Elect. Comm. in Probab. **10** (2005), 254–262

ELECTRONIC COMMUNICATIONS in PROBABILITY

SPHERICAL AND HYPERBOLIC FRACTIONAL BROWNIAN MOTION

JACQUES ISTAS
LMC-IMAG and LabSad
Département IMSS BSHM, Université Pierre Mendès-France
F-38000 Grenoble.
email: Jacques.Istas@upmf-grenoble.fr

Submitted 10 January 2005, *accepted in final form* 2 August 2005

AMS 2000 Subject classification: 60G15
Keywords: Fractional Brownian Motion, Manifold

Abstract

We define a Fractional Brownian Motion indexed by a sphere, or more generally by a compact rank one symmetric space, and prove that it exists if, and only if, $0 < H \leq 1/2$. We then prove that Fractional Brownian Motion indexed by an hyperbolic space exists if, and only if, $0 < H \leq 1/2$. At last, we prove that Fractional Brownian Motion indexed by a real tree exists when $0 < H \leq 1/2$.

1 Introduction

Since its introduction [10, 12], Fractional Brownian Motion has been used in various areas of applications (e.g. [14]) as a modelling tool. Its success is mainly due to the self-similar nature of Fractional Brownian Motion and to the stationarity of its increments. Fractional Brownian Motion is a field indexed by $\mathbb{R}^d$. Many applications, as texture simulation or geology, require a Fractional Brownian Motion indexed by a manifold. Many authors (e.g. [13, 8, 1, 7, 2]) use deformations of a field indexed by $\mathbb{R}^d$. Self-similarity and stationarity of the increments are lost by such deformations: they become only local self-similarity and local stationarity. We propose here to build Fractional Brownian Motion indexed by a manifold. For this purpose, the first condition is a stationarity condition with respect to the manifold. The second condition is with respect to the self-similar nature of the increments. Basically, the idea is that the variance of the Fractional Brownian Motion indexed by the manifold should be a fractional power of the distance. Let us be more precise.
The complex Brownian motion B indexed by $\mathbb{R}^d$, $d \geq 1$, can be defined [11] as a centered Gaussian field such that:

$$\begin{aligned} B(0) &= 0 \ \text{(a.s.)}\ , \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= ||M - M'|| \ \ M, M' \in \mathbb{R}^d\ , \end{aligned}$$

where $||M - M'||$ is the usual Euclidean distance in $\mathbb{R}^d$. The complex Fractional Brownian Motion B_H of index H, $0 < H < 1$, indexed by $\mathbb{R}^d$, $d \geq 1$, can be defined [10, 12] as a centered Gaussian field such that:

$$\begin{aligned} B_H(0) &= 0 \text{ (a.s.)} , \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= ||M - M'||^{2H} \quad M, M' \in \mathbb{R}^d . \end{aligned}$$

The complex Brownian motion B indexed by a sphere $\mathbb{S}_d$, $d \geq 1$, can be defined [11] as a centered Gaussian field such that:

$$\begin{aligned} B(O) &= 0 \text{ (a.s.)} , \\ \mathbb{E}|B(M) - B(M')|^2 &= d(M, M') \quad M, M' \in \mathbb{S}_d , \end{aligned}$$

where O is a given point of $\mathbb{S}_d$ and $d(M, M')$ the distance between M and M' on the sphere (that is, the length of the geodesic between M and M'). Our first aim is to investigate the fractional case on $\mathbb{S}_d$. We start with the circle $\mathbb{S}_1$. We first prove that there exists a centered Gaussian process (called Periodical Fractional Brownian Motion, in short PFBM) such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.)} , \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in \mathbb{S}_1 , \end{aligned}$$

where O is a given point of $\mathbb{S}_1$ and $d(M, M')$ the distance between M and M' on the circle, if and only if, $0 < H \leq 1/2$. We then give a random Fourier series representation of the PFBM. We then study the general case on $\mathbb{S}_d$. We prove that there exists a centered Gaussian field (called Spherical Fractional Brownian Motion, in short SFBM) such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.)} , && (1) \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in \mathbb{S}_d , && (2) \end{aligned}$$

where O is a given point of $\mathbb{S}_d$ and $d(M, M')$ the distance between M and M' on $\mathbb{S}_d$, if and only, if $0 < H \leq 1/2$. We then extend this result to compact rank one symmetric spaces (in short CROSS).

Let us now consider the case of a real hyperbolic space $\mathbb{H}_d$. We prove that there exists a centered Gaussian field (called Hyperbolic Fractional Brownian Motion, in short HFBM) such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.)} , && (3) \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in \mathbb{H}_d , && (4) \end{aligned}$$

where O is a given point of $\mathbb{H}_d$ and $d(M, M')$ the distance between M and M' on $\mathbb{H}_d$, if, and only if, $0 < H \leq 1/2$.

