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ON ASYMPTOTIC GROWTH OF THE SUPPORT OF FREE MULTIPLICATIVE CONVOLUTIONS

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Abstract

Let μ be a compactly supported probability measure on $\mathbb{R}^+$ with expectation 1 and variance V. Let μ_n denote the n-time free multiplicative convolution of measure μ with itself. Then, for large n the length of the support of μ_n is asymptotically equivalent to eVn, where e is the base of natural logarithms, $e = 2.71\ldots$

1 Preliminaries and the main result

First, let us recall the definition of the free multiplicative convolution. Let a_k denote the moments of a compactly-supported probability measure μ, $a_k = \int t^k d\mu$, and let the ψ-transform of μ be $\psi_\mu(z) = \sum_{k=1}^{\infty} a_k z^k$. The inverse ψ-transform is defined as the functional inverse of $\psi_\mu(z)$ and denoted as $\psi_\mu^{(-1)}(z)$. It is a well-defined analytic function in a neighborhood of $z = 0$, provided that $a_1 \neq 0$.

Suppose that μ and ν are two probability measures supported on $\mathbb{R}^+ = \{x | x \geq 0\}$ and let $\psi_\mu^{(-1)}(z)$ and $\psi_\nu^{(-1)}(z)$ be their inverse ψ-transforms. Then, as it was first shown by Voiculescu in [5], the function

$$f(z) := \left(1 + z^{-1}\right) \psi_\mu^{(-1)}(z)\, \psi_\nu^{(-1)}(z)$$

is the inverse ψ-transform of a probability measure supported on $\mathbb{R}^+$. (Voiculescu used a variant of the inverse ψ-transform, the S-transform.) This new probability measure is called the free multiplicative convolution of measures μ and ν, and denoted as $\mu \boxtimes \nu$.

The significance of this convolution operation can be seen from the fact that if μ and ν are the distributions of singular values of two free operators X and Y, then $\mu \boxtimes \nu$ is the distribution of singular values of the product operator XY (assuming that the algebra containing X and Y is tracial). For more details about free convolutions and free probability theory, the reader can consult [2], [4], or [6].

We are interested in the support of the n-time free multiplicative convolution of the measure

μ with itself, which we denote as μ_n:

$$\mu_n = \underbrace{\mu \boxtimes \ldots \boxtimes \mu}_{n\text{-times}}.$$

Let L_n denote the upper boundary of the support of μ_n.

Theorem 1. *Suppose that μ is a compactly-supported probability measure on $\mathbb{R}^+$, with the expectation 1 and variance V. Then*

$$\lim_{n\to\infty} \frac{L_n}{n} = eV,$$

where e denotes the base of natural logarithms, $e = 2.71\ldots$

Remarks: 1) Let X_i be operators in a von Neumann algebra $\mathcal{A}$ with trace E. Assume that X_i are free in the sense of Voiculescu and identically distributed, and let $\Pi_n = X_1 \ldots X_n$. It is known that if μ is the spectral probability measure of $X_i^* X_i$, then μ_n is the spectral probability measure of $\Pi_n^* \Pi_n$. Assume further that $E\left(X_i^* X_i\right) = 1$ and $E\left(\left(X_i^* X_i\right)^2\right) = 1 + V$, and define $\|\Pi_n\|_2 =: \left[E\left(\Pi_n^* \Pi_n\right)\right]^{1/2}$. Then our theorem implies that

$$\lim_{n\to\infty} \frac{\|\Pi_n\|}{\|\Pi_n\|_2} = \sqrt{eVn}$$

for all sufficiently large n. This result also holds if we relax the assumption $E\left(X_i^* X_i\right) = 1$ and define

$$V = \frac{E\left(\left(X_i^* X_i\right)^2\right)}{\left[E\left(X_i^* X_i\right)\right]^2} - 1.$$

2) Theorem 1 improves the author's result in [3], where it was shown that $L_n/n \leq cL$ where c is a certain absolute constant and L is the upper bound of the support of μ. Theorem 1 shows that the asymptotic growth in the support of free multiplicative convolutions μ_n is controlled by the variance of μ and not by the length of its support.

