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ON UNIQUENESS OF A SOLUTION OF $Lu = u^\alpha$ WITH GIVEN TRACE

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Abstract
A boundary trace (Γ, ν) of a solution of $\Delta u = u^\alpha$ in a bounded smooth domain in $\mathbb{R}^d$ was first constructed by Le Gall [12] who described all possible traces for $\alpha = 2, d = 2$ in which case a solution is defined uniquely by its trace. In a number of publications, Marcus, Véron, Dynkin and Kuznetsov gave analytic and probabilistic generalization of the concept of trace to the case of arbitrary $\alpha > 1, d \geq 1$. However, it was shown by Le Gall [13] that the trace, in general, does not define a solution uniquely in case $d \geq (\alpha+1)/(\alpha-1)$. He offered a sufficient condition for the uniqueness and conjectured that a uniqueness should be valid if the singular part Γ of the trace coincides with the set of all explosion points of the measure ν. Here, we establish a necessary condition for the uniqueness which implies a negative answer to the above conjecture.

1 Introduction and Results

1.1 Moderate solutions

Let L be a second order uniformly elliptic differential operator with smooth coefficients in $\mathbb{R}^d$ and let $E \subset \mathbb{R}^d$ be a bounded smooth domain. We consider a class $\mathcal{U}$ of all positive solutions of the equation

$$Lu = u^\alpha \quad \text{in } E \tag{1.1}$$

where $\alpha \in (1, 2]$ is a parameter. A solution u is called *moderate* if $u \leq h$ for an L-harmonic function h. The class of all moderate solutions is denoted by $\mathcal{U}_1$.
For every moderate solution u, there exists a minimal L-harmonic function that dominates u. It is called the minimal (L-harmonic) majorant of u. A solution u can be recovered from

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its majorant as the maximal solution to (1.1) dominated by h. Moreover, u is related to its minimal majorant h by the integral equation

$$u(x) + \Pi_x \int_0^\zeta u^\alpha(\xi_s)\, ds = h(x). \tag{1.2}$$

Here (ξ_t, Π_x) is the corresponding L-diffusion in E and ζ is its life time. See [5] for more detail. Every positive L-harmonic function h has a unique representation

$$h(x) = \int_{\partial E} k(x,y)\nu(dy) \tag{1.3}$$

where $k(x,y)$ is the Poisson kernel for L in E and ν is a finite measure on ∂E. We denote by h_ν the function given by (1.3). For a moderate solution $u \in \mathcal{U}_1$, we write $u = u_\nu$ if h_ν is the minimal majorant of u.

1.2 Superdiffusions and stochastic boundary values

An (L,α)-superdiffusion is a probabilistic model for an evolution of a random cloud of branching particles. A spatial movement of particles is described by an L-diffusion, and $\alpha \in (1,2]$ characterizes branching. See, for instance, [2]. To every open set D there corresponds a random measure (X_D, P_μ) on ∂D, called the exit measure from D. It represents the total accumulation of mass on ∂D assuming that the evolution starts from μ and particles are instantly frozen if they reach the complement of D. Relations between X_D and equation (1.1) can be described as follows. Let f be a positive continuous function on ∂E. The function

$$u(x) = -\log P_x e^{-\langle f, X_E\rangle}, \tag{1.4}$$

where P_x stands for P_{δ_x}, is the only solution of the boundary value problem

$$\begin{aligned} Lu &= u^\alpha \quad \text{in } E,\\ u &= f \quad \text{on } \partial E. \end{aligned} \tag{1.5}$$

An arbitrary solution u of (1.1) can also be represented in a form similar to (1.4) in terms of its *stochastic boundary value* Z_u (cf. [3]). It can be defined as a limit

$$Z_u = \lim\langle u, X_{D_n}\rangle \tag{1.6}$$

where D_n is an increasing sequence of bounded smooth domains approximating E. A solution u can be recovered from its stochastic boundary value by the formula

$$u(x) = -\log P_x e^{-Z_u}. \tag{1.7}$$

We write Z_ν instead of Z_{u_ν}. See [3] for more detail.
We define the range $\mathcal{R}$ of a superdiffusion in E as the minimal closed set that supports all X_D for $D \subset E$. A set $\Gamma \subset \partial E$ is called a polar set for the superdiffusion if, for any x, $P_x\{\mathcal{R} \cap \Gamma \neq \emptyset\} = 0$. According to [6], the class of polar sets coincides with the class of all removable boundary singularities for the equation (1.1). By [4], the equation (1.2) has a solution if and only if the corresponding measure ν does not charge polar sets. Therefore the mapping $\nu \to u_\nu$ defines a 1-1 correspondence between the class $\mathcal{N}_1$ of all finite measures on

