Elect. Comm. in Probab. **13** (2008), 482–493

ELECTRONIC COMMUNICATIONS in PROBABILITY

ERROR BOUNDS ON THE NON-NORMAL APPROXIMATION OF HERMITE POWER VARIATIONS OF FRACTIONAL BROWNIAN MOTION

JEAN-CHRISTOPHE BRETON
Laboratoire Mathématiques, Image et Applications, Université de La Rochelle, Avenue Michel Crépeau, 17042 La Rochelle Cedex, France
email: `jean-christophe.breton@univ-lr.fr`

IVAN NOURDIN
Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie (Paris VI), Boîte courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, France
email: `ivan.nourdin@upmc.fr`

Submitted April 15, 2008, *accepted in final form* August 25, 2008

AMS 2000 Subject classification: 60F05; 60G15; 60H05; 60H07.
Keywords: Total variation distance; Non-central limit theorem; Fractional Brownian motion; Hermite power variation; Multiple stochastic integrals; Hermite random variable.

Abstract
Let $q \geq 2$ be a positive integer, B be a fractional Brownian motion with Hurst index $H \in (0,1)$, Z be an Hermite random variable of index q, and H_q denote the qth Hermite polynomial. For any $n \geq 1$, set $V_n = \sum_{k=0}^{n-1} H_q(B_{k+1}-B_k)$. The aim of the current paper is to derive, in the case when the Hurst index verifies $H > 1-1/(2q)$, an upper bound for the total variation distance between the laws $\mathscr{L}(Z_n)$ and $\mathscr{L}(Z)$, where Z_n stands for the correct renormalization of V_n which converges in distribution towards Z. Our results should be compared with those obtained recently by Nourdin and Peccati (2007) in the case where $H < 1-1/(2q)$, corresponding to the case where one has normal approximation.

1 Introduction

Let $q \geq 2$ be a positive integer and B be a fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$. The asymptotic behavior of the q-Hermite power variations of B with respect to $\mathbb{N}$, defined as

$$V_n = \sum_{k=0}^{n-1} H_q(B_{k+1} - B_k), \quad n \geq 1, \tag{1}$$

has recently received a lot of attention, see e.g. [9], [10] and references therein. Here, H_q stands for the Hermite polynomial with degree q, given by $H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}\big(e^{-x^2/2}\big)$. We have $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, and so on. The analysis of the asymptotic behavior of (1)

is motivated, for instance, by the traditional applications of quadratic variations to parameter estimation problems (see e.g. [1, 4, 8, 15] and references therein).
In the particular case of the standard Brownian motion (that is when $H = \frac{1}{2}$), the asymptotic behavior of (1) can be immediately deduced from the classical central limit theorem. When $H \neq \frac{1}{2}$, the increments of B are not independent anymore and the asymptotic behavior of (1) is consequently more difficult to understand. However, thanks to the seminal works of Breuer and Major [3], Dobrushin and Major [6], Giraitis and Surgailis [7] and Taqqu [14], it is well-known that we have, as $n \to \infty$:

1. If $0 < H < 1 - 1/(2q)$ then
$$Z_n := \frac{V_n}{\sigma_{q,H}\sqrt{n}} \overset{\text{Law}}{\longrightarrow} \mathscr{N}(0,1). \tag{2}$$

2. If $H = 1 - 1/(2q)$ then
$$Z_n := \frac{V_n}{\sigma_{q,H}\sqrt{n\log n}} \overset{\text{Law}}{\longrightarrow} \mathscr{N}(0,1). \tag{3}$$

3. If $H > 1 - 1/(2q)$ then
$$Z_n := \frac{V_n}{n^{1-q(1-H)}} \overset{\text{Law}}{\longrightarrow} Z \sim \text{“Hermite random variable”}. \tag{4}$$

Here, $\sigma_{q,H} > 0$ denotes an (explicit) constant depending only on q and H. Moreover, the Hermite random variable Z appearing in (4) is defined as the value at time 1 of the Hermite process, i.e.
$$Z = I_q^W(L_1), \tag{5}$$
where I_q^W denotes the q-multiple stochastic integral with respect to a Wiener process W, while L_1 is the symmetric kernel defined as
$$L_1(y_1,\ldots,y_q) = \frac{1}{q!}\mathbf{1}_{[0,1]^q}(y_1,\ldots,y_q)\int_{y_1\vee\cdots\vee y_q}^{1}\partial_1 K_H(u,y_1)\ldots\partial_1 K_H(u,y_q)du,$$
with K_H the square integrable kernel given by (9). We refer to [10] for a complete discussion of this subject.
When $H \neq 1/2$, the exact expression of the distribution function (d.f.) of Z_n is very complicated. For this reason, when n is large, it is customary to use (2)–(4) as a sort of heuristic argument, implying that one can always replace the distribution function of Z_n with the one of the corresponding limit. Of course, if one applies this strategy without providing any estimate of the error for a fixed n, then there is in principle no reason to believe that such an approximation of the d.f. of Z_n is a good one. To the best of our knowledge, such a problem has not been considered in any of the works using relations (2)–(4) with statistical applications in mind (for instance [1, 4, 8, 15]). The current paper, together with [11], seem to be the first attempt in such a direction.
Recall that the *total variation distance* between the laws of two real-valued random variables Y and X is defined as
$$d_{TV}\big(\mathscr{L}(Y),\mathscr{L}(X)\big) = \sup_{A\in\mathscr{B}(\mathbb{R})}\big|P(Y\in A) - P(X\in A)\big|$$
where $\mathscr{B}(\mathbb{R})$ denotes the class of Borel sets of $\mathbb{R}$. In [11], by combining Stein's method with Malliavin calculus (see also Theorem 1.3 below), the following result is shown:

