \leq 1\}$, we obtain

$$\begin{aligned}\nu_{0,0}(\mathcal{A}_4) \geq & C_5 P_0\big(-3N^{-\kappa} \leq B(t_1 + N^{-\frac{1}{2}-\kappa}) \leq -2N^{-\kappa}\big) \\ & \times P_0\big(-3N^{-\kappa} \leq B(t_2 + N^{-\frac{1}{2}-\kappa}) \leq -2N^{-\kappa}\big) \\ \geq & C_6 N^{-2\kappa},\end{aligned}$$

with some $C_5, C_6 > 0$. This completes the proof of the lemma. □

Since Lemma 2.2 shows

$$\begin{aligned}\nu_{0,0}(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) &\geq \nu_{0,0}(\mathcal{A}_2 \cap \mathcal{A}_3) - \nu_{0,0}(\mathcal{A}_1^c) \\ &\geq \nu_{0,0}(\mathcal{A}_2 \cap \mathcal{A}_3) - e^{-\delta^2 N/4} \\ &\geq \exp\{-40N^{\frac{1}{2}-\kappa}\},\end{aligned}$$

for sufficiently large N (recall $\frac{1}{2} - \kappa < 1$), we have from (2.4)

$$\begin{aligned}& Z_N \mu_N(\|h^N - \hat{h}\|_\infty \leq \delta) \\ & \quad \geq \exp\big\{(\lambda_1 - \epsilon)C_1(\delta)N^{1-\alpha_1} - 2AN^{\frac{1}{2}-\kappa} - \lambda_2 N^{1-\alpha_2(\frac{1}{2}-\kappa)}|\tilde{I}_2| - 40N^{\frac{1}{2}-\kappa}\big\} \\ & \quad \geq \exp\big\{(\lambda_1 - 2\epsilon)C_1(\delta)N^{1-\alpha_1}\big\},\end{aligned} \tag{2.5}$$

for sufficiently large N if $1 - \alpha_1 > 0$ (i.e. $\alpha_1 < 1$), $\frac{1}{2} - \kappa < 1 - \alpha_1$ (i.e. $\kappa > \alpha_1 - \frac{1}{2}$) and $1 - \alpha_2(\frac{1}{2} - \kappa) < 1 - \alpha_1$ (i.e. $\kappa < \frac{1}{2} - \frac{\alpha_1}{\alpha_2}$). One can choose such $\kappa : \alpha_1 - \frac{1}{2} < \kappa < \frac{1}{2} - \frac{\alpha_1}{\alpha_2}$ under the first condition in (W.6), which implies that $\alpha_1(1 + \frac{1}{\alpha_2}) < 1$ and $\frac{1}{2} - \frac{\alpha_1}{\alpha_2} > 0$.
On the other hand, we have

$$Z_N \mu_N(\|h^N - \bar{h}\|_\infty \leq \delta) = E^{\nu_{0,0}} \Big[\exp\big\{\bar{F}_N(x)\big\}, \|x\|_\infty \leq \sqrt{N}\delta\Big], \tag{2.6}$$

where

$$\bar{F}_N(x) = -N \int_I W(\sqrt{N}x(t) + N\bar{h}(t))\, dt.$$

However, since $\sqrt{N}x(t) + N\bar{h}(t) \geq N(\bar{h}(t) - \delta)$ on the event $\mathcal{A}_1$, the condition (W.4) shows

$$\bar{F}_N \leq (\lambda_1 + \epsilon)N^{1-\alpha_1} \int_I (\bar{h}(t) - \delta)^{-\alpha_1}\, dt =: (\lambda_1 + \epsilon)C_2(\delta)N^{1-\alpha_1}. \tag{2.7}$$

Comparing (2.5) and (2.6) with (2.7), since $(\lambda_1 - 2\epsilon)C_1(\delta) > (\lambda_1 + \epsilon)C_2(\delta)$ for sufficiently small δ and $\epsilon > 0$ by the second condition in (W.6), the proof of Theorem 1.2 is concluded.

Remark 2.1. *In the proof of Theorem* 1.2, *the conditions* (W.1) *and* (W.4) *are used to show that* $F_N^{(1)} \geq (\lambda_1 - \epsilon)C_1(\delta)N^{1-\alpha_1}$ *and* $\bar{F}_N \leq (\lambda_1 + \epsilon)C_2(\delta)N^{1-\alpha_1}$, *while the conditions* (W.5) *and* (W.6) *are necessary to prove that the other terms, like* $F_N^{(3)}, \nu_{0,0}(\mathcal{A}_2 \cap \mathcal{A}_3)$ *are negligible.*

3 Discussions

Finally, this section makes a remark on the relation between the probability measure μ_N defined by (1.2) and the so-called $\nabla\varphi$ interface model in one dimension.
When a symmetric convex potential $V : \mathbb{R} \to \mathbb{R}$ is given, to each (microscopic) interface height variable $\phi = \{\phi_i\}_{i=1}^{N-1} \in \mathbb{R}^{N-1}$ satisfying the boundary condition $\phi_0 = aN$ and $\phi_N = bN$, an interfacial energy $H_N(\phi) = H_N^W(\phi)$ called a Hamiltonian is assigned by

$$H_N(\phi) = \sum_{i=1}^{N} V(\phi_i - \phi_{i-1}) + \sum_{i=1}^{N-1} W(\phi_i).$$

Then the statistical ensemble for ϕ is defined by the (finite volume) Gibbs measure

$$\tilde{\mu}_N(d\phi) = \frac{1}{\tilde{Z}_N} \exp\big\{ - H_N(\phi)\big\} \prod_{i=1}^{N-1} d\phi_i, \tag{3.1}$$

where $\tilde{Z}_N$ is the normalizing constant. We associate a macroscopic height variable $\{h^N(t); t \in I\}$ with the microscopic one ϕ by the linear interpolation of $h^N(i/N) = N^{-1}\phi_i, i = 0, 1, \ldots, N$. Note that, under this scaling, if we especially take $V(\eta) = \eta^2/2$, $H_N(\phi)$ is transformed into

$$\tilde{H}_N(h) = \frac{1}{2}\sum_{i=1}^{N} N^2\big(h(i/N) - h((i-1)/N)\big)^2 + \sum_{i=1}^{N-1} W(Nh(i/N)),$$

where we write h^N as h. One can thus expect that $\tilde{H}_N(h)$ behaves as

$$N\left[\frac{1}{2}\int_I (\dot{h})^2(t)\,dt + \int_I W(Nh(t))\,dt\right]$$

under the limit $N \to \infty$. In other words, μ_N defined by (1.2) may be regarded as the continuous analog of $\tilde{\mu}_N$ introduced in (3.1) under the scaling mentioned above. In fact, this is true in the sense that the errors in the probabilities in the discrete and continuous settings are superexponentially small and behave like $e^{-CN^2}, C > 0$ as $N \to \infty$ (see [7] or the proof of Lemma 6.6 in [2]).

Remark 3.1. *The LDP was studied by* [4] *for the* $\nabla\varphi$ *interface model on a d-dimensional large lattice domain with general convex potential* V *and the weak self potential* W *satisfying the condition* (W.1). *The variational problem minimizing the corresponding rate functional* Σ^W *naturally leads to the free boundary problem.*

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