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EULER'S FORMULAE FOR $\zeta(2n)$ AND PRODUCTS OF CAUCHY VARIABLES

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Abstract
We show how to recover Euler's formula for $\zeta(2n)$, as well as $L_{\chi_4}(2n+1)$, for any integer n, from the knowledge of the density of the product $\mathbb{C}_1, \mathbb{C}_2 \dots, \mathbb{C}_k$, for any $k \geq 1$, where the $\mathbb{C}_i$'s are independent standard Cauchy variables.

1 Introduction

Consider both the zeta function

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s} \quad (\Re s > 1)$$

and the L function associated with the quadratic character χ_4 :

$$L_{\chi_4}(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^s} \quad (\Re s > 0)\,.$$

The following formulae are very classical (see for example [9]) :

$$L_{\chi_4}(2n+1) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+1} \frac{A_n^{(1)}}{\Gamma(2n+1)}, \tag{1}$$

$$\left(1 - \frac{1}{2^{2n+2}}\right)\zeta(2n+2) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+2} \frac{A_n^{(2)}}{\Gamma(2n+2)}. \tag{2}$$

Here, the coefficients $(A_n^{(t)})$, $t = 1, 2$, are featured in the series developments

$$\frac{1}{(\cos(\theta))^t} = \sum_{n=0}^{\infty} \frac{A_n^{(t)}}{(2n)!}\, \theta^{2n} \qquad \left(|\theta| < \frac{\pi}{2}\right).$$

These coefficients $\left(A_n^{(1)}, n \geq 0\right)$ and $\left(A_n^{(2)}, n \geq 0\right)$ are well known to be $A_n^{(1)} = A_{2n}$ and $A_n^{(2)} = A_{2n+1}$, respectively the Euler or secant numbers, and the tangent numbers (more information about A_{2n} and A_{2n+1} can be found in [7]).
The most popular ways to prove (1) and (2) make use of Fourier inversion and Parseval's theorem, or of non trivial expansions of functions such as cotan (see for example [9]). In this paper, we show that formulae (1) and (2) may be obtained simply via either of the following methods :

(M1) In section 2, we compute in two different ways the moments $\mathbb{E}\left((\Lambda_1)^{2n}\right)$ and $\mathbb{E}\left((\Lambda_2)^{2n}\right)$, where $\Lambda_1 = \log(|\mathbb{C}_1|)$ and $\Lambda_2 = \log(|\mathbb{C}_1\mathbb{C}_2|)$, with $\mathbb{C}_1$ and $\mathbb{C}_2$ two independent standard Cauchy variables.

- On one hand, these moments can be computed explicitly in terms of L_{χ_4} and ζ respectively, thanks to explicit formulae for the densities of Λ_1 and Λ_2.
- On the other hand, these moments may be obtained via the representation

$$|\mathbb{C}_1| \overset{\text{law}}{=} e^{\frac{\pi}{2}\hat{C}_1}, \tag{3}$$

where $\hat{C}_1$ is a random variable whose distribution is characterized by

$$\mathbb{E}\left(e^{i\lambda\hat{C}_1}\right) = \frac{1}{\cosh\lambda} \qquad (\lambda \in \mathbb{R})$$

or

$$\mathbb{E}\left(e^{\theta\hat{C}_1}\right) = \frac{1}{\cos\theta} \qquad \left(|\theta| < \frac{\pi}{2}\right). \tag{4}$$

More properties about $\hat{C}_1$ or even the Lévy process $(\hat{C}_t, t \geq 0)$ can be found in [7]. This process $(\hat{C}_t, t \geq 0)$ entertains deep relations with, but is different from, the Cauchy process (see, e.g., [8], for such relations).

(M2) In section 3, we derive the formulae for $\zeta(2n)$ and $L_{\chi_4}(2n+1)$ from the identification of the density of the law of the product $\Pi_k = \mathbb{C}_1\mathbb{C}_2\ldots\mathbb{C}_k$ of k independent standard Cauchy variables, by exploiting the fact that the integral of this density is equal to 1.

Section 4 is devoted to an interpretation of (3) and (4) in terms of planar Brownian motion. In a final appendix, we indicate briefly how the preceding discussion may be generalized when the (square of a) Cauchy variable is replaced by a ratio of two independent unilateral stable(μ) variables ($0 < \mu < 1$).

