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PERCOLATION BEYOND $\mathbb{Z}^d$, MANY QUESTIONS AND A FEW ANSWERS

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Abstract

A comprehensive study of percolation in a more general context than the usual $\mathbb{Z}^d$ setting is proposed, with particular focus on Cayley graphs, almost transitive graphs, and planar graphs. Results concerning uniqueness of infinite clusters and inequalities for the critical value p_c are given, and a simple planar example exhibiting uniqueness and non-uniqueness for different $p > p_c$ is analyzed. Numerous varied conjectures and problems are proposed, with the hope of setting goals for future research in percolation theory.

1 Introduction

Percolation has been mostly studied in the lattices $\mathbb{Z}^d$, or in $\mathbb{R}^d$. Recently, several researchers have looked beyond this setting. For instance, Lyons (1996) gives an overview of current knowledge about percolation on trees, while Grimmett and Newman (1990) study percolation on (regular tree) $\times\, \mathbb{Z}$.

The starting point for a study of percolation on the Euclidean lattices is the fact that the critical probability for percolation, denoted p_c, is smaller than one. (See below for exact definitions and see Grimmett (1989) for background on percolation). The first step in a study of percolation on other graphs, for instance Cayley graphs of finitely generated groups, will be to prove that the critical probability for percolation on these graphs is smaller than one. In this note, we will show that for a large family of graphs, indeed, $p_c < 1$. In particular, this holds for graphs satisfying a strong isoperimetric inequality (positive Cheeger constant).

The second part of the paper discusses non-uniqueness of the infinite open cluster. A criterion is given for non-uniqueness, which is proved by dominating the percolation cluster by a branching random walk. This criterion is useful for proving non-uniqueness in "large" graphs.

Numerous open problems and conjectures are presented, probably of variable difficulty. The main questions are about the relation between geometric or topological properties of the graph, on the one hand, and the value of p_c, uniqueness and structure of the infinite cluster, on the other. It seems that there are many interesting features of percolation on planar graphs. One proposed conjecture is that a planar graph which has infinite clusters for $p = 1/2$ site

percolation has infinitely many such clusters. This is proved for graphs that are locally finite in $\mathbb{R}^2$ and are disjoint from the positive x axis.
In a forthcoming paper, we shall discuss a *Voronoi percolation* model. A principle advantage of this model is its generality, which allows to extend various percolation questions beyond $\mathbb{R}^d$, to arbitrary Riemannian manifolds. In this model, the cells of a Voronoi tiling generated by a point Poisson process are taken to be open with probability p, independently. It turns out that this model is advantageous for the study of the conformal invariance conjecture for critical percolation, introduced in Langlands et al (1994). See Benjamini-Schramm (1996). Additionally, we currently study Voronoi percolation in the hyperbolic plane.
We wish to express thanks to Uriel Feige, Rick Kenyon, Ronald Meester, Yuval Peres, and Benjamin Weiss for helpful conversations, and the anonymous referee, for a very careful review.

2 Notations and Definitions

The graphs we shall consider will always be locally finite, that is, each vertex has finitely many neighbors.

Cayley Graphs, Given a finite set of generators $S = \langle g_1^{\pm 1}, \ldots, g_n^{\pm 1} \rangle$ for a group Γ, the *Cayley graph* is the graph $G(\Gamma) = (V, E)$ with $V = \Gamma$ and $\{g, h\} \in E$ iff $g^{-1}h$ is a generator. $G(\Gamma)$ depends on the set of generators. Note that any two Cayley Graphs of the same group are roughly isometric (quasi-isometric). (See Magnus et al (1976) or Ghys et al (1991)).

Almost Transitive Graphs, A Graph G is *transitive* iff for any two vertices u, v in G, there is an automorphism of G mapping u onto v. In particular, Cayley graphs are transitive graphs. $G = (V, E)$ is *almost transitive*, if there is a finite set of vertices $V_0 \subset V$ such that any $v \in V$ is taken into V_0 by some automorphism of G.
For example, the lift to $\mathbb{R}^2$ of any finite graph drawn on the torus $\mathbb{R}^2/\mathbb{Z}^2$ is almost transitive.

