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# SMOOTHNESS OF THE LAW OF THE SUPREMUM OF THE FRACTIONAL BROWNIAN MOTION

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*Abstract*

This note is devoted to prove that the supremum of a fractional Brownian motion with Hurst parameter $H \in (0,1)$ has an infinitely differentiable density on $(0,\infty)$. The proof of this result is based on the techniques of the Malliavin calculus.

## 1 Introduction

A fractional Brownian motion (fBm for short) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B = \{B_t, t \in [0,1]\}$ with the covariance function

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1}$$

Notice that if $H = \frac{1}{2}$, the process $B$ is a standard Brownian motion. From (1) it follows that

$$E|B_t - B_s|^2 = |t-s|^{2H},$$

and, as consequence, $B$ has $\alpha$-Hölder continuous paths for any $\alpha < H$.
The Malliavin calculus is a suitable tool for the study of the regularity of the densities of functionals of a Gaussian process. We refer to [7] and [8] for a detailed presentation of this theory. This approach is particularly useful when analytical methods are not available. In [5] the Malliavin calculus has been applied to derive the smoothness of the law of the supremum

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of the Brownian sheet. In order to obtain this result, the authors establish a general criterion for the smoothness of the density, assuming that the random variable is locally in $\mathbb{D}^{\infty}$. The aim of this paper is to study the smoothness of the law of the supremum of a fBm using the general criterion obtained in [5].
The organization of this note is as follows. In Section 2 we present some preliminaries on the fBm and we review the basic facts on the Malliavin calculus and on the fractional calculus that will be used in the sequel. In Section 3 we state the general criterion for the smoothness of densities and we apply it to the supremum of the fBm.

# 2 Preliminaries

## 2.1 Fractional Brownian motion

Fix $H \in (0,1)$ and let $B = \{B_t, t \in [0,1]\}$ be a fBm with Hurst parameter $H$. That is, $B$ is a zero mean Gaussian process with covariance function given by (1). Let $\{\mathcal{F}_t, t \in [0,1]\}$ be the family of sub-$\sigma$-fields of $\mathcal{F}$ generated by $B$ and the $P$-null sets of $\mathcal{F}$. We denote by $\mathcal{E} \subset \mathcal{H}$ the class of step functions on $[0,1]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(s,t).$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with $B$.
The covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r) K_H(s,r) dr,$$

where $K_H$ is a square integrable kernel given by (see [4]):

$$K_H(t,s) = \Gamma(H + \frac{1}{2})^{-1} (t-s)^{H-\frac{1}{2}} F(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}),$$

$F(a,b,c,z)$ being the Gauss hypergeometric function. Consider the linear operator $K_H^*$ from $\mathcal{E}$ to $L^2([0,1])$ defined by

$$(K_H^* \varphi)(s) = K_H(1,s)\varphi(s) + \int_s^1 (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r,s) dr. \tag{2}$$

For any pair of step functions $\varphi$ and $\psi$ in $\mathcal{E}$ we have (see [3])

$$\langle K_H^* \varphi, K_H^* \psi \rangle_{L^2([0,1])} = \langle \varphi, \psi \rangle_{\mathcal{H}}. \tag{3}$$

As a consequence, the operator $K_H^*$ provides an isometry between the Hilbert spaces $\mathcal{H}$ and $L^2([0,1])$. Hence, the process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t = B^H((K_H^*)^{-1} (\mathbf{1}_{[0,t]})) \tag{4}$$

is a Wiener process, and the process $B^H$ has an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s) dW_s, \tag{5}$$

because $\left(K_H^* \mathbf{1}_{[0,t]}\right)(s) = K_H(t,s)$.

### 2.2 Fractional calculus

We refer to [9] for a complete survey of the fractional calculus. Let us introduce here the main definitions. If $f \in L^1([0,1])$ and $\alpha > 0$, the right and left-sided fractional Riemann-Liouville integrals of $f$ of order $\alpha$ on $[0,1]$ are given almost surely for all $t \in [0,1]$ by

$$I_{0+}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \tag{6}$$

and

$$I_{1-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} f(s)\, ds \tag{7}$$

respectively, where $\Gamma$ denotes the Gamma function.