∂E which don't charge polar sets and the class $\mathcal{U}_1$ of all moderate solutions of (1.1); see [5], [4], [9], [7].
For every Borel subset $B \subset \partial E$,

$$w_B(x) = -\log P_x\{\mathcal{R} \cap B = \emptyset\} \tag{1.8}$$

is a solution of (1.1). Its stochastic boundary value is given by the formula $Z_B = Z_{w_B} = \infty 1_{\{\mathcal{R}\cap B \neq \emptyset\}}$. If B is closed, then w_B is the maximal solution of (1.1) such that $w_B = 0$ on $\partial E \setminus B$. See [3], Sect. 6.

1.3 σ-moderate solutions

A solution u of (1.1) is called σ-moderate if there exists an increasing sequence of moderate solutions u_n such that $u_n \uparrow u$ as $n \to \infty$. It follows from (1.2) that the corresponding measures ν_n also increase to some measure ν. The measure ν does not charge polar sets, but it may be not finite and not even σ-finite. However, it is always Σ-finite. We denote by $\mathcal{N}_0$ the class of all Σ-finite measures that don't charge polar sets. Every measure $\nu \in \mathcal{N}_0$ can be represented as a limit of an increasing sequence of finite measures ν_n and therefore defines a σ-moderate solution $u = \lim u_{\nu_n}$. We denote this solution by u_ν and we write Z_ν for its stochastic boundary value. (It follows from [9], Theorem 4.2 that u_ν and Z_ν do not depend on the choice of $\nu_n \uparrow \nu$.) Every σ-moderate solution can be represented this way. However, in contrast to moderate solutions, this representation is not unique. σ-moderate solutions have been studied in Section 4 of [9] by means of continuous linear additive functionals.
The class of all σ-moderate solutions is denoted by $\mathcal{U}_0$. Existence of non-σ-moderate solutions remains an open question: all known elements of $\mathcal{U}$ either belong to $\mathcal{U}_0$ or, at least, it is not proved that this is not true. See [11], [7].

1.4 Sweeping and the trace

First definition of the trace was introduced by Le Gall [12], [14], [13], who used it to describe all solutions of the equation $\Delta u = u^2$ in a smooth planar domain. In a more general setting, a definition of a trace was introduced by Marcus and Véron [15], [16], [17], [18] and, in a probabilistic way, by Dynkin and Kuznetsov [9], [8].
Let $u \in \mathcal{U}$. For a closed set $B \subset \partial E$, we define $Q_B(u)$ as the maximal element of $\mathcal{U}$ such that $Q_B(u) \le u$ and $Q_B(u) = 0$ on $\partial E \setminus B$. We consider the maximal open subset O of ∂E such that $Q_B(u)$ is moderate for every compact $B \subset O$ and we set $\Gamma = O^c$. It can be shown that there exists a Radon measure ν on O such that $Q_B(u) = u_{\nu_B}$ for every compact $B \subset O$ where ν_B stands for the restriction of ν to B. The pair (Γ, ν) is called the trace of u. Cf. [9].
Let ν be a measure on ∂E. A point $x \in \partial E$ is called an explosion point for ν if $\nu(O) = \infty$ for every open set O containing x. The collection of all explosion points of ν is denoted by $Ex(\nu)$. Clearly, $Ex(\nu)$ is a closed set. Let Γ be a closed subset of ∂E and ν be a Radon measure on Γ^c not charging polar sets. The pair (Γ, ν) is called *normal* if there exists no nontrivial relatively open polar subset $B \subset \Gamma \setminus Ex(\nu)$.