Theorem 1.1. *If $H < 1 - 1/(2q)$ then, for some constant $c_{q,H} > 0$ depending uniquely on q and H, we have:*

$$d_{TV}\big(\mathscr{L}(Z_n), \mathscr{N}(0,1)\big) \leq c_{q,H} \begin{cases} n^{-1/2} & \text{if } H \in (0, \frac{1}{2}] \\ n^{H-1} & \text{if } H \in [\frac{1}{2}, \frac{2q-3}{2q-2}] \\ n^{qH-q+\frac{1}{2}} & \text{if } H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q}) \end{cases}$$

for Z_n defined by (2).

Here, we deal with the remaining cases, that is when $H \in [1 - \frac{1}{2q}, 1)$. Our main result is as follows:

Theorem 1.2.

1. *If $H = 1 - 1/(2q)$ then, for some constant $c_{q,H} > 0$ depending uniquely on q and H, we have*
$$d_{TV}\big(\mathscr{L}(Z_n), \mathscr{N}(0,1)\big) \leq \frac{c_{q,H}}{\sqrt{\log n}} \tag{6}$$
for Z_n defined by (3).

2. *If $H \in (1 - 1/(2q), 1)$ then, for some constant $c_{q,H} > 0$ depending uniquely on q and H, we have*
$$d_{TV}\big(\mathscr{L}(Z_n), \mathscr{L}(Z)\big) \leq c_{q,H}\, n^{1-\frac{1}{2q}-H} \tag{7}$$
for Z_n and Z defined by (4).

Actually, the case when $H = 1 - 1/(2q)$ can be tackled by mimicking the proof of Theorem 1.1. Only minor changes are required: we will also conclude thanks to the following general result by Nourdin and Peccati.

Theorem 1.3. *(cf. [11]) Fix an integer $q \geq 2$ and let $\{f_n\}_{n\geq 1}$ be a sequence of $\mathfrak{H}^{\odot q}$. Then we have*

$$d_{TV}\left(\mathscr{L}\big(I_q(f_n)\big), \mathscr{N}(0,1)\right) \leq 2\sqrt{E\left(1 - \frac{1}{q}\|DI_q(f_n)\|^2_{\mathfrak{H}}\right)^2},$$

where D stands for the Malliavin derivative with respect to X.

Here, and for the rest of the paper, X denotes a centered Gaussian isonormal process on a real separable Hilbert space $\mathfrak{H}$ and, as usual, $\mathfrak{H}^{\odot q}$ (resp. I_q) stands for the qth symmetric tensor product of $\mathfrak{H}$ (resp. the multiple Wiener-Itô integral of order q with respect to X). See Section 2 for more precise definitions and properties.

When $H \in (1 - 1/(2q), 1)$, Theorem 1.3 can not be used (the limit in (4) being not Gaussian), and another argument is required. Our new idea is as follows. First, using the scaling property (8) of fBm, we construct, for every *fixed* n, a copy S_n of Z_n that converges in L^2. Then, we use the following result by Davydov and Martynova.

Theorem 1.4. *(cf. [5]; see also [2]) Fix an integer $q \geq 2$ and let $f \in \mathfrak{H}^{\odot q} \setminus \{0\}$. Then, for any sequence $\{f_n\}_{n\geq 1} \subset \mathfrak{H}^{\odot q}$ converging to f, their exists a constant $c_{q,f}$, depending only on q and f, such that:*

$$d_{TV}\left(\mathscr{L}\big(I_q(f_n)\big), \mathscr{L}\big(I_q(f)\big)\right) \leq c_{q,f}\|f_n - f\|^{1/q}_{\mathfrak{H}^{\odot q}}.$$

The rest of the paper is organized as follows. In Section 2, some preliminary results on fractional Brownian motion and Malliavin calculus are presented. Section 3 deals with the case $H \in (1 - 1/(2q), 1)$, while the critical case $H = 1 - 1/(2q)$ is considered in Section 4.