Fractional differentiation can be introduced as an inverse operation. For any $p > 1$ and $\alpha > 0$, $I_{0+}^{\alpha}(L^p)$ (resp. $I_{1-}^{\alpha}(L^p)$) will denote the class of functions $f \in L^p([0,1])$ which may be represented as an $I_{0+}^{\alpha}$(resp. $I_{1-}^{\alpha}$)- integral of some function $\Phi$ in $L^p([0,1])$. If $f \in I_{0+}^{\alpha}(L^p)$ (resp. $I_{1-}^{\alpha}(L^p)$), the function $\Phi$ such that $f = I_{0+}^{\alpha}\Phi$ (resp. $I_{1-}^{\alpha}\Phi$) is unique in $L^p([0,1])$ and is given by

$$D_{0+}^{\alpha} f(t) = \frac{(-1)^{\alpha+1}}{\Gamma(1-\alpha)} \left( \frac{f(s)}{s^{\alpha}} - \alpha \int_0^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}} ds \right) \tag{8}$$

$$\left( D_{1-}^{\alpha} f(t) = \frac{(-1)^{\alpha+1}}{\Gamma(1-\alpha)} \left( \frac{f(s)}{(1-s)^{\alpha}} - \alpha \int_t^1 \frac{f(s)-f(t)}{(s-t)^{\alpha+1}} ds \right) \right), \tag{9}$$

where the convergence of the integrals at the singularity $t = s$ holds in the $L^p$- sense.

When $\alpha p > 1$ any function in $I_{a+}^{\alpha}(L^p)$ is $\left(\alpha - \frac{1}{p}\right)$- Hölder continuous. On the other hand, any Hölder continuous function of order $\beta > \alpha$ has fractional derivative of order $\alpha$. That is, $C^{\beta}([a,b]) \subset I_{a+}^{\alpha}(L^p)$ for all $p > 1$.

Recall that by construction for $f \in I_{a+}^{\alpha}(L^p)$,

$$I_{a+}^{\alpha}(D_{a+}^{\alpha} f) = f$$

and for general $f \in L^1([a,b])$ we have

$$D_{a+}^{\alpha}(I_{a+}^{\alpha} f) = f.$$

The operator $K_H^*$ can be expressed in terms of fractional integrals or derivatives. In fact, if $H > \frac{1}{2}$, we have

$$(K_H^* \varphi)(s) = c_H \Gamma(H - \frac{1}{2}) s^{\frac{1}{2}-H} (I_{1-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \varphi(u))(s), \tag{10}$$

where $c_H = \left[ \frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right]^{1/2}$, and if $H < \frac{1}{2}$, we have

$$(K_H^* \varphi)(s) = d_H \; s^{\frac{1}{2}-H} (D_{1-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \varphi(u))(s), \tag{11}$$

where $d_H = c_H \Gamma(H + \frac{1}{2})$.

### 2.3 Malliavin calculus

We briefly recall some basic elements of the stochastic calculus of variations with respect to the fBm $B$. For more complete presentation on the subject, see [7] and [8].
The process $B = \{B_t, t \in [0,1]\}$ is Gaussian and, hence, we can develop a stochastic calculus of variations (or Malliavin calculus) with respect to it. Let $C_b^\infty(\mathbb{R})$ be the class of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f$ and all its partial derivatives are bounded. We denote by $\mathcal{S}$ the class of smooth cylindrical random variables $F$ of the form

$$F = f(B(h_1), \ldots, B(h_n)), \tag{12}$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ and $h_1$, ..., $h_n \in \mathcal{H}$.
The derivative operator $D$ of a smooth and cylindrical random variable $F$ of the form (12) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (B(h_1), \ldots, B(h_n)\, h_i.$$

In this way the derivative $DF$ is an element of $L^2(\Omega; \mathcal{H})$. The iterated derivative operator of $D$ is denoted by $D^k$. It is a closable unbounded operator from $L^p(\Omega)$ into $L^p\left(\Omega; \mathcal{H}^{\otimes k}\right)$ for each $k \geq 1$, and each $p \geq 1$. We denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\| F \|_{k,p}^p = E(|F|^p) + E \sum_{j=1}^{k} \left\| D^j F \right\|_{\mathcal{H}^{\otimes j}}^p .$$