Percolation, We assume throughout that the graph G is connected. In Bernoulli site percolation, the vertices are open (respectively closed) with probability p (respectively $1 - p$) independently. The corresponding product measure on the configurations of vertices is denoted by $\mathbb{P}_p$. Let $C(v)$ be the (open) cluster of v. In other words, $C(v)$ is the connected component of the set of open vertices in G containing v, if v is open, and $C(v) = \emptyset$, otherwise. We write

$$\theta^v(p) = \theta^v_G(p) = \mathbb{P}_p\{C(v) \text{ is infinite}\}.$$

When G is transitive, we may write $\theta(p)$ for $\theta^v(p)$. If $C(v)$ is infinite, for some v, we say that *percolation* occurs. Clearly, if $\theta^v(p) > 0$, then $\theta^u(p) > 0$, for any vertices v, u. Let

$$p_c = \sup\{p : \theta^v(p) = 0\}$$

be the critical probability for percolation. See Grimmett (1989).
Throughout the paper, site percolation is discussed. Except when dealing with planar graphs, the results and questions remain equally valid, with only minor modifications, for bond percolation.

Uniqueness of the infinite open cluster, In $\mathbb{Z}^d$, when $\theta(p) > 0$, there exists with probability one a unique infinite open cluster. See Grimmett (1989), the charming proof in Burton and Keane (1989), and Meester's (1994) survey article. As was shown by Grimmett and Newman (1990), this is not the case for percolation on (some regular tree) $\times\, \mathbb{Z}$. For some values of p, uniqueness holds and for others, there are infinitely many infinite disjoint open clusters.

Therefore, it is natural to define

$$p_u = \inf\Big\{p : \mathbb{P}_p\{\text{there is exactly one infinite open cluster}\} = 1\Big\}.$$

Cheeger's constant, Let G be a graph. The Cheeger constant, $h(G)$, of G is

$$h(G) = \inf_S \frac{|\partial S|}{|S|},$$

where S is a finite nonempty set of vertices in G, and ∂S, the *boundary* of S, consists of all vertices in $V \setminus S$ that have a neighbor in S.

3 The percolation critical probability p_c

The starting point of the study of percolation on groups is the following "obvious" conjecture.

Conjecture 1 *If G is the Cayley graph of an infinite (finitely generated) group, which is not a finite extension of $\mathbb{Z}$, then $p_c(G) < 1$.*

Lyons (1994) covers the case of groups with Cayley graphs of exponential volume growth. Babson and Benjamini (1995) covers finitely presented groups with one end. Note that if a group Γ contains as a subgroup another group Γ' for which $p_c(G(\Gamma')) < 1$ then $p_c(G(\Gamma)) < 1$. By results mentioned below, if a group admits a quotient with p_c smaller than 1 for its Cayley graphs, then the same is true for Cayley graphs of the group itself. Thus, the conjecture holds for any group that has a $\mathbb{Z}^2$ quotient.

Suppose that Γ is a group of automorphisms of a graph G. The *quotient graph* G/Γ is the graph whose vertices, $V(G/\Gamma)$, are the equivalence classes $V(G)/\Gamma = \big\{\Gamma v : v \in V(G)\big\}$, or Γ-orbits, and an edge $\{\Gamma u, \Gamma v\}$ appears in G/Γ if there are representatives $u_0 \in \Gamma u, v_0 \in \Gamma v$ that are neighbors in G, $\{u_0, v_0\} \in E(G)$. The map $v \to \Gamma v$ from $V(G)$ to $V(G/\Gamma)$ is called the *quotient map*. If every $\gamma \in \Gamma$, except the identity, has no fixed point in $V(G)$, then G is also called a *covering graph* of G/Γ, and the quotient map is also called a *covering map*.