At last, we consider the case of a real tree (X, d). We prove that there exists a centered Gaussian field such that:

$$\begin{aligned} B_H(O) &= 0 \text{ (a.s.)} , \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \quad M, M' \in X , \end{aligned}$$

where O is a given point of X, for $0 < H \leq 1/2$.

2 Periodical Fractional Brownian Motion

Theorem 2.1

1. *The PFBM exists, if and only if,* $0 < H \leq 1/2$.
2. *Assume* $0 < H \leq 1/2$. *Let us parametrize the points* M *of the circle* $\mathbb{S}_1$ *of radius* r *by their angles* x. B_H *can be represented as:*

$$B_H(x) = \sqrt{r} \sum_{n \in \mathbb{Z}^\star} d_n \varepsilon_n . (e^{inx} - 1) , \tag{5}$$

where

$$d_n = \frac{\sqrt{- \int_0^{|n|\pi} u^{2H} \cos(u) du}}{\sqrt{2\pi} |n|^{1/2+H}} , \tag{6}$$

and $(\varepsilon_n)_{n \in \mathbb{Z}^\star}$ *is a sequence of i.i.d. complex standard normal variables.*

Proof of Theorem 2.1
Without loss of generality, we work on the unit circle $\mathbb{S}_1$: $r = 1$. Let M and M' be parametrized by $x, x' \in [0, 2\pi[$. We then have:

$$\begin{aligned} d(M, M') &= d(x, x') \\ &= \inf(|x - x'|, 2\pi - |x - x'|) . \\ d_H(x, x') &\overset{def}{=} d^{2H}(x, x') \\ &= \inf(|x - x'|^{2H}, (2\pi - |x - x'|)^{2H}) . \end{aligned}$$

The covariance function of B_H, if there exists, is:

$$R_H(x, x') = \frac{1}{2}(d_H(x, 0) + d_H(x', 0) - d_H(x, x')) .$$

Let us expand the function $x \to d_H(x, 0)$ in Fourier series:

$$d_H(x, 0) = \sum_{n \in \mathbb{Z}} f_n e^{inx} . \tag{7}$$

We will see that the series $\sum_{n \in \mathbb{Z}} |f_n|$ converges. It follows that equality (7) holds pointwise. Since $d_H(0, 0) = 0$, $\sum_{n \in \mathbb{Z}} f_n = 0$. The function d_H is odd: $f_{-n} = f_n$. We can therefore write, no matter if $x - x'$ is positive or negative:

$$\begin{aligned} d_H(x, x') &= \sum_{n \in \mathbb{Z}} f_n e^{in(x - x')} \\ &= \sum_{n \in \mathbb{Z}^\star} f_n (e^{in(x - x')} - 1) . \end{aligned}$$

We now prove that R_H is a covariance function if and only if $0 < H \leq 1/2$.

$$\begin{aligned}
\sum_{i,j=1}^{p} \lambda_i \overline{\lambda_j} R_H(x_i, x_j) &= \frac{1}{2} \sum_{n \in \mathbb{Z}} f_n \left[\sum_{i,j=1}^{p} \lambda_i \overline{\lambda_j} \left(e^{inx_i} + e^{-inx_j} - e^{-in(x_i - x_j)} \right) \right] \\
&= -\frac{1}{2} \sum_{n \in \mathbb{Z}^\star} f_n \left| \sum_{i=1}^{p} \lambda_i (1 - e^{inx_i}) \right|^2 . \qquad (8)
\end{aligned}$$

Let us study the sign of $f_n, n \in \mathbb{Z}^\star$. Since $f_{-n} = f_n$, let us only consider $n > 0$.

$$\begin{aligned}
f_n &= \frac{1}{\pi} \int_0^{\pi} x^{2H} \cos(nx) dx \\
&= \frac{1}{\pi n^{1+2H}} \int_0^{n\pi} u^{2H} \cos(u) du .
\end{aligned}$$

1. n odd.

$$\int_{2k\pi}^{2k\pi+\pi} u^{2H} \cos(u) du = \int_0^{\pi/2} \cos(v) [(v + 2k\pi)^{2H} - (2k\pi + \pi - v)^{2H}] dv \leq 0 .$$

2. n even.

$$\begin{aligned}
\int_{2k\pi}^{2(k+1)\pi} u^{2H} \cos(u) du &= \int_0^{\pi/2} \cos(v) \left[(v + 2k\pi)^{2H} - (2k\pi + \pi - v)^{2H} \right. \\
&\quad \left. -(2k\pi + \pi + v)^{2H} + (2k\pi + 2\pi - v)^{2H} \right] dv .
\end{aligned}$$

Using the concavity/convexity of the functions $x \to x^{2H}$, one sees that $[(v + 2k\pi)^{2H} - (2k\pi + \pi - v)^{2H} - (2k\pi + \pi + v)^{2H} + (2k\pi + 2\pi - v)^{2H}]$ is negative when $H \leq 1/2$ and positive when $H > 1/2$.