We set $\mathbb{D}^\infty = \cap_{k,p} \mathbb{D}^{k,p}$.
For any given Hilbert space $V$, the corresponding Sobolev space of $V$-valued random variables can also be introduced. More precisely, let $\mathcal{S}_V$ denote the family of $V$-valued smooth random variables of the form

$$F = \sum_{j=1}^{n} F_j v_j, \quad (v_j, F_j) \in V \times \mathcal{S}.$$

We define

$$D^k F = \sum_{j=1}^{n} D^k F_j \otimes v_j, \;\; k \geq 1.$$

Then $D^k$ is a closable operator from $\mathcal{S}_V \subset L^p(\Omega; V)$ into $L^p(\Omega; \mathcal{H}^{\otimes k} \otimes V)$ for any $p \geq 1$. For any integer $k \geq 1$ and for any real number $p \geq 1$, a norm is defined on $\mathcal{S}_V$ by

$$\|F\|_{k,p,V}^p = E(\|F\|_V^p) + \sum_{j=1}^{k} E\left( \left\| D^j F \right\|_{\mathcal{H}^{\otimes j} \otimes V}^p \right).$$

We denote by $\mathbb{D}^{k,p}(V)$ the completion of $\mathcal{S}_V$ with respect to the norm $\|.\|_{k,p,V}$. We set $\mathbb{D}^\infty(V) = \cap_{k,p} \mathbb{D}^{k,p}(V)$.
Our main result will be based on the application of the following general criterion for smoothness of densities for one-dimensional random variable established in [5].

**Theorem 1** *Let $F$ be a random variable in $\mathbb{D}^{1,2}$. Let $A$ be an open subset of $\mathbb{R}$. Suppose that there exist an $\mathcal{H}$-valued random variable $u_A$ and a random variable $G_A$ such that* *(i)* $u_A \in \mathbb{D}^{\infty}(\mathcal{H})$,

*(ii)* $G_A \in \mathbb{D}^{\infty}$ *and* $G_A^{-1} \in L^p(\Omega)$ *for any* $p \geq 2$ *and,*

*(iii)* $\langle DF, u_A \rangle_{\mathcal{H}} = G_A$ *on* $\{F \in A\}$.

*Then the random variable* $F$ *possesses an infinitely differentiable density on the set* $A$.

## 3 Supremum of the fractional Brownian motion

The process $B$ has a version with continuous paths as result of being $\alpha$-Hölder continuous for any $\alpha < H$. Set

$$M = \sup_{0 \leq s \leq 1} B_s.$$

From results of [10] we know that $M$ possesses an absolutely continuous density on $(0, \infty)$. In order to apply Theorem 1, we will first recall some results on this supremum .

**Lemma 2** *The process* $B$ *attains its maximum on a unique random point* $T$.

**Proof.** The proof of this lemma would follow by the same arguments as the proof of Lemma 3.1 of [5], applying the criterion for absolute continuity of the supremum of a Gaussian process established in [10]. ■

The following lemma will ensure the weak differentiability of the supremum of the fBm and give the value of its derivative.

**Lemma 3** *The random variable* $M$ *belongs to* $\mathbb{D}^{1,2}$ *and it holds* $D_t M = \mathbf{1}_{[0,T]}(t)$, *for any* $t \in [0,1]$, *where* $T$ *is the point where the supremum is attained.*

**Proof.** Similar to the proof of Lemma 3.2. in [5]. ■

With the above results in hands, we are in position to prove our main result.

**Proposition 4** *The random variable* $M = \sup_{0 \leq s \leq 1} B_s$ *possesses an infinitely differentiable density on* $(0, \infty)$.

**Proof.** Fix $a > 0$ and set $A = (a, \infty)$. Define the following random variable

$$T_a = \inf \left\{ t \in [0,1] \text{ such that } \sup_{0 \leq s \leq t} B_s > a \right\}.$$

Recall that $T_a$ is a stopping time with respect to the filtration $\{\mathcal{F}_t, t \in [0,1]\}$ and notice that $T_a \leq T$ on the set $\{M > a\}$. Hence, by Lemma 3, it holds that

$$\{M > a, t \leq T_a\} \subset \{D_t M = 1\}. \tag{13}$$

Set

$$\Delta = \left\{ (p, \gamma) \in \mathbb{N}^* \times (0, \infty) \text{ such that } \frac{1}{2p} < \gamma < H \right\}.$$