The idea of proof of Theorem 1 is based on the fact that the radius of convergence of Taylor series for $\psi_n(z)$ is $1/L_n$. Therefore the function $\psi_n(z)$ must have a singularity at the boundary of the disc $|z| = 1/L_n$. Since all the coefficients in this Taylor series are real and positive, the singularity is $z_n = 1/L_n$. Therefore, the study of L_n is equivalent to the study of the singularity of $\psi_n(z)$ which is located on $\mathbb{R}^+$ and which is closest to 0.

By Proposition 5.2 in [1], we know that for all sufficiently large n, the measure μ_n is absolutely continuous on $\mathbb{R}^+ \setminus \{0\}$, and its density is analytic at all points where it is different from zero. For these n, the singularity of $\psi_n(z)$ is neither an essential singularity nor a pole. Hence, the problem is reduced to finding a branching point of $\psi_n(z)$ which is on $\mathbb{R}^+$ and closest to zero. The branching point of $\psi_n(z)$ equals a critical value of $\psi_n^{(-1)}(u)$. Since by Voiculescu's theorem,

$$\psi_n^{(-1)}(u) = \left(\frac{1+u}{u}\right)^{n-1} \left[\psi^{(-1)}(u)\right]^n,$$

therefore we can find critical points of $\psi_n^{(-1)}(u)$ from the equation

$$\frac{d}{du}\left[n \log \psi^{(-1)}(u) + (n-1)\log\left(\frac{1+u}{u}\right)\right] = 0,$$

or

$$\frac{d}{du}\log\psi^{(-1)}(u) = \left(1-\frac{1}{n}\right)\frac{1}{u(1+u)}. \tag{1}$$

Thus, our task is to estimate the root u_n of this equation which is real, positive and closest to 0, and then study the asymptotic behavior of $z_n = \psi_n^{(-1)}(u_n)$ as $n \to \infty$. This study will be undertaken in the next section.

2 Proof of Theorem 1

Notation: L and L_n are the least upper bounds of the support of measures μ and μ_n, respectively; V and V_n are variances of these measures; $\psi(z)$ and $\psi_n(z)$ are ψ-transforms for measures μ and μ_n, and $\psi^{(-1)}(u)$ and $\psi_n^{(-1)}(u)$ are functional inverses of these ψ-transforms. When we work with ψ-transforms, we use letters t, x, y, z to denote variables in the domain of ψ-transforms, and b, u, v, w to denote the variables in their range.

In our analysis we need some facts about functions $\psi(z)$ and $\psi^{(-1)}(u)$. Let the support of a measure μ be inside the interval $[0, L]$, and let μ have expectation 1 and variance V. Note that for $z \in (0, 1/L)$, the function $\psi(z)$ is positive, increasing, and convex. Correspondingly, for $u \in (0, \psi(1/L))$, the function $\psi^{(-1)}(u)$ is positive, increasing and concave.

Lemma 2. *For all positive z such that $z < 1/(2L)$, it is true that*

$$\begin{aligned} \left|\psi(z) - z - (1+V)z^2\right| &\leq c_1 z^3, \\ \left|\psi'(z) - 1 - 2(1+V)z\right| &\leq c_2 z^2, \end{aligned}$$

where c_1 and c_2 depend only on L.

Proof: Clearly, $E\left(X^k\right) \leq L^k$. Using the Taylor series for $\psi(z)$ and $\psi'(z)$, we find that for all positive z such that $z < 1/(2L)$,

$$\left|\psi(z) - z - (1+V)z^2\right| \leq \frac{L^3}{1-Lz}z^3,$$

and

$$\left|\psi'(z) - 1 - 2(1+V)z\right| \leq L^3\frac{3-2Lz}{(1-Lz)^2}z^2,$$

which implies the statement of this lemma. QED.