The following theorem follows from Campanino-Russo (1985). We bring an easy proof here, since it introduces a method which will also be useful below. The proof is reminiscent of the coupling argument of Grimmett and Wierman, which was used by Wierman (1989) in the study of AB-percolation.

Theorem 1 *Assume that G_2 is a quotient graph of G_1, $G_2 = G_1/\Gamma$. Let $v' \in G_1$, and let v be the projection of v' to G_2. Then for any $p \in [0,1]$,*

$$\theta^{v'}_{G_1}(p) \geq \theta^{v}_{G_2}(p),$$

and consequently,

$$p_c(G_1) \leq p_c(G_2).$$

Proof:

We will construct a coupling between percolation on G_2 and on G_1. Consider the following inductive procedure for constructing the percolation cluster of $v \in V(G_2)$. If v is closed, set $C^2_n = \emptyset$ for each n. Otherwise, set $C^2_1 = \{v\}$ and $W^2_1 = \emptyset$. Now let $n \geq 2$. If ∂C^2_{n-1} is

contained in W^2_{n-1}, set $C^2_n = C^2_{n-1}$, $W^2_n = W^2_{n-1}$. Otherwise, choose a vertex $w_n \in V(G_2)$ which is not in $C^2_{n-1} \cup W^2_{n-1}$, but is adjacent to a vertex $z_n \in C^2_{n-1}$. If w_n is open, then let $C^2_n = C^2_{n-1} \cup \{w\}$ and $W^2_n = W^2_{n-1}$, if closed, let $C^2_n = C_{n-1}$ and $W^2_n = W^2_{n-1} \cup \{w_n\}$. $C^2 = \bigcup_n C^2_n$ is $C(v)$ the percolation cluster of v.
We will now describe the coupling with the percolation process in G_1. Let f be the quotient map from G_1 to G_2. Recall that $f(v') = v$. If v is closed, let v' be closed. Otherwise, let v' be open, and let $C^1_1 = \{v'\}$, $W^1_1 = \emptyset$. Assume that $n \geq 2$ and C^1_{n-1}, W^1_{n-1} were defined, and satisfy $f(C^1_{n-1}) = C^2_{n-1}$, $f(W^1_{n-1}) = W^2_{n-1}$. If the construction of $C(v)$ in G_2 stopped at stage n, that is, if $C^2_n = C^2_{n-1}$, $W^2_n = W^2_{n-1}$, then let $C^1_n = C^1_{n-1}$, $W^1_n = W^1_{n-1}$. Otherwise, let z'_n be some vertex in $f^{-1}(z_n) \cap C^1_{n-1}$, and let w'_n be some vertex in $f^{-1}(w_n)$ that neighbors with z'_n. Let w'_n be open iff w_n is open, and define C^1_n and W^1_n accordingly. Then $\bigcup_n C^1_n$ is a connected set of open vertices contained in the percolation cluster $C(v')$ of v'. Hence, $f\big(C(v')\big) \supset C(V)$, and the theorem follows. $\blacksquare$

Question 1 *When does strict inequality hold? We believe that if both G_1 and G_2 are connected almost transitive graphs, G_1 covers but is not isomorphic to G_2 and $p_c(G_2) < 1$, then $p_c(G_1) < p_c(G_2)$.*

Compare with Men'shikov (1987) and Aizenman-Grimmett (1991). Other conjectures regarding p_c for graphs:

Conjecture 2 *Assume that G is an almost transitive graph with ball volume growth faster then linear. Then $p_c(G) < 1$.*

Let G be a graph, define the isoperimetric dimension,

$$\mathrm{Dim}(G) = \sup\left\{ d > 0 : \inf_S \frac{|\partial S|}{|S|^{\frac{d-1}{d}}} > 0 \right\},$$

where S is a finite nonempty set of vertices in G.