For any $(p, \gamma) \in \Delta$, we define the process $Y$ on $[0,1]$ by setting, for any $t \in [0,1]$

$$Y_t = \int_0^t \int_0^t \frac{|B_s - B_r|^{2p}}{|s - r|^{2p\gamma + 1}} ds dr.$$

We will need the following property: There exists a constant $R$ depending on $a, \gamma$ and $p$ such that

$$Y_t < R \text{ implies that } \sup_{0\leq s\leq t} B_s \leq a. \tag{14}$$

To prove this fact we use the Garsia, Rodemich and Rumsey Lemma in [6]. This lemma applied to the function $s \in [0,t] \to B_s$, with the hypothesis that $Y_t < R$, implies

$$|B_s - B_r| \leq C_{p,\gamma} R^{\frac{1}{2p}} |s-r|^{\gamma - \frac{1}{2p}} \text{ for all } s, r \text{ in } [0,t].$$

This implies that $\sup_{0\leq s\leq t} |B_s| \leq C_{p,\gamma} R^{\frac{1}{2p}}$ . It suffices to choose $R$ in such a way that $C_{p,\gamma} R^{\frac{1}{2p}} < a$.
Let $\psi : \mathbb{R}^+ \to [0,1]$ be an infinitely differentiable function such that

$$\psi(x) = \begin{cases} 0 & \text{if } \; x > R, \\ \psi(x) \in [0,1] & \text{if } \; x \in \left[\frac{R}{2}, R\right], \\ 1 & \text{if } \; x \leq \frac{R}{2}. \end{cases}$$

Consider the $\mathcal{H}$-valued random variable given by

$$u_A = (K_H^*)^{-1} \left(K_H^{*,adj}\right)^{-1} (\psi(Y_\cdot)), \tag{15}$$

where $K_H^*$ is the operator defined in (2) and $K_H^{*,adj}$ denotes its adjoint in $L^2([0,1])$. We claim that the random element $u_A$ introduced in (15) and the random variable $G_A = \int_0^1 \psi(Y_t)\, dt$ satisfy the conditions of Theorem 1.
Let us first show that $u_A$ belongs to $\mathbb{D}^\infty(\mathcal{H})$. Fix an integer $j \geq 0$. It suffices to show that for any $q \geq 1$,

$$E \left\| D^j u_A \right\|^q_{\mathcal{H}^{\otimes(j+1)}} < \infty. \tag{16}$$

The $j$-th order derivative $D^j$ of the function $\psi(Y_t)$ is evaluated with the help of the Faà di Bruno formula, see formula [24.1.2] in [1], as follows

$$D^j \psi(Y_t) = \sum_{n=1}^{j} \psi^{(n)}(Y_t) \sum_{i, l_i : \sum_{i=1}^j l_i = n, \sum_{i=1}^j i l_i = j} \prod_{i=1}^{j} \frac{1}{i!} \left(\frac{D^i Y_t}{l_i!}\right)^{l_i}.$$

Hence, in order to show (16) it suffices to check that

$$E \left\| (K_H^*)^{-1} \left(K_H^{*,adj}\right)^{-1} \left[\psi^{(n)}(Y_t) \prod_{i=1}^{j} \left(D^i Y_t\right)^{l_i}\right] \right\|^q_{\mathcal{H}^{\otimes(j+1)}} < \infty. \tag{17}$$

for all $1 \leq n \leq j$, $\sum_{i=1}^j l_i = n, \sum_{i=1}^j i l_i = j$. Set

$$\Lambda_t = \psi^{(n)}(Y_t) \prod_{i=1}^{j} \left(D^i Y_t\right)^{l_i}.$$

By (3)

$$\left\| (K_H^*)^{-1} \left(K_H^{*,adj}\right)^{-1} \Lambda_t \right\|_{\mathcal{H}^{\otimes(j+1)}} = \left\| \left(K_H^{*,adj}\right)^{-1} \Lambda_t \right\|_{\mathcal{H}^{\otimes j} \otimes L^2([0,1])}. \tag{18}$$