Proposition 1.1 (See [9]). *The trace (Γ, ν) of a solution $u \in \mathcal{U}$ is always a normal pair. Each normal pair (Γ, ν) is the trace of some solution u. The maximal solution with the given trace (Γ, ν) is given by the formula*

$$w_{\Gamma,\nu}(x) = -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-Z_\nu}\}. \tag{1.9}$$

1.5 Essential explosion points

For an arbitrary Borel set $B \subset \partial E$, put

$$\mathrm{Cap}_R(B) = P_c\{\mathcal{R} \cap B \neq \emptyset\} \tag{1.10}$$

where c is a reference point and $\mathcal{R}$ is the range of the (L, α)-superdiffusion in E. According to [1], Theorem III.32, $\mathrm{Cap}_R(B)$ is a Choquet capacity. By [3], Sect. 6.2, $\mathrm{Cap}_R(B) = 0$ if and only if B is polar.
Let $x \in Ex(\nu)$. We call x a point of *non-essential explosion* if there exists a neighborhood U of x and a sequence of open sets $O_n \subset U$ such that $\mathrm{Cap}_R(O_n) \downarrow 0$ as $n \to \infty$ and $\nu(U \setminus O_n) < \infty$ for all n. Otherwise x is called a point of *essential explosion*. We denote the set of all essential explosion points by $Ess(\nu)$. Note that $Ex(\nu) = Ess(\nu)$ if single points on the boundary are not polar (this happens if $d < (\alpha+1)/(\alpha-1)$; see [10], [6]).
Properties of $Ess(\nu)$ can be summarized as follows.

Theorem 1.1. *Let ν be a Σ-finite measure that doesn't charge polar sets. Then:*

(i) The set $Ess(\nu)$ is always closed;

(ii) There exist open sets $U_n \supset Ess(\nu)$ such that $\mathrm{Cap}_R(U_n) \downarrow \mathrm{Cap}_R(Ess(\nu))$ and $\nu(U_n^c) < \infty$;

(iii) $Ess(\nu)$ is either empty or non-polar;

(iv) ν is σ-finite on the complement of $Ess(\nu)$.

The following result clarifies the difference between $Ex(\nu)$ and $Ess(\nu)$.

Theorem 1.2. *Let $d \geq (\alpha+1)/(\alpha-1)$. For every closed set $\Gamma \subset \partial E$, there exists a σ-finite measure $\nu \in \mathcal{N}_0$ such that $Ex(\nu) = \Gamma$ and $Ess(\nu)$ is empty. The measure ν can be chosen to have a density with respect to the surface measure.*

Main result of the paper is given by the following

Theorem 1.3. *Let $d \geq (\alpha+1)/(\alpha-1)$ and let (Γ, ν) be a normal pair. If $\Gamma \setminus Ess(\nu)$ is not polar, then there exist at least two solutions with the trace (Γ, ν).*

Remarks. 1. For $\nu = 0$ this is, essentially, Proposition 3 in [13] (to be precise, [13] is devoted to the initial trace of the corresponding semilinear parabolic equation. However, the arguments can be easily extended to an elliptic case.)
2. Le Gall proved in [13] that the uniqueness takes place if Γ is polar and that there is no uniqueness if $\nu = 0$ and Γ is not polar. He also conjectured that the uniqueness is valid if $\Gamma = Ex(\nu)$. The following example shows that it is not true. Let Γ be a non-polar closed subset of ∂E with the surface measure 0 and let ν be the measure constructed in Theorem 1.2. Since ν has the density with respect to the surface measure, $\nu(\Gamma) = 0$. By Theorem 1.2, $\Gamma = Ex(\nu)$ and therefore (Γ, ν) is a normal pair. However, $Ess(\nu)$ is empty. By Theorem 1.3, there exist at least two solutions with the trace (Γ, ν). This implies a negative answer to the above conjecture.

1.6 Remarks on fine trace

To overcome the difficulties related to the non-uniqueness, Dynkin and Kuznetsov have defined in [11], [7] a fine trace of a solution and they have shown that σ-moderate solutions are defined uniquely by their fine traces. The fine trace is a pair (Γ, ν) where Γ is closed in a certain topology (fine topology) related to the equation (1.1) and ν is a σ-finite measure on the complement of Γ not charging polar sets. See [11], [7] for all details.