2 From the even moments of Λ_1 and Λ_2 to the derivation of Euler's formulae

As is well known, the density of $\mathbb{C}_1$ is

$$\Psi_1(x) = \frac{1}{\pi(1+x^2)}.$$

It is not difficult to show that Ψ_2, the density of $\mathbb{C}_1\mathbb{C}_2$, is

$$\Psi_2(x) = \frac{2\log|x|}{\pi^2(x^2-1)}.$$

From the knowledge of Ψ_1 and Ψ_2 we deduce the following result.

Proposition 1. *The even moments of* Λ_1 *and* Λ_2 *are given by*

$$\mathbb{E}\left[(\Lambda_1)^{2n}\right] = \frac{4}{\pi}\Gamma(2n+1)L_{\chi_4}(2n+1), \tag{5}$$

$$\mathbb{E}\left[(\Lambda_2)^{2n}\right] = \frac{8}{\pi^2}\Gamma(2n+2)\left(1-\frac{1}{2^{2n+2}}\right)\zeta(2n+2). \tag{6}$$

Proof. The LHS of (5) equals

$$\frac{2}{\pi}\int_0^\infty \frac{(\log x)^{2n}\mathrm{d}x}{1+x^2} = \frac{4}{\pi}\int_1^\infty \frac{(\log x)^{2n}\mathrm{d}x}{1+x^2}.$$

Then, making the change of variables $x = e^u$, followed by the series expansion $\frac{1}{1+e^{-2u}} = \sum_{k=0}^\infty (-1)^k e^{-2ku}$, we obtain formula (5).
The proof of formula (6) relies on the same argument, starting from the expression of Ψ_2. $\square$

Let us now assume formula (3), and define a variable $\hat{C}_2$ such that

$$e^{\frac{\pi}{2}\hat{C}_2} \overset{\text{law}}{=} |\mathbb{C}_1\mathbb{C}_2|.$$

We note that $\hat{C}_1 \overset{\text{law}}{=} \frac{2}{\pi}\log|\mathbb{C}_1| \overset{\text{law}}{=} \frac{2}{\pi}\Lambda_1$ and likewise $\hat{C}_2 \overset{\text{law}}{=} \frac{2}{\pi}\Lambda_2$. Then, from formula (4) and the definition of the coefficients $A_n^{(t)}$, we see that the even moments of $\hat{C}_1$ and $\hat{C}_2$ are given by

$$\mathbb{E}\left[(\hat{C}_t)^{2n}\right] = A_n^{(t)} \quad (t=1,2)$$

so that, from the relations between $\hat{C}_t$ and Λ_t, we get

$$\mathbb{E}\left[(\Lambda_t)^{2n}\right] = \left(\frac{\pi}{2}\right)^{2n} A_n^{(t)} \quad (t=1,2). \tag{7}$$

Putting together formulae (7)-(8) on one hand, and formula (9) on the other hand, we obtain the desired results (1) and (2).

To finish completely our proof, it now remains to show formula (3), that is, starting with $\mathbb{C}_1$, to show that

$$\mathbb{E}\left[e^{i\lambda\frac{2}{\pi}\log|\mathbb{C}_1|}\right] = \frac{1}{\cosh\lambda} \quad (\lambda\in\mathbb{R}). \tag{8}$$

The LHS of (8) is $\mathbb{E}\left[|\mathbb{C}_1|^{\frac{2i\lambda}{\pi}}\right]$. To compute this quantity we use the fact that $\mathbb{C}_1 \overset{\text{law}}{=} N/N'$, where N and N' are two standard independent Gaussian variables. We shall also use the fact that $N^2 \overset{\text{law}}{=} 2\gamma_{1/2}$ where γ_a is a gamma(a) variable. Thus, we have

$$\mathbb{E}\left[|\mathbb{C}_1|^{\frac{2i\lambda}{\pi}}\right] = \left|\mathbb{E}\left[\left(\gamma_{1/2}\right)^{\frac{i\lambda}{\pi}}\right]\right|^2 = \frac{\left|\Gamma\left(\frac{1}{2}+i\frac{\lambda}{\pi}\right)\right|^2}{\left(\Gamma(\frac{1}{2})\right)^2} = \frac{1}{\cosh(\lambda)} \quad (\lambda\in\mathbb{R}).$$

For a proof of this last identity see [5], Problem 1 p. 14.

3 Another proof for Euler's formulae

In this section, we first give the density of the law of $\Pi_k = \mathbb{C}_1\mathbb{C}_2\ldots\mathbb{C}_k$ for any $k \geq 0$. We need to distinguish the odd and even cases.

Proposition 2.