Lemma 3. *For all positive u such that $u < 1/(12L)$, it is true that*

$$\left|\psi^{(-1)}(u) - u + (1+V)u^2\right| \leq c_3 u^3,$$

where c_3 depends only on L.

Proof: Let the Taylor series for $\psi^{(-1)}(u)$ be

$$\psi^{(-1)}(u) = u - (1+V)u^2 + \sum_{k=3}^{\infty} d_k u^k.$$

Using the Lagrange inversion formula, it is possible to prove that

$$|d_k| \leq \frac{3}{2} (6L)^{k-1},$$

see, e.g., proof of Lemmas 3 and 4 in [3]. This implies that the Taylor series for $\psi^{(-1)}(u)$ are convergent in the disc $|u| < (6L)^{-1}$. Hence, in this disc,

$$\left| \sum_{k=3}^{\infty} d_k u^k \right| \leq \left| \frac{54L^2}{1 - 6Lu} u^3 \right|,$$

which implies the statement of this lemma. QED.
The proof of Theorem 1 uses the following proposition. Its purpose is to estimate the critical point of $\psi_n^{(-1)}(u)$ from below. Later, we will see that this estimate gives the asymptotically correct order of magnitude of the critical point.

Proposition 4. *Let u_n be the critical point of $\psi_n^{(-1)}(u)$ which belongs to $\mathbb{R}^+$ and which is closest to* 0. *Then for all $\varepsilon > 0$, there exists such $n_0(L, V, \varepsilon)$, that for all $n > n_0$,*

$$u_n \geq \frac{1}{n(1 + 2V + \varepsilon)}.$$

Proof of Proposition 4:
Claim: *Let ε be an arbitrary positive constant. Let $x_n = (n(1 + 2V + 2\varepsilon))^{-1}$ and $b_n = \psi(x_n)$. Then for all $n \geq n_0(\varepsilon, L, V)$ and all $u \in [0, b_n]$, the following inequality is valid:*

$$\frac{d}{du} \log \psi^{(-1)}(u) > \frac{n-1}{n} \frac{1}{u(1+u)}. \tag{2}$$

If this claim is valid, then since u_n is the smallest positive root of equation (1), therefore we can conclude that $u_n > b_n = \psi(x_n)$. By Lemma 2, it follows that for all sufficiently large n

$$u_n > \psi\left(\frac{1}{n(1 + 2V + 2\varepsilon)}\right) > \frac{1}{n(1 + 2V + \varepsilon)}.$$

(Indeed, note that the last inequality has 2ε and ε on the left-hand and right-hand side, respectively. Since Lemma 2 implies that $\psi(z) \sim z$ for small z, therefore this inequality is valid for all sufficiently large n.)
Hence, Proposition 4 follows from the claim, and it remains to prove the claim.
Proof of Claim: Let us re-write inequality (2) as

$$\frac{1}{z\psi'(z)} > \frac{n-1}{n} \frac{1}{\psi(z)(1 + \psi(z))}, \tag{3}$$

where $z = \psi^{(-1)}(u)$.
Using Lemma 2, we infer that inequality (3) is implied by the following inequality:

$$\frac{1}{z} \frac{1}{1 + 2(1+V)z + c_2 z^2} > \frac{n-1}{n} \frac{1}{\psi(z)(1 + \psi(z))},$$

where c_2 depends only on L. Note that $\psi(z) \geq z$ because the first moment of μ is 1 and all other moments are positive. Therefore, it is enough to show that

$$\frac{1}{1+2(1+V)z+c_2z^2} > \frac{n-1}{n}\frac{1}{1+z}.$$

for $z \leq (n(1+2V+2\varepsilon))^{-1}$ and all sufficiently large n. Let us write this inequality as

$$\frac{1}{n-1} + \frac{1}{n-1}z > (1+2V)z + c_2z^2.$$

If we fix an arbitrary $\varepsilon > 0$, then clearly for all $z \leq (n(1+2V+2\varepsilon))^{-1}$ this inequality holds if n is sufficiently large. QED.