Theorem 1.4. *Let ν be a Σ-finite measure that doesn't charge polar sets. If $Ess(\nu)$ is empty, then the fine trace of u_ν is equal to (Γ, ν) with polar Γ.*

By applying this result to measures ν constructed in Theorem 1.2, we see that the corresponding solutions u_ν are not determined by their traces and they can be recovered from their fine traces.

2 Proofs

2.1 Some Lemmas

We start with an important lemma.

Lemma 2.1. *Let $u \leq v$ be two solutions of (1.1). If $u(c) = v(c)$ at some interior point $c \in E$, then $u = v$ everywhere in E.*

Proof. It is sufficient to show that $u = v$ in any bounded smooth domain D such that $c \in D$ and $\bar{D} \subset E$. Solutions u and v are bounded and continuous in D and therefore they admit a representation

$$u(x) = -\log P_x e^{-\langle u, X_D \rangle}, \qquad v(x) = -\log P_x e^{-\langle v, X_D \rangle}. \tag{2.1}$$

Since $u \leq v$, we conclude from (2.1) that $\langle u, X_D \rangle = \langle v, X_D \rangle$ P_c-a.s. Therefore

$$\Pi_c u(\xi_{\tau_D}) = P_c \langle u, X_D \rangle = P_c \langle v, X_D \rangle = \Pi_c v(\xi_{\tau_D})$$

and $u = v$ on ∂D. By (2.1), this yields $u = v$ in D. □

As a next step, we compute the trace of w_B for Borel B.

Lemma 2.2. *Let B be a Borel subset of ∂E. The trace of w_B is equal to $(\Gamma, 0)$ where Γ is the smallest closed set such that $B \setminus \Gamma$ is polar.*

Proof. Let (Γ, ν) be the trace of w_B. Suppose $B \setminus \Gamma$ is not polar. There exists a non-polar compact $K \subset B \setminus \Gamma$. Since $K \subset B$, $w_B \geq w_K$ and therefore the sweeping $Q_K(w_B) \geq Q_K(w_K) = w_K$. But, w_K is not moderate.
Suppose now that a closed $\tilde{\Gamma}$ is such that $B \setminus \tilde{\Gamma}$ is polar. Then $w_B \leq w_{\tilde{\Gamma}}$ by (1.8) and therefore $Q_K(w_B) \leq Q_K(w_{\tilde{\Gamma}})$ for every K. This implies $Q_K(w_B) = 0$ for every compact $K \subset \tilde{\Gamma}^c$, and therefore $\Gamma \subset \tilde{\Gamma}$. Same argument applied to Γ instead of $\tilde{\Gamma}$ shows that $\mu = 0$. □

Lemma 2.3. *Let $\nu \in \mathcal{N}_0$. The trace of u_ν is equal to (Γ, μ) where $\Gamma = Ex(\nu)$ and μ coincides with the restriction of ν to Γ^c.*

Proof. Let $B \subset \partial E$ be a compact. Clearly, $u_\nu \geq u_{\nu_B}$ where ν_B is the restriction of ν to B. Therefore

$$Q_B(u_\nu) \geq Q_B(u_{\nu_B}) \geq u_{\nu_B}. \tag{2.2}$$

Suppose now that $B \cap Ex(\nu) = \emptyset$. Show that

$$Q_B(u_\nu) = u_{\nu_B} \tag{2.3}$$

for such B. There exists a relatively open $U \subset \partial E$ such that $B \subset U$ and $\nu(U) < \infty$. Let λ and κ be the restrictions of ν to U and U^c, respectively. Let $u_1 = u_\lambda$ and $u_2 = u_\kappa$. By [3], Theorem 2.3,

$$u \leq u_1 + u_2$$

and therefore

$$Q_B(u) \leq Q_B(u_1) + Q_B(u_2). \tag{2.4}$$

However, solution u_1 is moderate and therefore $Q_B(u_1) = u_{\lambda_B} = u_{\nu_B}$. On the other hand, κ vanishes on U and therefore $Q_B(u_2) = 0$ by [9], 4.4.A and Theorem 3.1. Combining (2.2) and (2.4), we get (2.3).