2 Preliminaries

The reader is referred to [12] or [13] for any unexplained notion discussed in this section. Let $B = \{B_t,\, t \geq 0\}$ be a fBm with Hurst index $H \in (0,1)$, that is a centered Gaussian process, started from zero and with covariance function $E(B_s B_t) = R(s,t)$, where

$$R(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right); \quad s,t \geq 0.$$

In particular, it is immediately shown that B has stationary increments and is selfsimilar of index H. Precisely, for any $h, c > 0$, we have

$$\{B_{t+h} - B_h,\, t \geq 0\} \overset{\text{Law}}{=} \{B_t,\, t \geq 0\} \quad \text{and} \quad \{c^{-H} B_{ct},\, t \geq 0\} \overset{\text{Law}}{=} \{B_t,\, t \geq 0\}. \tag{8}$$

For any choice of the Hurst parameter $H \in (0,1)$, the Gaussian space generated by B can be identified with an isonormal Gaussian process of the type $B = \{B(h) : h \in \mathfrak{H}\}$, where the real and separable Hilbert space $\mathfrak{H}$ is defined as follows: (i) denote by $\mathscr{E}$ the set of all $\mathbb{R}$-valued step functions on $[0,\infty)$, (ii) define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathscr{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = R(t,s).$$

In particular, with such a notation, one has that $B_t = B(\mathbf{1}_{[0,t]})$.

From now, assume on one hand that B is defined on $[0,1]$ and on the other hand that the Hurst index verifies $H > \frac{1}{2}$. The covariance kernel R can be written as

$$R(t,s) = \int_0^{s\wedge t} K_H(t,r) K_H(s,r) dr,$$

where K_H is the square integrable kernel given by

$$K_H(t,s) = \Gamma\left(H + \frac{1}{2}\right)^{-1} (t-s)^{H-\frac{1}{2}} F\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\right), \tag{9}$$

$F(a,b,c,z)$ being the classical Gauss hypergeometric function. Consider the linear operator K_H^* from $\mathscr{E}$ to $L^2([0,1])$ defined by

$$(K_H^* \varphi)(s) = K_H(1,s)\varphi(s) + \int_s^1 \big(\varphi(r) - \varphi(s)\big) \partial_1 K_H(r,s) dr.$$

For any pair of step functions φ and ψ in $\mathscr{E}$, we have $\langle K_H^*\varphi, K_H^*\psi \rangle_{L^2} = \langle \varphi, \psi \rangle_{\mathfrak{H}}$. As a consequence, the operator K_H^* provides an isometry between the Hilbert spaces $\mathfrak{H}$ and $L^2([0,1])$. Hence, the process $W = (W_t)_{t\in[0,1]}$ defined by

$$W_t = B\left((K_H^*)^{-1}(\mathbf{1}_{[0,t]})\right) \tag{10}$$

is a Wiener process, and the process B has an integral representation of the form $B_t = \int_0^t K_H(t,s) dW_s$, because $(K_H^* \mathbf{1}_{[0,t]})(s) = K_H(t,s)$.

The elements of $\mathfrak{H}$ may be not functions but distributions. However, $\mathfrak{H}$ contains the subset $|\mathfrak{H}|$ of all measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\int_{[0,1]^2} |f(u)||f(v)||u-v|^{2H-2} du dv < \infty.$$

Moreover, for $f, g \in |\mathfrak{H}|$, we have

$$\langle f, g\rangle_{\mathfrak{H}} = H(2H-1)\int_{[0,1]^2} f(u)\, g(v)\, |u-v|^{2H-2} du dv.$$

In the sequel, we note $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$, respectively, the tensor space and the symmetric tensor space of order $q \geq 1$. Let $\{e_k : k \geq 1\}$ be a complete orthogonal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \dots, p \wedge q$, the rth contraction of f and g is the element of $\mathfrak{H}^{\odot(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1=1,\dots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}.$$

In particular, note that $f \otimes_0 g = f \otimes g$ and, when $p = q$, that $f \otimes_p g = \langle f, g\rangle_{\mathfrak{H}}$. Since, in general, the contraction $f \otimes_r g$ is not a symmetric element of $\mathfrak{H}^{\otimes(p+q-2r)}$, we define $f \widetilde{\otimes}_r g$ as the canonical symmetrization of $f \otimes_r g$. When $f \in \mathfrak{H}^{\odot q}$, we write $I_q(f)$ to indicate its qth multiple integral with respect to B. The following formula is useful to compute the product of such integrals: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g). \tag{11}$$

Let $\mathscr{S}$ be the set of cylindrical functionals F of the form

$$F = \varphi(B(h_1), \dots, B(h_n)), \tag{12}$$

where $n \geq 1$, $h_i \in \mathfrak{H}$ and the function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ is such that its partial derivatives have polynomial growth. The Malliavin derivative DF of a functional F of the form (12) is the square integrable $\mathfrak{H}$-valued random variable defined as

$$DF = \sum_{i=1}^{n} \partial_i \varphi(B(h_1), \dots, B(h_n)) h_i,$$

where $\partial_i \varphi$ denotes the ith partial derivative of φ. In particular, one has that $D_s B_t = \mathbf{1}_{[0,t]}(s)$ for every $s, t \in [0,1]$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{1,2}$, defined by the relation $\|F\|_{1,2}^2 = E\big|F\big|^2 + E\|DF\|_{\mathfrak{H}}^2$. Note that every multiple integral belongs to $\mathbb{D}^{1,2}$. Moreover, we have

$$D_t\big(I_q(f)\big) = q I_{q-1}\big(f(\cdot, t)\big)$$

for any $f \in \mathfrak{H}^{\odot q}$ and $t \geq 0$. The Malliavin derivative D also satisfies the following chain rule formula: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded derivatives and if $(F_1, \dots, F_n)$ is a random vector such that each component belongs to $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, F_n)$ is itself an element of $\mathbb{D}^{1,2}$, and moreover

$$D\varphi(F_1, \dots, F_n) = \sum_{i=1}^{n} \partial_i \varphi(F_1, \dots, F_n) DF_i.$$