Question 2 *Does $Dim(G) > 1$ imply $p_c(G) < 1$?*

Conjecture 3 *Assume that G is a (bounded degree) triangulation of a disc. Any of the following list of progressively weaker assumptions should be sufficient to guarantee $p_c(G) \leq 1/2$:*

(1) $Dim(G) \geq 2$,

(2) $Dim(G) > 1$,

(3) for any finite set A of vertices in G the inequality $|\partial A| \geq f\big(|A|\big) \log |A|$ holds, where f is some function satisfying $\lim_{n\to\infty} f(n) = \infty$.

Moreover, if $h(G) > 0$, then $p_c(G) < 1/2$. The latter might be easier to establish under the assumption of non-positive curvature, that is, minimal degree ≥ 6.

The statement that (1) is sufficient to guarantee $p_c \leq 1/2$ would generalize the fact that $p_c = 1/2$ for the triangular lattice. See Wierman (1989).
So far, we can only show that $p_c < 1$ for graphs with positive Cheeger constant.

Theorem 2

$$p_c(G) \leq \frac{1}{h(G)+1}.$$

Remark 1 *For a degree k regular tree, $p_c(T_k) = (k-1)^{-1} = \big(h(T_k)+1\big)^{-1}$, because $h(T_k) = k-2$. So the theorem is sharp.*

Proof:
Let C_n and W_n be defined as C_n^2, W_n^2 were in the proof of Theorem 1, and let $C = \bigcup_n C_n$. If C is finite (and nonempty), then there is some smallest N such that the boundary of C_N is W_N. By the definition of the Cheeger constant $|W_N| = |\partial C_N| \geq h(G)|C_N|$. That is, we flipped N independent $(p, 1-p)$-coins and $|W_N| \geq Nh(G)/\big(h(G)+1\big)$ turned out closed. But if $p > 1 - h(G)/\big(h(G)+1\big)$, then, with positive probability, a random infinite sequence of independent Bernoulli$(p, 1-p)$ variables does not have an N such that at least $Nh(G)\big(h(G)+1\big)^{-1}$ zeroes appear among the first N elements. In particular, with positive probability we have percolation. ■

Remark 2 *Suppose that $G = (V, E)$ is a finite graph, which is an α-expander; that is, $|\partial A| \geq \alpha|A|$ for any $A \subset V$ with $|A| < |V|/2$. Then the above proof shows that with probability bounded away from zero, the percolation process on G with $p > 1 - \alpha/(\alpha+1)$ will have a cluster with at least half of the vertices of G.*

It has been one of the outstanding challenges of percolation theory to prove that critical percolation in $\mathbb{Z}^d$ $(d > 2)$ dies at p_c, that is, $\theta(p_c) = 0$. This has been proved for $d = 2$ by Kesten (1980) and Russo (1981), and for sufficiently high d by Hara and Slade (1989). It might be beneficial to study the problem in other settings.

Conjecture 4 *Critical percolation dies in every almost transitive graph (assuming $p_c < 1$).*

It is not hard to construct a tree where critical percolation lives, compare Lyons (1996). This shows that the assumption of almost transitivity is essential.

4 Uniqueness and non-uniqueness in almost transitive graphs

In this section, the number and structure of the infinite clusters is discussed, and conditions that guarantee $p_c < p_u$ or $p_u < 1$, are given.

Definition 1 *Let G be a graph. An end of G is a map e that assigns to every finite set of vertices $K \subset V(G)$ a connected component $e(K)$ of $G \setminus K$, and satisfies the consistency condition $e(K) \subset e(K')$ whenever $K' \subset K$.*

It is not hard to see that if $K \subset V(G)$ is finite and F is an infinite component of $G \setminus K$, then there is an end of G satisfying $e(K) = F$.

Definition 2 *The percolation subgraph of a graph G is the graph spanned by all open vertices.*

By Kolmogorov's 0-1 law, the probability of having an infinite component in the percolation subgraph is either 0 or 1. Following is an elementary extension of this 0-1 law in the setting of almost transitive graphs.