The statement of the lemma follows from (2.2), (2.3) and the definition of the trace. □

Lemma 2.4. *For every Borel set $B \subset \partial E$, and every $\nu \in \mathcal{N}_0$,*

$$w_{B,\nu}(x) = -\log P_x\{\mathcal{R} \cap B = \emptyset, e^{-Z_\nu}\}$$

is a solution of (1.1)*. Its trace (Γ, μ) can be characterized by the following properties. The set Γ is the smallest closed set such that $\Gamma \supset Ex(\nu)$ and $B \setminus \Gamma$ is polar, and μ is the restriction of ν to Γ^c.*

Proof. Note that $w_{B,\nu}(x) = -\log P_x e^{-Z_B - Z_\nu}$ and therefore the first part follows easily from Theorems 2.3 and 6.1 in [3]. Second part follows easily from Lemmas 2.2 and 2.3 and from the inequalities

$$w_B \leq w_{B,\nu}, \quad u_\nu \leq w_{B,\nu}, \quad w_{B,\nu} \leq u_\nu + w_B$$

(for the last inequality, see, e.g., [3], Theorem 2.3). □

Lemma 2.5. *Let Γ be a closed subset of ∂E and let $B \supset \Gamma$ be such that $B \setminus \Gamma$ is not polar. Then* $\mathrm{Cap}_R(B) > \mathrm{Cap}_R(\Gamma)$.

Proof. Suppose $\mathrm{Cap}_R(B) = \mathrm{Cap}_R(\Gamma)$. Then $w_B(c) = w_\Gamma(c)$ and therefore $w_B = w_\Gamma$ everywhere by Lemma 2.1. By assumption, there exists a compact $K \subset B \setminus \Gamma$ such that $\mathrm{Cap}_R(K) > 0$. Clearly, $w_K \leq w_B$ and therefore $w_K \leq w_\Gamma$. However, $w_K = 0$ on $\partial E \setminus K$ and $w_\Gamma = 0$ on $\partial E \setminus \Gamma$, which implies $w_K = 0$ on ∂E and therefore $w_K = 0$ in E, that is $\mathrm{Cap}_R(K) = 0$. □

2.2 Proof of Theorem 1.1

1°. Let $x \notin Ess(\nu)$. If $x \notin Ex(\nu)$, then there exists an open set $U \subset \partial E$ such that $x \in U$ and $\nu(U) < \infty$. By definition of explosion points, all points of U do not belong to $Ex(\nu) \supset Ess(\nu)$. Suppose now that x is a point of non-essential explosion and U is as in definition. Clearly, any $y \in Ex(\nu) \cap U$ must also be a point of non-essential explosion. Therefore each point $x \notin Ess(\nu)$ has a neighborhood that does not intersect with $Ess(\nu)$. Hence $Ess(\nu)$ is closed.

2°. To prove (ii), it is enough to show that, for every $\varepsilon > 0$, there exists an open set $O \supset Ess(\nu)$ such that and $\mathrm{Cap}_R(O) < \mathrm{Cap}(Ess(\nu)) + \varepsilon$ and $\nu(O^c) < \infty$.
Let $U \supset Ess(\nu)$ be an open set such that $\mathrm{Cap}_R(U) \le \mathrm{Cap}_R(Ess(\nu)) + \varepsilon$. Denote $B = U^c$ and $F = B \cap Ex(\nu)$. By construction, F consists of non-essential explosion points. For $x \in F$, denote by U_x the neighborhood of x described in the definition of a non-essential explosion point. Open sets U_x cover a compact set F and therefore there exists a finite set $x_1, \dots, x_k \in F$ such that $F \subset U_{x_1} \cup \dots \cup U_{x_k}$. For each x_j, there exists an open set $O_j \subset U_{x_j}$ such that $\nu(U_{x_j} \setminus O_j)$ is finite and $\mathrm{Cap}_R(O_j) \le \varepsilon/k$. Put $O = U \cup O_1 \cup \dots O_k$. By construction,

$$\mathrm{Cap}_R(O) \le \mathrm{Cap}_R(U) + \sum_j \mathrm{Cap}_R(O_j) \le \mathrm{Cap}_R(Ess(\nu)) + 2\varepsilon$$