From (10), if $H > \frac{1}{2}$, we obtain

$$\begin{aligned}\left(K_H^{*,adj}\right)^{-1}\Lambda_t &= d_H t^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} t^{\frac{1}{2}-H}\Lambda_t \\ &= \frac{d_H}{\Gamma\left(\frac{3}{2}-H\right)}\left(t^{\frac{1}{2}-H}\Lambda_t - \left(H-\frac{1}{2}\right)t^{H-\frac{1}{2}}\int_0^t \frac{t^{\frac{1}{2}-H}\Lambda_t - s^{\frac{1}{2}-H}\Lambda_s}{(t-s)^{H+\frac{1}{2}}}ds\right)\end{aligned}$$

where $d_H = \left(c_H\Gamma(H-\frac{1}{2})\right)^{-1}$. After some computations we get

$$\left(K_H^{*,adj}\right)^{-1}\Lambda_t = \beta(t)\Lambda_t + \int_0^t R(t,\theta)\Lambda'_\theta d\theta, \tag{19}$$

where

$$\beta(t) = \frac{d_H}{\Gamma\left(\frac{3}{2}-H\right)}\left(t^{\frac{1}{2}-H} - \left(H-\frac{1}{2}\right)t^{H-\frac{1}{2}}\int_0^t \frac{t^{\frac{1}{2}-H} - s^{\frac{1}{2}-H}}{(t-s)^{H+\frac{1}{2}}}ds\right),$$

and

$$R(t,\theta) = -\frac{d_H\left(H-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}\int_0^\theta s^{\frac{1}{2}-H}(t-s)^{-H-\frac{1}{2}}ds.$$

On the other hand, if $H < \frac{1}{2}$, from (11) we obtain

$$\left(K_H^{*,adj}\right)^{-1}\Lambda_t = e_H t^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} t^{\frac{1}{2}-H}\Lambda_t, \tag{20}$$

where $e_H = \left(c_H\Gamma(H+\frac{1}{2})\right)^{-1}$.
In the sequel $C_H$ will denote a generic constant depending on $H$. If $H > \frac{1}{2}$, (19) yields

$$\begin{aligned}\left\|\left(K_H^{*,adj}\right)^{-1}\Lambda_t\right\|^2_{\mathcal{H}^{\otimes j}\otimes L^2([0,1])} &= \left\|\beta(t)\Lambda_t + \int_0^t R(t,\theta)\Lambda'_\theta d\theta\right\|^2_{\mathcal{H}^{\otimes j}\otimes L^2([0,1])} \\ &\leq 2\int_0^1 \beta(t)^2 \left\|\Lambda_t\right\|^2_{\mathcal{H}^{\otimes j}} dt \\ &\quad + C_H\int_0^1 \left\|\Lambda'_t\right\|^2_{\mathcal{H}^{\otimes j}} dt, \end{aligned}\tag{21}$$

and for $H < \frac{1}{2}$, (20) yields

$$\left\|\left(K_H^{*,adj}\right)^{-1}\Lambda_t\right\|^2_{\mathcal{H}^{\otimes j}\otimes L^2([0,1])} \leq C_H\int_0^1 \left\|\Lambda_t\right\|^2_{\mathcal{H}^{\otimes j}} dt. \tag{22}$$

We have

$$\left\|\Lambda_t\right\|_{\mathcal{H}^{\otimes j}} \leq \prod_{i=1}^j \left\|D^i Y_t\right\|^{l_i}_{\mathcal{H}^{\otimes i}}. \tag{23}$$

Taking into account that

$$D^i Y_t = \int_{[0,t]^2} \frac{(B_r - B_s)^{2p-i}}{|r-s|^{2p\gamma+1}} \mathbf{1}_{[r,s]^i} drds,$$

we obtain

$$\left\|D^i Y_t\right\|_{\mathcal{H}^{\otimes i}} \leq \int_{[0,t]^2} \frac{|B_r - B_s|^{2p-i}}{|r-s|^{2p\gamma+1-iH}} drds,$$

and this implies that

$$\sup_{0\leq t\leq 1} E\left\|D^i Y_t\right\|_{\mathcal{H}^{\otimes i}}^q < \infty, \tag{24}$$

for any $q \geq 1$.
On the other hand, from

$$\begin{aligned}
\Lambda_t' &= \frac{d}{dt}\left(\psi^{(n)}\left(Y_t\right)\prod_{i=1}^{j}\left(D^i Y_t\right)^{l_i}\right) \\
&= \psi^{(n)}\left(Y_t\right)\sum_{m=1}^{j} l_m\left(D^m Y_t\right)^{l_m-1} D^m Y_t' \prod_{\substack{i=1\\ i\neq m}}^{j}\left(D^i Y_t\right)^{l_i} \\
&\quad +\psi^{(n+1)}\left(Y_t\right) Y_t' \prod_{i=1}^{j}\left(D^i Y_t\right)^{l_i}
\end{aligned}$$