- *The density of* $\Pi_{2n+1} := \mathbb{C}_1\mathbb{C}_2\ldots\mathbb{C}_{2n+1}$ *is equal to*

$$\Psi_{2n+1}(x) = \frac{2^{2n}}{\pi(2n)!}\left(\prod_{j=1}^{n}\left(\left(j-\frac{1}{2}\right)^2+\frac{(\log|x|)^2}{\pi^2}\right)\right)\frac{1}{1+x^2}. \tag{9}$$

- *The density of* $\Pi_{2n} := \mathbb{C}_1\mathbb{C}_2\ldots\mathbb{C}_{2n}$ *is equal to*

$$\Psi_{2n}(x) = \frac{2^{2n-1}}{\pi^2(2n-1)!}\left(\prod_{j=1}^{n-1}\left(j^2+\frac{(\log|x|)^2}{\pi^2}\right)\right)\frac{\log|x|}{x^2-1}. \tag{10}$$

Proof. From the formula

$$\mathbb{E}\left[e^{i\lambda\frac{2}{\pi}\log|\mathbb{C}_1|}\right] = \frac{1}{\cosh\lambda} \quad (\lambda\in\mathbb{R}),$$

we easily deduce the Mellin transform of Π_k, and once inverted and integrated twice by parts, we get a recurrence relation between $\Psi_{k+2}(x)$ and $\Psi_k(x)$:

$$\Psi_{k+2}(x) = \frac{4}{k(k+1)}\left(\left(\frac{k}{2}\right)^2+\left(\frac{\log|x|}{\pi}\right)^2\right)\Psi_k(x).$$

As we know Ψ_1 and Ψ_2 (see the previous section), an easy induction gives (9) and (10). □

The explicit densities of Proposition 2 allow us to obtain very simply the following recurrence relations for the $\zeta(2n)$'s and the $L_{\chi_4}(2n+1)$'s.

Proposition 3. *Let the coefficients* $p_{n,k}^{(t)}$ *(*$t=1$ *or* 2*) be defined through the expansion*

$$\prod_{j=0}^{n-1}\left(\left(j+\frac{t}{2}\right)^2+X\right) = \sum_{k=0}^{n}p_{n,k}^{(t)}X^k.$$

Then the following recurrence relations for $\zeta(2n)$ *and* $L_{\chi_4}(2n+1)$ *hold :*

$$\frac{2^{2n+2}}{(2n)!}\sum_{j=0}^{n}p_{n,j}^{(1)}\frac{(2j)!}{\pi^{2j+1}}L_{\chi_4}(2j+1) = 1, \tag{11}$$

$$\frac{2^{2n+3}}{(2n+1)!}\sum_{j=0}^{n}p_{n,j}^{(2)}\frac{(2j+1)!}{\pi^{2(j+1)}}\left(1-\frac{1}{2^{2(j+1)}}\right)\zeta(2j+2) = 1. \tag{12}$$

Proof. Knowing the density of Π_{2n} from (10), and the moments of $\log|\Pi_2|$ in terms of ζ, equation (12) is just the transcription of the relation

$$1 = \int_{\mathbb{R}} \Psi_{2n+2}(x)\mathrm{d}x = \frac{2^{2n}}{(2n+1)!}\sum_{j=0}^{n} \frac{p_{n,j}^{(2)}}{\pi^{2j}}\,\mathbb{E}\left((\log|\Pi_2|)^{2j}\right).$$

Equation (11) is a transcription of the similar identity,

$$1 = \int_{\mathbb{R}} \Psi_{2n+1}(x)\mathrm{d}x = \frac{2^{2n}}{(2n)!}\sum_{j=0}^{n} \frac{p_{n,j}^{(1)}}{\pi^{2j}}\,\mathbb{E}\left((\log|\Pi_1|)^{2j}\right).$$

with the moments of $\log|\Pi_1|$ then written in terms of L_{χ_4}. □

From the previous recurrence relations one can easily deduce Euler's formulae (2) as well as (1). Indeed, as relations (11) (resp (12)) determine the values of $L_{\chi_4}(2n+1)$ (resp $\zeta(2n)$) for all n, it is sufficient to check that the $A_n^{(t)}$'s ($t = 1$ or 2) satisfy the relation

$$\frac{2^{2n}}{\Gamma(2n+t)}\sum_{j=0}^{n} p_{n,j}^{(t)}\frac{A_j^{(t)}}{2^{2j}} = 1.$$

This is implied by the more general relation, evaluated for $\theta = 0$, where $f_t(\theta) = \frac{1}{(\cos\theta)^t}$:

$$\prod_{j=0}^{n-1}\left[(2j+t)^2 + \partial_\theta^2\right]\ f_t(\theta) = (t)_{2n}\ f_t(\theta)^{1+\frac{2n}{t}}.$$

Here $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol notation. The previous relation can easily be shown by induction on n.