Theorem 3 *Let G be an almost transitive infinite graph, and consider a percolation process for some fixed $p \in (0,1)$. Then precisely one of the following situations occurs.*

(1) With probability 1, every component of the percolation graph is finite.

(2) With probability 1, the percolation graph has exactly one infinite component, and has exactly one end.

(3) With probability 1, the percolation graph has infinitely many infinite components, and for every finite n there is some infinite component with more than n ends.

The proof is virtually identical to the proof of a similar theorem for dependent percolation in $\mathbb{Z}^d$, by Newman-Schulman (1981); it is therefore omitted.

Conjecture 5 *Let G be a connected almost transitive graph, and fix a $p \in (0,1)$. Suppose that a.s. there is more than one infinite component in the percolation subgraph. Then, with probability 1, each infinite component in the percolation subgraph has precisely $2^{\aleph_0}$ ends.*

The conjecture is equivalent to saying that in the case of non-uniqueness, there cannot be an infinite component with exactly one end. To demonstrate this, recall that the collection of sets of the form $\{e : e(K) = F\}$ is a sub-basis for a topology on the set of ends of a connected graph, and the set of ends with this topology is a compact (totally disconnected) Hausdorff space. A compact Hausdorff space with no isolated points has cardinality $\geq 2^{\aleph_0}$. Hence, the conjecture follows if one shows that there are no isolated ends in the percolation subgraph. It is easy to see, using the argument of Newman-Schulman (1981), that if there are isolated ends, then there are also infinite clusters with just one end.
Häggström (1996) studies similar questions in the setting of dependent percolation on trees.

It is easy to verify that the proof by Burton and Keane (1989) of the uniqueness of the infinite open cluster for $\mathbb{Z}^d$ works as well for almost transitive graphs with Cheeger constant zero.

Conjecture 6 *Assume that the almost transitive graph G has positive Cheeger constant. Then $p_c(G) < p_u(G)$.*

The conjecture, if true, gives a percolation characterization of amenability. We now present a comparison between the connectivity function and hitting probabilities for a branching random walk on G. This will be useful in showing that $p_c < p_u$, for some graphs. For background on branching random walks on graphs, see Benjamini and Peres (1994).
Let G be a connected graph, and let $p^n(v,u)$ denote the n-step transition probability between v and u, for the simple random walk on G.

$$\rho(G) = \limsup_{n\to\infty} \left(p^n(v,u)\right)^{1/n}$$

is the *spectral radius* of G, and does not depend on v and u. By Dodziuk (1984), for a bounded degree graph, $\rho(G) < 1$ iff the Cheeger constant of G is positive. We have the following

Theorem 4 *Let G be an almost transitive graph, with maximal degree k. Let p be such that percolation occurs at p, that is, $\theta^v(p) > 0$ for some v. If, additionally,*

$$\rho(G)kp < 1,$$

then $\mathbb{P}_p$-almost surely there are infinitely many infinite open clusters. In particular, if $\rho(G)kp_c < 1$, then $p_c < p_u$.

Proof:
Given $p \in [0,1]$, consider the following branching random walk (BRW) on G. Start with a particle at v at time 0. At time 1, at every neighbor of v, a particle is born with probability p, and the particle at v is deleted. Continue inductively: if at time n we have some particles located on G, then at time $n+1$ each one of them gives birth to a particle on each of its neighbors with probability p, independently from the other neighbors, and then dies. We claim that

$$\mathbb{P}_p\{u \in C(v)\} \leq \mathbb{P}\{\text{the BRW starting at } v \text{ hits } u\}.$$