3 Case $H \in (1-1/(2q),1)$

In this section, we fix $q \geq 2$, we assume that $H > 1 - \frac{1}{2q}$ and we consider Z defined by (5) for W the Wiener process defined by (10). By the scaling property (8) of fBm, remark first that Z_n, defined by (4), has the same law, for any fixed $n \geq 1$, as

$$S_n = n^{q(1-H)-1} \sum_{k=0}^{n-1} H_q\big(n^H (B_{(k+1)/n} - B_{k/n})\big) = I_q(f_n), \tag{13}$$

for $f_n = n^{q-1} \sum_{k=0}^{n-1} \mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q} \in \mathfrak{H}^{\odot q}$. In [10], Theorem 1 (point 3), it is shown that the sequence $\{S_n\}_{n\geq 1}$ converges in L^2 towards Z, or equivalently that $\{f_n\}_{n\geq 1}$ is Cauchy in $\mathfrak{H}^{\odot q}$. Here, we precise the rate of this convergence:

Proposition 3.1. *Let f denote the limit of the Cauchy sequence $\{f_n\}_{n\geq 1}$ in $\mathfrak{H}^{\odot q}$. We have*

$$E\big|S_n - Z\big|^2 = E\big|I_q(f_n) - I_q(f)\big|^2 = \|f_n - f\|^2_{\mathfrak{H}^{\odot q}} = O(n^{2q-1-2qH}), \quad \textit{as } n \to \infty.$$

Proposition 3.1, together with Theorem 1.4 above, immediately entails (7) so that the rest of this section is devoted to the proof of the proposition.

Proof of Proposition 3.1. We have

$$\begin{aligned} \|f_n\|^2_{\mathfrak{H}^{\odot q}} &= n^{2q-2} \sum_{k,l=0}^{n-1} \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]} \rangle^q_{\mathfrak{H}} \\ &= H^q(2H-1)^q\, n^{2q-2} \sum_{k,l=0}^{n-1} \left(\int_{k/n}^{(k+1)/n} du \int_{l/n}^{(l+1)/n} dv |u-v|^{2H-2} \right)^q. \end{aligned} \tag{14}$$

By letting n go to infinity, we obtain

$$\begin{aligned} \|f\|^2_{\mathfrak{H}^{\odot q}} &= H^q(2H-1)^q \int_{[0,1]^2} |u-v|^{2qH-2q} du dv \\ &= H^q(2H-1)^q \sum_{k,l=0}^{n-1} \int_{k/n}^{(k+1)/n} du \int_{l/n}^{(l+1)/n} dv |u-v|^{2qH-2q}. \end{aligned} \tag{15}$$

Now, let $\phi \in |\mathfrak{H}|$. We have

$$\begin{aligned} \langle f_n, \phi^{\otimes q} \rangle_{\mathfrak{H}^{\odot q}} &= n^{q-1} \sum_{l=0}^{n-1} \langle \mathbf{1}_{[l/n,(l+1)/n]}, \phi \rangle^q_{\mathfrak{H}} \\ &= H^q(2H-1)^q\, n^{q-1} \sum_{l=0}^{n-1} \left(\int_{l/n}^{(l+1)/n} dv \int_0^1 du\, \phi(u) |u-v|^{2H-2} \right)^q. \end{aligned}$$

By letting n go to infinity, we obtain

$$\langle f, \phi^{\otimes q} \rangle_{\mathfrak{H}^{\odot q}} = H^q(2H-1)^q \int_0^1 dv \left(\int_0^1 du\, \phi(u) |u-v|^{2H-2} \right)^q.$$

Hence, we have

$$\begin{aligned}\langle f, f_n\rangle_{\mathfrak{H}^{\odot q}} &= H^q(2H-1)^q\, n^{q-1} \sum_{k=0}^{n-1}\int_0^1 dv \left(\int_{k/n}^{(k+1)/n} du|u-v|^{2H-2}\right)^q \\ &= H^q(2H-1)^q\, n^{q-1} \sum_{k,l=0}^{n-1}\int_{l/n}^{(l+1)/n} dv \left(\int_{k/n}^{(k+1)/n} du|u-v|^{2H-2}\right)^q. \quad (16)\end{aligned}$$