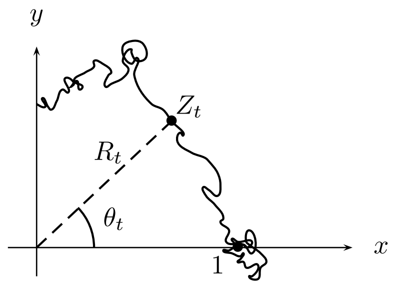
Remark. We have been looking for a generalization of our approach for continuous values of $t \in [1,2]$, which would yield Euler-kind of expressions for certain functions of " type L". This would be possible if "elementary" expressions for the density of $(\hat{C}_t, t \in]1,2[)$ (see [7] for more details about this process) were known. In fact, that density is known to be (see, e.g. Pitman-Yor [7])

$$\frac{2^{t-2}}{\pi\Gamma(t)}\left|\Gamma\left(\frac{t+ix}{2}\right)\right|^2.$$

This simplifies only for $t = 1$ and $t = 2$ (see [1]) hence with the help of the functional equation of the gamma function, for any integer t, which corresponds to the above formulae (9) and (10).

4 Understanding the relation (3) in terms of planar Brownian motion

Since our derivation of the identity (8) is rather analytical, it seems of interest to provide a more probabilistic proof of it.



Consider $Z_t = X_t + iY_t$ a $\mathbb{C}-$valued Brownian motion, starting from $1+i0$. Denote $R_t = |Z_t| = (X_t^2+Y_t^2)^{1/2}$, and $(\theta_t, t \geq 0)$ a continuous determination of the argument of $(Z_u, u \leq t)$ around 0, with $\theta_0 = 0$.
Recall that there exist two independent one-dimensional Brownian motions $(\beta_u, u \geq 0)$ and $(\gamma_u, u \geq 0)$ such that

$$\log R_t = \beta_{H_t}, \text{ and } \theta_t = \gamma_{H_t}. \tag{13}$$

Next, we consider $T = \inf \{t : X_t = 0\} = \inf \left\{t : |\theta_t| = \frac{\pi}{2}\right\}$.

Now, from (13) we obtain, on the one hand,

$$H_T = \inf \left\{u : |\gamma_u| = \frac{\pi}{2}\right\} \stackrel{\text{def}}{=} T^{\gamma,*}_{\pi/2},$$

and, on the other hand, it is well known that Y_T is distributed as $\mathbb{C}_1$; therefore, using (13), we obtain $\log|\mathbb{C}_1| \stackrel{\text{law}}{=} \beta_{T^{\gamma,*}_{\pi/2}}$, so that

$$\frac{2}{\pi}\log|\mathbb{C}_1| \stackrel{\text{law}}{=} \beta_{T^{\gamma,*}_1}.$$

Consequently, thanks to the independence of β and γ, we obtain

$$\mathbb{E}\left[e^{i\lambda\frac{2}{\pi}\log|\mathbb{C}_1|}\right] = \mathbb{E}\left[e^{i\lambda\beta_{T^{\gamma,*}_1}}\right] = \mathbb{E}\left[e^{-\frac{\lambda^2}{2}T^{\gamma,*}_1}\right] = \frac{1}{\cosh\lambda},$$

as is well known.

5 Conclusion

This paper gives two new probabilistic proofs of the celebrated formulae (1) and (2), in relation with the process $\hat{C}_t$ for $t \in \mathbb{N}$. More details and applications to the asymptotic study of jumps of the Cauchy process are provided in [8].
Another discussion about the links between some probability laws and L-functions can be found in [2]. In a similar vein, the reader will find some closely related computations by Paul Lévy [6] who, for the same purpose as ours, uses Fourier inversion of the characteristic functions $1/\cosh\lambda$, $\lambda\sinh\lambda$ and $1/(\cosh\lambda)^2$.

Appendix : a slight generalization in terms of the stable one-sided laws

Let $X_\mu = \frac{T_\mu}{T'_\mu}$, with T_μ and T'_μ two independent, unilateral, stable variables with exponent μ :

$$\mathbb{E}\left[e^{-\lambda T_\mu}\right] = e^{-\lambda^\mu}.$$

Although, except for $\mu = 1/2$, the density of T_μ does not admit a simple expression, we know from Lamperti [4] (see also Chaumont-Yor [3] exercise 4.21) that

$$\mathbb{E}\left[(X_\mu)^s\right] = \frac{\sin \pi s}{\mu \sin\left(\frac{\pi s}{\mu}\right)}, \tag{14}$$

$$\mathbb{P}\left((X_\mu)^\mu \in \mathrm{d}y\right) = \frac{\sin(\pi\mu)}{\pi\mu} \frac{\mathrm{d}y}{y^2 + 2y\cos(\pi\mu) + 1}. \tag{15}$$

As in the previous sections, we may calculate $\mathbb{E}\left[(\log X_\mu^\mu)^{2n}\right]$ in two different ways.