Say that $u \in G$ is in the *support* of the BRW if u is visited by a particle at some time. We will show inductively that the support of the BRW dominates $C(v)$. Consider the following inductive procedure for a coupling of the BRW with the percolation process. Let $C_0 = v$ and $W_0 = \emptyset$. For each $n \geq 1$, choose a vertex w, which is not in $C_{n-1} \cup W_{n-1}$, but is adjacent to a vertex $z \in C_{n-1}$. If, at least once, a particle located at z gave birth to a particle at w, then let $C_n = C_{n-1} \cup w$ and $W_n = W_{n-1}$. Otherwise let $C_n = C_{n-1}$ and $W_n = W_{n-1} \cup \{w\}$. It follows that at each step, the new vertex w is added to C_{n-1} with probability $\geq p$. If at some point there is no vertex w as required, the process stops and we have generated a cluster, which we denote by C. If the process continues indefinitely, we set $C = \bigcup_n C_n$. Note that C is contained in the support of the BRW. Now view the process from a different perspective. Consider C_n to be the set of open vertices and W_n to be the set of closed vertices in the percolation model. Each vertex in $\bigcup_n (W_n \cup C_n)$ is tested only once, and added to C with probability $\geq p$. Thus, the cluster C dominates the open percolation cluster containing v.
The population size of the BRW is dominated by the population size of a Galton-Watson branching process with binomial(p,k) offspring distribution. The mean number of offsprings for that branching process is pk. Hence, by Borel-Cantelli, when $pk < \rho(G)^{-1}$, the BRW is transient, that is, almost surely only finitely many particles will visit v. (Compare Benjamini and Peres (1994)).
Now suppose that p satisfies $\theta^v(p) > 0$ and $\rho(G)pk < 1$. We claim that the transience of the corresponding BRW implies that the probability that two vertices x, y are in the same percolation cluster goes to zero as the distance from x to y tends to infinity. Indeed, suppose that there is an $\epsilon > 0$ and a sequence of vertices x_n, y_n with the distance $d(x_n, y_n)$ tending to infinity, but the probability that they are in the same cluster is greater than ϵ. That would mean that the BRW starting at x_n has probability at least ϵ to reach y_n and the BRW starting at y_n has probability at least ϵ to reach x_n. Consequently, with probability at least ϵ^2, the BRW starting at x_n will reach x_n again at some time after $d(x_n, y_n)$ steps. Since G is almost transitive, by passing to a subsequence and applying an automorphism of G, we may assume that all x_n are the same. This contradicts the transience of the BRW.
Let $r > 0$, and consider m balls in the graph with radius r. If r is large, then with probability arbitrarily close to 1 each of these balls will intersect an infinite open cluster. But the probability that any cluster will intersect two such balls goes to zero as the distances between the balls goes to infinity. Hence, with probability 1, there are more than m infinite open clusters. The theorem follows. ■

Suppose that G is not almost transitive, but has bounded degree. Then the above argument can be modified to show that for p as above, the probability of having at least m infinite open clusters is at least $\Big(\inf_v \theta^v(p)\Big)^m$.
Grimmett and Newman (1990) showed that $\mathbb{Z} \times$ (some regular tree) satisfies $p_c < p_u < 1$. We now show that $p_c < p_u$ for many products.

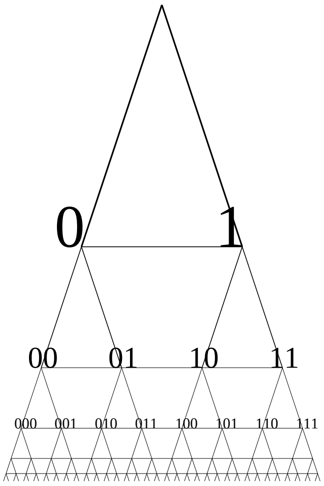

Figure 1: The enhanced binary tree.