Finally, by combining (14), (15) and (16), and by using among others elementary change of variables, we can write:

$$\begin{aligned}\|f_n - f\|^2_{\mathfrak{H}^{\odot q}} &= H^q(2H-1)^q \sum_{k,l=0}^{n-1}\Bigg\{ n^{2q-2}\left(\int_{k/n}^{(k+1)/n} du \int_{l/n}^{(l+1)/n} dv|u-v|^{2H-2}\right)^q \\ &\quad -2\, n^{q-1}\int_{l/n}^{(l+1)/n} dv\left(\int_{k/n}^{(k+1)/n} du|u-v|^{2H-2}\right)^q \\ &\quad +\int_{k/n}^{(k+1)/n} dw \int_{l/n}^{(l+1)/n} dz|w-z|^{2qH-2q}\Bigg\} \\ &= H^q(2H-1)^q\, n^{2q-2-2qH} \sum_{k,l=0}^{n-1}\Bigg\{\left(\int_0^1 du\int_0^1 dv|k-l+u-v|^{2H-2}\right)^q \\ &\quad -2\int_0^1 dv\left(\int_0^1 du|k-l+u-v|^{2H-2}\right)^q + \int_0^1 du\int_0^1 dv|k-l+u-v|^{2qH-2q}\Bigg\} \\ &\leq H^q(2H-1)^q\, n^{2q-1-2qH}\sum_{r\in\mathbb{Z}}\Bigg|\left(\int_0^1 du\int_0^1 dv|r+u-v|^{2H-2}\right)^q \\ &\quad -2\int_0^1 dv\left(\int_0^1 du|r+u-v|^{2H-2}\right)^q + \int_0^1 du\int_0^1 dv|r+u-v|^{2qH-2q}\Bigg|. \quad (17)\end{aligned}$$

Consequently, to achieve the proof of Theorem 3.1, it remains to ensure that the sum over $\mathbb{Z}$ in (17) is finite. For $r>1$, elementary computations give

$$\begin{aligned}\left(\int_0^1 du\int_0^1 dv|r+u-v|^{2H-2}\right)^q &= \big(2H(2H-1)\big)^{-q}\big((r+1)^{2H}-2r^{2H}+(r-1)^{2H}\big)^q \\ &= \big(r^{2H-2}+O(r^{2H-4})\big)^q \\ &= r^{2qH-2q}+O(r^{2qH-2q-2}) \quad (18)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 du\int_0^1 dv|r+u-v|^{2qH-2q} &= \frac{(r+1)^{2qH-2q+2}-2r^{2qH-2q+2}+(r-1)^{2qH-2q+2}}{(2qH-2q+1)(2qH-2q+2)} \\ &= r^{2qH-2q}+O(r^{2qH-2q-2}). \quad (19)\end{aligned}$$

Moreover, using the following inequality (for all $x \geq 0$):

$$\begin{aligned}\left|(1+x)^{2H-1} - 1 - (2H-1)x\right| &= (2H-1)(2-2H)\int_0^x du \int_0^u \frac{dv}{(1+v)^{3-2H}} \\ &\leq (2H-1)(2-2H)\int_0^x du \int_0^u dv = (2H-1)(1-H)x^2,\end{aligned}$$

we can write

$$\begin{aligned}\int_0^1 dv \left(\int_0^1 du|r+u-v|^{2H-2}\right)^q &= (2H-1)^{-q}\int_0^1 \left((r+1-v)^{2H-1}-(r-v)^{2H-1}\right)^q dv \\ &= (2H-1)^{-q}\int_0^1 (r-v)^{2qH-q}\left(\left(1+\frac{1}{r-v}\right)^{2H-1}-1\right)^q dv \\ &= \int_0^1 (r-v)^{2qH-q}\left(\frac{1}{r-v}+R\left(\frac{1}{r-v}\right)\right)^q dv\end{aligned}$$

where the remainder term R verifies $|R(u)| \leq (1-H)u^2$. In particular, for any $v \in [0,1]$, we have

$$(r-v)\left|R\left(\frac{1}{r-v}\right)\right| \leq \frac{1-H}{r-1}.$$

Hence, we deduce:

$$\begin{aligned}\int_0^1 dv \left(\int_0^1 du|r+u-v|^{2H-2}\right)^q &= \int_0^1 (r-v)^{2qH-2q}\left(1+O(1/r)\right)^q dv \\ &= r^{2qH-2q+1}\frac{1-(1-1/r)^{2qH-2q+1}}{2qH-2q+1}(1+O(1/r)) \\ &= r^{2qH-2q+1}(1/r+O(1/r^2))(1+O(1/r)) \\ &= r^{2qH-2q}+O(r^{2qH-2q-1}). \qquad (20)\end{aligned}$$

By combining (18), (19) and (20), we obtain (since similar arguments also apply to the case $r < -1$) that

$$\left(\int_0^1 du \int_0^1 dv|r+u-v|^{2H-2}\right)^q - 2\int_0^1 dv\left(\int_0^1 du|r+u-v|^{2H-2}\right)^q + \int_0^1 du\int_0^1 dv|r+u-v|^{2qH-2q}$$

is $O(|r|^{2qH-2q-1})$, so that the sum over $\mathbb{Z}$ in (17) is finite. The proof of Proposition 3.1 is done. □

4 Case $H = 1 - 1/(2q)$

As we already pointed out in the Introduction, the proof of (6) is a slight adaptation of that of Theorem 1.1 (that is Theorem 4.1 in [11]) which dealt with the case $H < 1 - 1/(2q)$. That is why we will only focus, here, on the differences between the cases $H < 1 - 1/(2q)$ and $H = 1 - 1/(2q)$. In particular, we will freely refer to [11] each time we need an estimate already computed therein.