On the other hand, the set O^c is contained in the union of the sets $K_j = U_{x_j} \setminus O_j$ and the set $K = E \setminus \{U \cup U_{x_1} \cup \dots \cup U_{x_j}\}$. Sets K_j have finite measure by construction. The set K is a compact set disjoint from $Ex(\nu)$ and therefore $\nu(K) < \infty$. (Recall that ν is a Radon measure on the complement of $Ex(\nu)$ and therefore $\nu(K) < \infty$ for every compact K that contains no explosion points.)
3°. Suppose $Ess(\nu)$ is polar. Let O_n be the sequence constructed in (ii). Put $U = E$. Since $\mathrm{Cap}_R(O_n) \downarrow 0$ and $\nu(E \setminus O_n) < \infty$, all explosion points are non-essential.
4°. Again, let O_n be the sequence constructed in (ii). Denote $B = \cap_n O_n$. By construction, ν is σ-finite on B^c. Besides, $\mathrm{Cap}_R(B) = \mathrm{Cap}_R(Ess(\nu))$ and therefore $B \setminus Ess(\nu)$ is polar by Lemma 2.5. Hence $\nu(B \setminus Ess(\nu)) = 0$ and ν is σ-finite on the complement of $Ess(\nu)$ as well. □

2.3 Essential explosion points and stochastic boundary values

Lemma 2.6. *Let $\nu \in \mathcal{N}_0$. Then Z_ν is finite a.s. on the set $\{\mathcal{R} \cap Ess(\nu) = \emptyset\}$.*

Proof. Let K be a compact with $\nu(K) < \infty$ and let ν_K be the restriction of ν to K. The solution u_{ν_K} is moderate, and therefore Z_{ν_K} is finite a.s. As a first step, we prove that

$$Z_\nu = Z_{\nu_K} \quad \text{a.s. on the set } \{\mathcal{R} \cap K^c = \emptyset\}. \tag{2.5}$$

Indeed, let $B \subset K^c$ be a compact. Then $u_{\nu_B} \le w_B$ and

$$Z_{\nu_B} \le Z_{w_B} = \infty 1_{\{\mathcal{R} \cap B \neq \emptyset\}}$$

and therefore $Z_{\nu_B} = 0$ on $\{\mathcal{R} \cap K^c = \emptyset\}$. Let now B_n be an increasing sequence of compacts with the union K^c. Since ν is the increasing limit of $\nu_n = \nu_K + \nu_{B_n}$,

$$Z_\nu = \lim Z_{\nu_n} = \lim Z_{\nu_K} + Z_{\nu_{B_n}}$$

(see [3], Sect. 3.8) and therefore

$$Z_\nu = Z_{\nu_K} \quad \text{on } \{\mathcal{R} \cap K^c = \emptyset\}.$$

By Theorem 1.1(ii), there exists a decreasing sequence of open sets O_n such that $\nu(O_n^c) < \infty, Ess(\nu) \subset O_n$ and

$$P_c\{\mathcal{R} \cap O_n \neq \emptyset\} \downarrow P_c\{\mathcal{R} \cap Ess(\nu) \neq \emptyset\}. \tag{2.6}$$

Denote by K_n the complement of O_n. By (2.5), Z_ν is a.s. finite on the sets $A_n = \{\mathcal{R} \cap O_n = \emptyset\}$. On the other hand, $A_n \uparrow \{\mathcal{R} \cap Ess(\nu) = \emptyset\}$ by (2.6). □

2.4 Proof of Theorem 1.2

In order to construct the measure ν, take an arbitrary countable dense subset $\{x_k\}$ of Γ. For every k, let $\phi_k(x) = |x - x_k|^{-d}$. ϕ_k is a continuous positive function on $\partial D \setminus \{x_k\}$, which is not integrable over every neighborhood of x_k. The assumption $d \geq (\alpha+1)/(\alpha-1)$ implies $\mathrm{Cap}_R(x_k) = 0$. Hence $\lim_{n\to\infty} \mathrm{Cap}_R\{\phi_k > n\} = 0$ and therefore $\mathrm{Cap}_R\{\phi_k > C_k\} \leq 2^{-k}$ for sufficiently large C_k.
Put $f = \sup_k \phi_k / C_k$ and $\nu(dx) = f(x)\sigma(dx)$, where $\sigma(dx)$ is the surface measure. By increasing C_k, if necessary, we can make f bounded on a positive distance from Γ. For this reason, $Ex(\nu) \subset \Gamma$. On the other hand, x_n are dense in Γ. Therefore, if $x \in \Gamma$ and U is a neighborhood of x, then U contains at least one of the points x_n and therefore f is not integrable over U. Hence $\Gamma = Ex(\nu)$.
Denote $O_n = \{f > n\} = \cup_k \{\phi_k > nC_k\}$. Since ϕ_k are continuous, O_n is open for every n. Besides, $\nu(O_n^c) \leq n\sigma(O_n^c) < \infty$ for every n. By construction, $\mathrm{Cap}_R\{O_n\} \leq \sum \mathrm{Cap}_R\{\phi_k > nC_k\} \to 0$ as $n \to \infty$ by the dominated convergence theorem. Hence, ν has no essential explosion points. □