Corollary 1 *Let G be an almost transitive graph. Then there is a $k_0 = k_0(G)$ such that the product $G \times T_k$ of G with the k-regular tree satisfies $p_c < p_u$ whenever $k \geq k_0$.*

Proof:
Let m be the maximal degree in G. Note that $p_c(G \times T_k) \leq p_c(T_k) = (k-1)^{-1}$, and the maximal degree in $G \times T_k$ is $m + k$. Observe that $\rho(G \times T_k) \to 0$ as $k \to \infty$. Hence, the corollary follows from the theorem. ■

If one wishes to drop the assumption that G is almost transitive (but still has bounded degree), then the above arguments show that for k sufficiently large, there is a p such that the probability of having more than one infinite open cluster in $G \times T_k$ is positive.

Following is a simple example of a planar graph satisfying $p_c < p_u < 1$. The planarity will make the analysis easy.

Example 1 *Consider the graph obtained by adding to the binary tree edges connecting all vertices of same level along a line (see Figure 1). To be more precise, represent the vertices of the binary tree by sequences of zeros and ones in the usual way, and add to the binary tree an edge between v, w if v and w are at level n and $|0.v - 0.w| = 1/2^n$, where $0.v$ is the number in $[0,1]$ represented by the sequence corresponding to v. Note that this graph is roughly isometric to a sector in the hyperbolic plane.*

Proposition 1 *For this graph, $p_c < 1-p_c \leq p_u(G) < 1$, and for p in the range $p \in (p_c, 1-p_c)$ there are, with probability 1, infinitely many infinite open clusters.*

Proof:
First note that G contains the graph G_0, obtained by adding to the binary tree edges only between vertices that neighbor in G and have the same grandmother. G_0 is a "periodic refinement" of the binary tree. By comparing the number of vertices in G_0 that are the the cluster of the root to a Galton-Watson branching process, it is easy to see that $p_c(G) \leq$

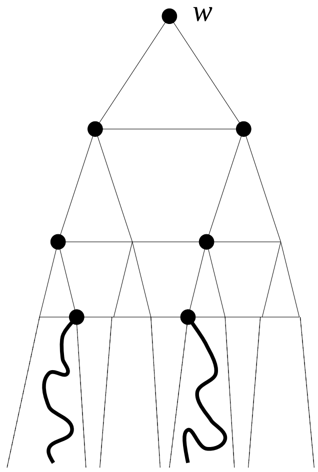

Figure 2: The infinite black clusters separate.

$p_c(G_0) < p_c(T_3) = 1/2$. Let $p \in (p_c, 1-p_c)$. For such p, both the open and the closed clusters percolate, and this is so in any subgraph of G spanned by the binary tree below any fixed vertex, as it is isomorphic to the whole graph. Pick some finite binary word w. Suppose that the vertices $w, w0, w00, w1, w10$ are closed, and each of $w00, w10$ percolates (in closed vertices) in the subgraph below it (see Figure 2). This implies that the open clusters intersecting the subgraph below $w01$ will be disjoint from those below $w11$, which gives non-uniqueness, because each of these subgraphs is sure to contain infinite open clusters. With probability 1, there is some such w. Hence, for $p \in (p_c, 1-p_c)$, there is no uniqueness, and $p_u \geq 1 - p_c$. It is easy to see that for $p \in (p_c, 1-p_c)$ there are, with probability 1, infinitely many infinite open clusters. The proof is completed by the following lemma, which gives $p_u < 1$. ∎

Lemma 1 *Let G be a bounded degree triangulation of a disk, or, more generally, the 1-skeleton of a (locally finite) tiling of a disk, where the number of vertices surrounding a tile is bounded. Then, for p sufficiently close to 1, there is a.s. at most 1 infinite open cluster.*

Proof:
Suppose that each tile is surrounded by at most k edges. Let G' be the k'th power of G; that is, $V(G') = V(G)$, and an edge appears in G' if the distance in G between its endpoints is at most k. Then G' has bounded degree, and therefore, for some $p^* > 0$ close to zero, there is no percolation in G'. Hence, $\mathbb{P}_{p^*}$ a.s., given any $n > 0$, there is a closed set of vertices in G' that separate a fixed basepoint from 'infinity', and all have distance at least n from the basepoint. If one now thinks of these as vertices in G, they contain the vertices of a loop separating the basepoint from infinity. The distance from this loop to the basepoint is arbitrarily large. Taking $p = 1 - p^*$, then shows that a.s. for $\mathbb{P}_p$ on G there are open loops separating the basepoint from infinity which are arbitrarily far away from the basepoint. Each infinite open cluster must intersect all but finitely many of these loops. Hence there is at most one infinite open cluster. ∎