From now, fix $H = 1 - 1/(2q)$ and let us evaluate the right-hand side in Theorem 1.3. Once again, instead of Z_n, we will rather use S_n defined by

$$S_n = \frac{1}{\sigma_H \sqrt{n \log n}} \sum_{k=0}^{n-1} H_q\big(n^H (B_{(k+1)/n} - B_{k/n})\big) = I_q\left(\frac{n^{q-1}}{\sigma_H \sqrt{\log n}} \sum_{k=0}^{n-1} \mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q}\right)$$

in the sequel, in order to facilitate the connection with [11]. First, observe that the covariance function ρ_H of the Gaussian sequence $\big(n^H (B_{(r+1)/n} - B_{r/n})\big)_{r\geq 0}$, given by

$$\rho_H(r) = \frac{1}{2}\big(|r+1|^{2-1/q} - 2|r|^{2-1/q} + |r-1|^{2-1/q}\big),$$

verifies the following straightforward expansion:

$$\rho_H(r)^q = \left((1-\frac{1}{2q})(1-\frac{1}{q})\right)^q |r|^{-1} + O(|r|^{-3}), \quad \text{as } |r|\to\infty. \tag{21}$$

Using $n^{2-1/q}\langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}} = \rho_H(k-l)$, note that

$$\begin{aligned}
\mathrm{Var}(S_n) &= \frac{n^{2q-2}}{\sigma_H^2 \log n} \sum_{k,l=0}^{n-1} E\big[I_q\big(\mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q}\big) I_q\big(\mathbf{1}_{[l/n,(l+1)/n]}^{\otimes q}\big)\big] \\
&= \frac{q!\, n^{2q-2}}{\sigma_H^2 \log n} \sum_{k,l=0}^{n-1} \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}}^q = \frac{q!}{\sigma_H^2 n \log n} \sum_{k,l=0}^{n-1} \rho_H(k-l)^q
\end{aligned}$$

from which, together with (21), we deduce the exact value of σ_H^2:

$$\sigma_H^2 := \lim_{n\to+\infty} \frac{q!}{n\log n} \sum_{k,l=0}^{n-1} \rho_H(k-l)^q = 2q! \left(\frac{(2q-1)(q-1)}{2q^2}\right)^q. \tag{22}$$

In order to apply Theorem 1.3, we compute the Malliavin derivative of S_n:

$$DS_n = \frac{qn^{q-1}}{\sigma_H \sqrt{\log n}} \sum_{k=0}^{n-1} I_{q-1}\big(\mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q-1}\big) \mathbf{1}_{[k/n,(k+1)/n]}.$$

Hence

$$\|DS_n\|_{\mathfrak{H}}^2 = \frac{q^2 n^{2q-2}}{\sigma_H^2 \log n} \sum_{k,l=0}^{n-1} I_{q-1}\big(\mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q-1}\big) I_{q-1}\big(\mathbf{1}_{[l/n,(l+1)/n]}^{\otimes q-1}\big) \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}}.$$

The multiplication formula (11) yields

$$\begin{aligned}
\|DS_n\|_{\mathfrak{H}}^2 &= \frac{q^2 n^{2q-2}}{\sigma_H^2 \log n} \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 \times \\
&\quad \times \sum_{k,l=0}^{n-1} I_{2q-2-2r}\big(\mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q-1-r} \tilde{\otimes} \mathbf{1}_{[l/n,(l+1)/n]}^{\otimes q-1-r}\big) \langle \mathbf{1}_{[k/n,(k+1)/n]}, \mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}}^{r+1}.
\end{aligned}$$

We can rewrite

$$1-\frac{1}{q}\|DS_n\|_{\mathfrak{H}}^2 = 1-\sum_{r=0}^{q-1}A_r(n)$$

where

$$\begin{aligned} A_r(n) &= \frac{q\,r!\binom{q-1}{r}^2}{\sigma_H^2}\frac{n^{2q-2}}{\log n}\times \\ &\quad\times\sum_{k,l=0}^{n-1} I_{2q-2-2r}\big(\mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q-1-r}\tilde{\otimes}\mathbf{1}_{[l/n,(l+1)/n]}^{\otimes q-1-r}\big)\langle \mathbf{1}_{[k/n,(k+1)/n]},\mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}}^{r+1}. \end{aligned}$$

For the term $A_{q-1}(n)$, we have:

$$\begin{aligned} 1-A_{q-1}(n) &= 1-\frac{q!}{\sigma_H^2\log n}n^{2q-2}\sum_{k,l=0}^{n-1}\langle \mathbf{1}_{[k/n,(k+1)/n]},\mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}}^{q} \\ &= 1-\frac{q!}{\sigma_H^2 n\log n}\sum_{k,l=0}^{n-1}\rho_H(k-l)^q = 1-\frac{q!}{\sigma_H^2 n\log n}\sum_{|r|<n}(n-|r|)\rho_H(r)^q \\ &= 1-\frac{q!}{\sigma_H^2\log n}\sum_{|r|<n}\rho_H(r)^q+\frac{q!}{\sigma_H^2 n\log n}\sum_{|r|<n}|r|\rho_H(r)^q = O(1/\log n) \end{aligned}$$

where the last estimate comes from the development (21) of ρ_H and from the exact value (22) of σ_H^2.