2.5 Proof of Theorem 1.3

1°. Suppose $\Gamma = Ex(\nu)$ and $Ex(\nu) \setminus Ess(\nu)$ is not polar. By Theorem 1.1(i), both $Ex(\nu)$ and $Ess(\nu)$ are closed. Therefore Lemma 2.5 implies that $\mathrm{Cap}_R(Ex(\nu)) > \mathrm{Cap}_R(Ess(\nu))$ and therefore

$$P_c\{\mathcal{R} \cap Ex(\nu) \neq \emptyset, \mathcal{R} \cap Ess(\nu) = \emptyset\} > 0. \tag{2.7}$$

Let $v = w_{\Gamma,\nu}$ and $u = u_\nu$. By Proposition 1.1, v is the maximal solution with the trace (Γ, ν). By Lemma 2.3, u also has the trace (Γ, ν). However,

$$Z_v \geq Z_\Gamma = \infty 1_{\mathcal{R}\cap\Gamma\neq\emptyset}$$

and

$$Z_u < \infty \qquad \text{a.s. on } \mathcal{R} \cap Ess(\nu) = \emptyset$$

by Lemma 2.6. By (2.7), $Z_u \neq Z_v$ with positive probability.
2°. The proof in case $\Gamma \neq Ex(\nu)$ is essentially the same as the proof of Proposition 3 in [13]. Since both Γ and $Ex(\nu)$ are closed, $C = \Gamma \setminus Ex(\nu)$ is relatively open in Γ and therefore it is not polar (see Proposition 1.1 and the definition of a normal pair). For the same reason, for every $x \in C$ and every neighborhood U_ε of x, $\Gamma_\varepsilon(x) = U_\varepsilon \cap C$ is not polar.
Put

$$\mathrm{Cap}_{R,\nu}(B) = P_c\{\mathcal{R} \cap B \neq \emptyset, e^{-Z_\nu}\}.$$

Likewise Cap_R, it is a Choquet capacity by [1], Theorem III.32. Note that

$$\mathrm{Cap}_{R,\nu}(B) = e^{-u_\nu(c)} - e^{-w_{B,\nu}(c)}. \tag{2.8}$$

By Lemma 2.5,

$$P_c\{\mathcal{R} \cap B \neq \emptyset, \mathcal{R} \cap Ex(\nu) = \emptyset\} > 0$$

for every non-polar B disjoint from $Ex(\nu)$. In addition, $Z_\nu < \infty$ on $\{\mathcal{R} \cap Ex(\nu) = \emptyset\}$ by Lemma 2.6. Therefore $\mathrm{Cap}_{R,\nu}(B) > 0$ for every non-polar B disjoint from $Ex(\nu)$. In particular, $\mathrm{Cap}_{R,\nu}(\Gamma_\varepsilon(x)) > 0$ for every $x \in C$.