- If we define the sequence $(a_n^{(\mu)}, n \geq 0)$ via the Taylor series $\frac{\sin \pi s}{\mu \sin\left(\frac{\pi s}{\mu}\right)} = \sum_{n\geq 0} \frac{a_n^{(\mu)}}{(2n)!}(\pi s)^{2n}$ then, from (14),
$$\mathbb{E}\left[(\log(X_\mu)^\mu)^{2n}\right] = \pi^{2n} a_n^{(\mu)}. \tag{16}$$

- We rewrite (15) as $\mathbb{P}\left((X_\mu)^\mu \in \mathrm{d}y\right) = \frac{\mathrm{d}y}{2i\pi\mu}\left(\frac{1}{y+e^{-i\pi\mu}} - \frac{1}{y+e^{i\pi\mu}}\right)$. With the usual series expansion we get
$$\mathbb{E}\left[(\log(X_\mu)^\mu)^{2n}\right] = \frac{2\Gamma(2n+1)}{\pi\mu} \sum_{k\geq 1} \frac{(-1)^{k+1}\sin(k\mu\pi)}{k^{2n+1}}. \tag{17}$$

Formulae (16) and (17) give

$$\sum_{k\geq 1} \frac{(-1)^{k+1}\sin(k\mu\pi)}{k^{2n+1}} = \frac{\pi^{2n+1}\mu}{2\Gamma(2n+1)}\ a_n^{(\mu)}. \tag{18}$$

We now make somme comments, essentially about formula (18).

- Formula (18) with $\mu = 1/2$ gives $L_{\chi_4}(2n+1) = \frac{\pi^{2n+1}}{4\Gamma(2n+1)}\ a_n^{(1/2)}$, which is consistent with formula (1).

- Formula (2) about ζ may also be generalized via the random variable X_μ. We consider now the product of two independent copies X_μ and $\tilde{X}_\mu$. We then need to introduce the Taylor expansion of $\left(\frac{\sin \pi s}{\mu \sin\left(\frac{\pi s}{\mu}\right)}\right)^2$ and the density of $(X_\mu)^\mu(\tilde{X}_\mu)^\mu$, which is
$$\mathbb{P}\left((X_\mu)^\mu(\tilde{X}_\mu)^\mu \in \mathrm{d}y\right) = \frac{\mathrm{d}y}{(2\pi\mu)^2}\left(\frac{-\log y - 2i\pi\mu}{y - e^{-2i\pi\mu}} + \frac{-\log y + 2i\pi\mu}{y - e^{2i\pi\mu}} + \frac{2\log y}{y-1}\right).$$
The straightforward calculations for $\mathbb{E}\left[(\log((X_\mu)^\mu(\tilde{X}_\mu)^\mu))^{2n}\right]$ are left to the reader.

- Formula (18) looks like the famous formula
$$\sum_{k=0}^{\infty} \frac{\sin((2k+1)\mu\pi)}{(2k+1)^{2n+1}} = \frac{(-1)^n \pi^{2n+1}}{4(2n)!} E_{2n}(\mu), \tag{19}$$

where E_{2n} is the $2n^{th}$ Euler polynomial. Formula (18) (with μ replaced by 2μ) taken together with (19) gives the explicit expression (for all $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$)

$$\sum_{k\geq 1} \frac{\sin(k\mu\pi)}{k^{2n+1}} = \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!} B_{2n+1}(\mu/2), \tag{20}$$

where B_{2n+1} is the $(2n+1)^{th}$ Bernoulli polynomial. The derivative of (20) with respect to μ gives the explicit expression

$$\sum_{k\geq 1} \frac{\cos(k\mu\pi)}{k^{2n}} = \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n}(\mu/2). \tag{21}$$

For $\mu = 0$, we get an expression for $\zeta(2n)$. More details about formulae (19), (20) and (21) can be found, e.g., in [10].

To summarize, we have found a third way to prove formula (2) by making use of the one parameter family (X_μ) generalizing the Cauchy variable (or, more precisely, its square).

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