Next, we show that for any *fixed* $r\le q-2$, we have $E|A_r(n)|^2 = O(1/\log n)$. Indeed:

$$\begin{aligned} &E|A_r(n)|^2 \\ &= c(H,r,q)\frac{n^{4q-4}}{\log^2 n}\sum_{i,j,k,l=0}^{n-1}\langle \mathbf{1}_{[k/n,(k+1)/n]},\mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}}^{r+1}\langle \mathbf{1}_{[k/n,(k+1)/n]},\mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}}^{r+1} \\ &\times\langle \mathbf{1}_{[k/n,(k+1)/n]}^{\otimes q-1-r}\tilde{\otimes}\mathbf{1}_{[l/n,(l+1)/n]}^{\otimes q-1-r},\mathbf{1}_{[i/n,(i+1)/n]}^{\otimes q-1-r}\otimes\mathbf{1}_{[j/n,(j+1)/n]}^{\otimes q-1-r}\rangle_{\mathfrak{H}^{\otimes 2q-2-2r}} \\ &= \sum_{\substack{\gamma,\delta\ge 0\\ \gamma+\delta=q-r-1}}\ \sum_{\substack{\alpha,\beta\ge 0\\ \alpha+\beta=q-r-1}} c(H,r,q,\alpha,\beta,\gamma,\delta)B_{r,\alpha,\beta,\gamma,\delta}(n) \end{aligned}$$

where $c(\cdot)$ is a generic constant depending only on its arguments and

$$\begin{aligned} &B_{r,\alpha,\beta,\gamma,\delta}(n) = \frac{n^{4q-4}}{\log n^2}\sum_{i,j,k,l=0}^{n-1} \\ &\langle \mathbf{1}_{[k/n,(k+1)/n]},\mathbf{1}_{[l/n,(l+1)/n]}\rangle_{\mathfrak{H}}^{r+1}\langle \mathbf{1}_{[i/n,(i+1)/n]},\mathbf{1}_{[j/n,(j+1)/n]}\rangle_{\mathfrak{H}}^{r+1}\langle \mathbf{1}_{[k/n,(k+1)/n]},\mathbf{1}_{[i/n,(i+1)/n]}\rangle_{\mathfrak{H}}^{\alpha} \\ &\langle \mathbf{1}_{[k/n,(k+1)/n]},\mathbf{1}_{[j/n,(j+1)/n]}\rangle_{\mathfrak{H}}^{\beta}\langle \mathbf{1}_{[l/n,(l+1)/n]},\mathbf{1}_{[i/n,(i+1)/n]}\rangle_{\mathfrak{H}}^{\gamma}\langle \mathbf{1}_{[l/n,(l+1)/n]},\mathbf{1}_{[j/n,(j+1)/n]}\rangle_{\mathfrak{H}}^{\delta} \\ &= \frac{1}{n^2\log^2 n}\sum_{i,j,k,l=0}^{n-1}\rho_H(k-l)^{r+1}\rho_H(i-j)^{r+1}\rho_H(k-i)^{\alpha}\rho_H(k-j)^{\beta}\rho_H(l-i)^{\gamma}\rho_H(l-j)^{\delta}. \end{aligned}$$

As in [11], when $\alpha, \beta, \gamma, \delta$ are fixed, we can decompose the sum appearing in $B_{r,\alpha,\beta,\gamma,\delta}(n)$ as follows:

$$\sum_{i=j=k=l} + \left(\sum_{\substack{i=j=k\\ l\neq i}} + \sum_{\substack{i=j=l\\ k\neq i}} + \sum_{\substack{i=k=l\\ j\neq i}} + \sum_{\substack{j=k=l\\ i\neq j}} \right) + \left(\sum_{\substack{i=j,k=l\\ k\neq i}} + \sum_{\substack{i=k,j=l\\ j\neq i}} + \sum_{\substack{i=l,j=k\\ j\neq i}} \right)$$
$$+ \left(\sum_{\substack{i=j,k\neq l\\ k\neq l,l\neq i}} + \sum_{\substack{i=k,j\neq i\\ j\neq l,k\neq l}} + \sum_{\substack{i=l,k\neq i\\ k\neq j,j\neq i}} + \sum_{\substack{j=k,k\neq i\\ k\neq l,l\neq i}} + \sum_{\substack{j=l,k\neq i\\ k\neq l,l\neq i}} + \sum_{\substack{k=l,k\neq i\\ k\neq j,j\neq i}} \right) + \sum_{\substack{i,j,k,l\\ \text{are all distinct}}}$$