Let x_i be everywhere dense in C. Since $\mathrm{Cap}_{R,\nu}(x_i) = 0$, one can choose $\varepsilon_i > 0$ to have

$$\sum \mathrm{Cap}_{R,\nu}(\Gamma_{\varepsilon_i}(x_i)) < \mathrm{Cap}_{R,\nu}(C) \tag{2.9}$$

Put $B = \cup_i \Gamma_{\varepsilon_i}(x_i)$. By (2.9),

$$\mathrm{Cap}_{R,\nu}(B) < \mathrm{Cap}_{R,\nu}(C) \leq \mathrm{Cap}_{R,\nu}(\Gamma)$$

and, by (2.8) and Lemma 2.1,

$$w_{B,\nu} < w_{\Gamma,\nu}. \tag{2.10}$$

Since (Γ, ν) is a normal pair, the trace of $w_{\Gamma,\nu}$ coincides with (Γ, ν) by Proposition 1.1. Let now $(\tilde{\Gamma}, \mu)$ be the trace of $w_{B,\nu}$. By Lemma 2.4, $\tilde{\Gamma} \subset \Gamma$ and $\mu = \nu$.
Suppose $A = \Gamma \setminus \tilde{\Gamma}$ is not empty. Since A is relatively open in Γ, it contains at least on of x_i. The set $A \cap \Gamma_{\varepsilon_i}(x_i)$ is also relatively open in Γ, and therefore it is not polar by the definition of normal pair. But,

$$A \cap \Gamma_{\varepsilon_i}(x_i) \subset A \cap B,$$

and therefore $B \setminus \tilde{\Gamma}$ is not polar, in contradiction with Lemma 2.4. Therefore $\tilde{\Gamma} = \Gamma$ and the two solutions $w_{B,\nu}$ and $w_{\Gamma,\nu}$ do not coincide and have the trace (Γ, ν). □

2.6 Proof of Theorem 1.4

$1°$. Recall some notation from [7] and [3]. A point $y \in \partial E$ is a singular point of a solution u if

$$\int_0^\zeta u^{\alpha-1}(\xi_s)\, ds = \infty \qquad \Pi_c^y\text{-a.s.}$$

Here (ξ_t, Π_x^y) is the L-diffusion in E conditionned to exit from E at the point y, and ζ is its life time. The set of all singular points of u is denoted by $\mathrm{SG}(u)$.
Let $\Gamma = \mathrm{SG}(u_\nu)$. By [3], Theorem 1.1

$$P_x Z_\eta e^{-Z_\nu} = 0$$

for every measure $\eta \in \mathcal{N}_1$ concentrated on Γ. Since $Z_\nu < \infty$ a.s. by Lemma 2.6, this is possible only if $Z_\eta = 0$ a.s. and therefore $\eta = 0$. By [6], Theorem 1.2, this is equivalent to the polarity of Γ.
$2°$. Let B be a Borel subset of ∂E. As in [7], denote by u_B the supremum of all moderate solutions u_μ such that $\mu(B^c) = 0$. For two solutions u_1, u_2, we define $u_1 \oplus u_2$ as the maximal solution dominated by $u_1 + u_2$. See [7] for more detail.
Since Γ is polar, $u_\Gamma = 0$ and $u_\Gamma \oplus u_\nu = u_\nu$.
$3°$. The fine trace of a solution u is defined as a pair (Γ, μ) where $\Gamma = \mathrm{SG}(u)$ and μ is the maximal measure such that $\mu(\Gamma) = 0$, μ does not charge polar sets and $u_\mu \leq u$. According to [7], Theorem 1.3, the fine trace of any solution has the following properties:

(A) (See [7],1.10.A.) The set Γ is finely closed (that is, closed in fine topology introduced in [7]).

(B) (See [7],1.10.B.) The measure μ is a σ-finite measure on Γ^c not charging polar sets and such that $\mathrm{SG}(u_\mu) \subset \Gamma$.

Moreover (see [7], Theorem 1.4), if (Γ, μ) is any pair satisfying (A) and (B), then $v = u_\Gamma \oplus u_\mu$ has the fine trace (Γ', μ) where $\Gamma' = \mathrm{SG}(v)$ differs from Γ by a polar set.
The set $\Gamma = \mathrm{SG}(u_\nu)$ is finely closed by [7], Theorem 1.3. Since Γ is polar, ν does not charge Γ. By assumtion, $Ess(\nu)$ is empty and therefore ν is σ-finite by Theorem 1.1(iv). Hence the pair (Γ, ν) satisfies (A) and (B) and $u_\nu = u_\Gamma \oplus u_\nu$ has the fine trace $(\mathrm{SG}(u_\nu), \nu) = (\Gamma, \nu)$. Since Γ is polar, the statement follows. $\square$

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