we get

$$\begin{aligned}
\left\|\Lambda_t'\right\|_{\mathcal{H}^{\otimes j}} &\leq \sum_{m=1}^{j} l_m \left\|D^m Y_t\right\|_{\mathcal{H}^{\otimes m}}^{l_m-1}\left\|D^m Y_t'\right\|_{\mathcal{H}^{\otimes m}} \prod_{\substack{i=1\\ i\neq m}}^{j}\left\|D^i Y_t\right\|_{\mathcal{H}^{\otimes i}}^{l_i} \\
&\quad + \left|Y_t'\right| \prod_{i=1}^{j}\left\|D^i Y_t\right\|_{\mathcal{H}^{\otimes i}}^{l_i}.
\end{aligned} \tag{25}$$

From

$$D^i Y_t' = \int_0^t \frac{(B_t - B_s)^{2p-i}}{|t-s|^{2p\gamma+1}} \mathbf{1}_{[t,s]^i} ds,$$

we obtain

$$\left\|D^i Y_t'\right\|_{\mathcal{H}^{\otimes i}} \leq \int_0^t \frac{|B_t - B_s|^{2p-i}}{|t-s|^{2p\gamma+1-iH}} ds,$$

and this implies that

$$\sup_{0\leq t\leq 1} E\left\|D^i Y_t'\right\|_{\mathcal{H}^{\otimes i}}^q < \infty, \tag{26}$$

for any $q \geq 1$.
Finally, (24), (23), (21), (22), (18), (26) and (25) imply (17). This shows condition (i) of Theorem 1.
In order to show condition (iii) notice that

$$\begin{aligned}
\left\langle DM, u_A\right\rangle_{\mathcal{H}} &= \left\langle \mathbf{1}_{[0,T]}, u_A\right\rangle_{\mathcal{H}} = \left\langle K_H^* \mathbf{1}_{[0,T]}, K_H^* u_A\right\rangle_{L^2([0,1])} \\
&= \left\langle \mathbf{1}_{[0,T]}, K_H^{*,adj} K_H^* u_A\right\rangle_{L^2([0,1])} \\
&= \int_0^T \psi\left(Y_t\right) dt.
\end{aligned}$$

On the other hand, on the set $\{M > a\}$, taking into account (13) and (14), it holds that

$$\psi(Y_t) > 0 \Longrightarrow t \leq T,$$

and, as a consequence, $\int_0^T \psi(Y_t)\, dt = G_A$.
Finally, it remains to show condition (ii), that is, $G_A^{-1} \in L^q(\Omega)$ for any $q \geq 2$. We have

$$\begin{aligned} G_A &\geq \int_0^1 \psi(Y_t)\, \mathbf{1}_{\left\{Y_t < \frac{R}{2}\right\}} dt \\ &= \int_0^1 \mathbf{1}_{\left\{Y_t < \frac{R}{2}\right\}} dt \\ &= \lambda\left\{t \in [0,1] : Y_t < \frac{R}{2}\right\} \\ &= Y_t^{-1}\left(\frac{R}{2}\right), \end{aligned}$$

because $Y$ is non-decreasing and is continuous. For any $\varepsilon > 0$ we get

$$\begin{aligned} P\left(Y_t^{-1}\left(\frac{R}{2}\right) < \varepsilon\right) &= P\left(\frac{R}{2} < Y_\varepsilon\right) \\ &\leq \left(\frac{2}{R}\right)^p E\left|Y_\varepsilon\right|^p \\ &\leq \left(\frac{2}{R}\right)^p \left[\int_{[0,\varepsilon]^2} \frac{\left\| |B_r - B_s|^{2p} \right\|_{L^p(\Omega)}}{|r-s|^{2p\gamma+1}} drds\right]^p, \\ &\leq R^{-p} C_p \left[\int_{[0,\varepsilon]^2} |r-s|^{2pH-2p\gamma-1}\, drds\right]^p, \\ &= R^{-p} C_p \varepsilon^{(2p(H-\gamma)+1)p}. \end{aligned}$$

This completes the proof of the proposition. ■

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