where the indices run over $\{0, 1 \ldots, n-1\}$. Similar computations as in [11] show that the first, second and third sums are $O(1/(n \log^2 n))$; the fourth and fifth sums are $O(1/(n^{1/q} \log^2 n))$; the sixth, seventh and eighth sums are $O(1/(n^{2/q} \log^2 n))$; the ninth, tenth, eleventh, twelfth, thirteenth and fourteenth sums are also $O(1/(n^{1/q} \log^2 n))$. We focus only on the last sum. Actually, it is precisely its contribution which will indicate the true order in (6). Once again, we split this sum into 24 sums:

$$\sum_{k>l>i>j} + \sum_{k>l>j>i} + \cdots \tag{23}$$

We first deal with the first one, for which we have

$$\begin{aligned}
&\frac{1}{(n\log n)^2} \sum_{k>l>i>j} \rho_H(k-l)^{r+1} \rho_H(i-j)^{r+1} \rho_H(k-i)^{\alpha} \rho_H(k-j)^{\beta} \rho_H(l-i)^{\gamma} \rho_H(l-j)^{\delta} \\
&\leqslant \frac{1}{(n\log n)^2} \sum_{k>l>i>j} (k-l)^{-1} (i-j)^{-(r+1)/q} (l-i)^{-(q-r-1)/q} \\
&= \frac{1}{(n\log n)^2} \sum_{k} \sum_{l<k} (k-l)^{-1} \sum_{i<l} (l-i)^{-(q-r-1)/q} \sum_{j<i} (i-j)^{-(r+1)/q} \\
&\leqslant \frac{1}{n\log^2 n} \sum_{l=1}^{n-1} l^{-1} \sum_{i=1}^{n-1} i^{-(q-r-1)/q} \sum_{j=1}^{n-1} j^{-(r+1)/q} \leqslant \frac{1}{n\log^2 n} \sum_{l=1}^{n-1} l^{-1} n^{(r+1)/q} n^{1-(r+1)/q} \leqslant \frac{1}{\log n}
\end{aligned}$$

where the notation $a_n \leqslant b_n$ means that $\sup_{n\geq 1} |a_n|/|b_n| < +\infty$. Since the other sums in (23) are similarly bounded, the fifteenth sum is $O(1/\log n)$. Consequently:

$$\sum_{r=0}^{q-2} E[A_r(n)^2] = O(1/\log n).$$

Finally, together with (23), we obtain $E\big[\big(1 - \frac{1}{q}\|DS_n\|^2_{\mathfrak{H}}\big)^2\big] = O(1/\log n)$ and the proof of (6) is achieved thanks to Theorem 1.3.

Acknowledgments: We are grateful to an anonymous referee for helpful comments.

References

[1] A. Begyn. Asymptotic expansion and central limit theorem for quadratic variations of Gaussian processes. *Bernoulli*, vol. 13, no. 3, pp. 712–753, 2007. MR2348748

[2] J.-C. Breton. Convergence in variation of the joint laws of multiple Wiener-Itô integrals. *Stat. Probab. Letters*, vol. 76, pp. 1904–1913, 2006. MR2271186

[3] P. Breuer and P. Major. Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.*, vol. 13, no. 3, pp. 425–441, 1983. MR0716933

[4] J.-F. Coeurjolly. Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. *Statist. Infer. Stoch. Proc.*, vol. 4, pp. 199-227, 2001. MR1856174

[5] Y. A. Davydov and G. V. Martynova. Limit behavior of multiple stochastic integral. *Statistics and control of random process*. Preila, Nauka, Moscow, pp. 55–57, 1987 (in Russian). MR1079335

[6] R. L. Dobrushin and P. Major. Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. verw. Gebiete*, vol. 50, pp. 27–52, 1979. MR0550122

[7] L. Giraitis and D. Surgailis. CLT and other limit theorems for functionals of Gaussian processes. *Z. Wahrsch. verw. Gebiete*, vol. 70, pp. 191–212, 1985. MR0799146

[8] J. Istas and G. Lang. Quadratic variations and estimators of the Hölder index of a Gaussian process. *Ann. Inst. H. Poincaré Probab. Statist.*, vol. 33, pp. 407-436, 1997. MR1465796

[9] I. Nourdin. Asymptotic behavior of weighted quadratic and cubic variations of fractional Brownian motion. To appear in: *Ann. Probab.*

[10] I. Nourdin, D. Nualart and C.A. Tudor. Central and non-central limit theorems for weighted power variations of fractional Brownian motion. Preprint.

[11] I. Nourdin and G. Peccati. Stein's method on Wiener chaos. To appear in: *Probab. Th. Related Fields*.

[12] D. Nualart. The Malliavin Calculus and Related Topics. Springer Verlag. Second edition, 2006. MR2200233

[13] D. Nualart. Stochastic calculus with respect to the fractional Brownian motion and applications. *Contemp. Math.* **336**, 3-39, 2003. MR2037156

[14] M. Taqqu. Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. verw. Gebiete*, vol. 50, pp. 53-83, 1979. MR0550123

[15] C.A. Tudor and F. Viens. Variations and estimators for the selfsimilarity order through Malliavin calculus. Preprint.