This completes the proof of Proposition 4.

Now let us proceed with the proof of Theorem 1.

Let u_n be the critical point of $\psi_n^{(-1)}(u)$, which is positive and closest to zero, and let $y_n = \psi^{(-1)}(u_n)$. We know that y_n is a root of the equation

$$\frac{1}{z\psi'(z)} = \left(1 - \frac{1}{n}\right)\frac{1}{\psi(z)(1+\psi(z))}. \tag{4}$$

(This is equation (1) in a slightly different form.) After a re-arrangement, we can re-write this equation as

$$\frac{\psi(z)}{z}(1+\psi(z)) = \left(1-\frac{1}{n}\right)\psi'(z). \tag{5}$$

On the other hand, from the proof of Proposition 4 we know that $u_n \geq b_n = \psi(x_n)$, so that monotonicity of $\psi^{(-1)}$ implies

$$y_n = \psi^{(-1)}(u_n) \geq x_n = \frac{1}{n(1+2V+\varepsilon)}$$

Let us look for a root of equation (5) in the range $[x_n, c/n]$ where c is a fixed positive number. Let us make a substitution $z = t/n$ in equation (5) and use Lemma 2. We get:

$$\left(1+(1+V)\frac{t}{n}+O\left(n^{-2}\right)\right)\left(1+\frac{t}{n}+O\left(n^{-2}\right)\right) = \left(1-\frac{1}{n}\right)\left(1+2(1+V)\frac{t}{n}+O\left(n^{-2}\right)\right).$$

After a simplification, we get

$$t - \frac{1}{V} + O\left(n^{-1}\right) = 0.$$

Hence, for a fixed $c > 1$ and all sufficiently large n, the root is unique in the interval $[0, c]$ and given by the expression

$$t = \frac{1}{V} + O\left(n^{-1}\right).$$

Therefore,

$$y_n = \frac{1}{Vn} + O\left(n^{-2}\right).$$

By Lemma 2, this implies that

$$u_n = \psi(y_n) = \frac{1}{Vn} + O\left(n^{-2}\right).$$

This is the critical point of $\psi_n^{(-1)}(u)$.
The next step is to estimate the critical value of $\psi_n^{(-1)}(u)$, which is $z_n = \psi_n^{(-1)}(u_n)$. We write:

$$z_n = u_n \left[\frac{\psi^{(-1)}(u_n)}{u_n}\right]^n (1+u_n)^{n-1}.$$

Using Lemma 3, we infer that

$$\begin{aligned} z_n &= u_n \left[1-(1+V)\,u_n + O\left(n^{-2}\right)\right]^n (1+u_n)^{n-1} \\ &= \left(\frac{1}{Vn} + O\left(n^{-2}\right)\right) \\ &\quad\times \left[1-(1+V)\frac{1}{Vn} + O\left(n^{-2}\right)\right]^n \\ &\quad\times \left[1+\frac{1}{Vn} + O\left(n^{-2}\right)\right]^n \\ &\sim \frac{1}{eVn}, \end{aligned}$$

as $n \to \infty$. Here e denotes the base of the natural logarithm: $e = 2.71\ldots$.
Hence,

$$\lim_{n\to\infty} \frac{L_n}{n} = \lim_{n\to\infty} \frac{1}{nz_n} = eV.$$

QED.

3 Conclusion

Let me conclude with a slightly different formulation of the main result. Suppose that X_i are free, identically distributed random variables in a tracial non-commutative W^*-probability space with a faithful trace E. We proved that if $E\left(X_i^* X_i\right) = 1$, then the asymptotic growth in the square of the norm of products $\Pi_n = X_n \ldots X_1$ is linear in n with the rate equal to $e\left(E\left(X_i^* X_i X_i^* X_i\right) - 1\right)$.

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