Question 3 *Give general conditions that guarantee $p_u < 1$. For example, is $p_u < 1$ for any transitive graph with one end?*

The proof of the proposition suggests the following conjecture and question.

Conjecture 7 *Suppose G is planar, and the minimal degree in G is at least 7. Then at every p in the range $(p_c, 1-p_c)$, there are infinitely many infinite open clusters. Moreover, we conjecture that $p_c < 1/2$, so the above interval is nonempty.*

Such a graph has positive Cheeger constant, and spectral radius less than 1.

Conjecture 8 *Let G be planar, set $p = 1/2$ and assume that a.s. percolation occurs. Then a.s. there are infinitely many infinite open clusters.*

It is not clear if there is any analogous statement in the setting of bond percolation.
There is a large class of graphs in which we can prove the conjecture.

Theorem 5 *Let G be a planar graph, which is disjoint from the positive x axis, $\{(x,0) : x \geq 0\}$. Suppose that every bounded set in the plane meets finitely many vertices and edges of G. Set $p = 1/2$, and assume that almost surely percolation occurs in G. Then, almost surely, there are infinitely many infinite open clusters.*

Proof:
Let X be the collection of all infinite open or closed clusters. Suppose that X is finite, and let $R > 0$ be sufficiently large so that the disk $x^2 + y^2 < R^2$ intersects each cluster in X. For any $r > R$, and $A \in X$, let $t(A,r)$ be the least $t \in [0, 2\pi]$ such that the point $(r\cos t, r\sin t)$ is on an edge connecting two vertices in A (or is a vertex of A). Let $A, B \in X$ be distinct. Suppose that $R < r_1 < r_2$, $t(A,r_1) < t(B,r_1)$ and $t(A,r_2) < t(B,r_2)$. Take some r in the range $r_1 < r < r_2$. If $t(B,r) < t(A,r)$, then it follows that B is contained in the domain bounded by the arcs $\{(x,0) : x \in [r_1,r_2]\}$, $\{(r_1\cos t, r_1\sin t) : t \in [0, t(A,r_1)]\}$, $\{(r_2\cos t, r_2\sin t) : t \in [0, t(A,r_2)]\}$ and by A. This is impossible, because B has infinitely many vertices, and therefore, $t(A,r) < t(B,r)$. Consequently, the inequality between $t(A,s)$ and $t(B,s)$ changes at most once in the interval $R < s < \infty$. So either $t(A,s) > t(B,s)$ for all s sufficiently large, or $t(A,s) < t(B,s)$ for all s sufficiently large. In the latter case, we write $A < B$. It is clear that this defines a linear order on X.
Because X is finite, it has a minimal element. Let E be the event that the minimal element in X is an open cluster. By symmetry, $\mathbb{P}\{E\} = 1/2$. But Kolmogorov's 0-1 law implies that $\mathbb{P}\{E\}$ is either 0 or 1. The contradiction implies that X is infinite. Consequently, a.s. there are infinitely many open clusters, or there are infinitely many closed cluster. Consequently, there are infinitely many open clusters, again by Kolmogorov's 0-1 law. ■

Other questions regarding p_u are

Question 4 *Let G, G' be two Cayley graphs of the same group (or, more generally, two roughly isometric almost transitive graphs). Does $p_c(G) < p_u(G)$ imply $p_c(G') < p_u(G')$?*

Question 5 *Assume that G is an almost transitive graph. Is there uniqueness for every $p > p_u$? Is there uniqueness at $p = p_u$?*

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