1. $H \leq 1/2$ All the f_n are negative and (8) is positive.

2. $H > 1/2$. We check that, if B_H exists, then we should have:

$$\begin{aligned}
\mathbb{E} \left| \int_0^{2\pi} B_H(t) e^{int} dt \right|^2 &= \int_0^{2\pi} \int_0^{2\pi} (d_H(t,0) + d_H(s,0) - d_H(s,t)) e^{in(t-s)} dt ds \\
&= -4\pi^2 f_n .
\end{aligned}$$

All the f_n, with n even, are positive, which constitutes a contradiction.

In order to prove the representation (5), we only need to compute the covariance:

$$\begin{aligned}
\mathbb{E} B_H(x) \overline{B_H(x')} &= \frac{1}{2} \left(f_0 - \sum_{n \in \mathbb{Z}^\star} d_n^2 e^{inx} - \sum_{n \in \mathbb{Z}^\star} d_n^2 e^{inx'} + \sum_{n \in \mathbb{Z}^\star} d_n^2 e^{in(x-x')} \right) \\
&= R_H(x, x') .
\end{aligned}$$

3 Spherical Fractional Brownian Motion

Theorem 3.1

1. *The SFBM, defined by (1), (2), exists if, and only if,* $0 < H \leq 1/2$.
2. *The same holds for CROSS: the Fractional Brownian Motion, indexed by a CROSS, exists if, and only if,* $0 < H \leq 1/2$.

Corollary 3.1
Let $(\mathcal{M}, \tilde{d})$ *be a complete Riemannian manifold such that* $\mathcal{M}$ *and a CROSS are isometric. Then the Fractional Brownian Motion indexed by* $\mathcal{M}$ *and defined by:*

$$\begin{aligned} B_H(\widetilde{O}) &= 0 \ \ (a.s.)\,, \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= \widetilde{d}^{2H}(M, M') \ \ M, M' \in \mathcal{M}\,, \end{aligned}$$

exists if and only if $0 < H \leq 1/2$.

Proof of Theorem 3.1
Let us first recall the classification of the CROSS, also known as two points homogeneous spaces [9, 17]: spheres $\mathbb{S}_d$, $d \geq 1$, real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d \geq 2$, complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 2k$, $k \geq 2$, quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 4k$, $k \geq 2$ and Cayley projective plane P^{16}. [6] has proved that Brownian Motion indexed by CROSS can be defined. The proof of Theorem 3.1 begins with the following Lemma, which implies, using [6], the existence of the Fractional Brownian Motion indexed by a CROSS for $0 < H \leq 1/2$.

Lemma 3.1
Let (X, d) *be a metric space. If the Brownian Motion* B *indexed by* X *and defined by:*

$$\begin{aligned} B(O) &= 0 \ \ (a.s.)\,, \\ \mathbb{E}|B(M) - B(M')|^2 &= d(M, M') \ \ M, M' \in X\,, \end{aligned}$$

exists, then the Fractional Brownian Motion B_H *indexed by* X *and defined by:*

$$\begin{aligned} B_H(O) &= 0 \ \ (a.s.)\,, \\ \mathbb{E}|B_H(M) - B_H(M')|^2 &= d^{2H}(M, M') \ \ M, M' \in X\,, \end{aligned}$$

exists for $0 < H \leq 1/2$.

Proof of Lemma 3.1
For $\lambda \geq 0$, $0 < \alpha < 1$, one has:

$$\lambda^\alpha = -\frac{1}{C_\alpha} \int_0^{+\infty} \frac{e^{-\lambda x} - 1}{x^{1+\alpha}} dx\,,$$

with

$$C_\alpha = \int_0^{+\infty} \frac{1 - e^{-u}}{u^{1+\alpha}} du\,.$$

We then have, for $0 < H < 1/2$:

$$d^{2H}(M, M') = -\frac{1}{C_{2H}} \int_0^{+\infty} \frac{e^{-x d(M, M')} - 1}{x^{1+2H}} dx\,.$$

Let us remark that:

$$e^{-xd(M,M')} = \mathbb{E}\left(e^{i\sqrt{2x}(B(M)-B(M'))}\right) ,$$

so that:

$$d^{2H}(M,M') = -\frac{1}{C_{2H}}\int_0^{+\infty}\frac{\mathbb{E}\left(e^{i\sqrt{2x}(B(M)-B(M'))}\right)-1}{x^{1+2H}}dx .$$

Denote by $R_H(M,M')$ the covariance function of B_H, if exists:

$$R_H(M,M') = \frac{1}{2}\left(d_H(O,M)+d_H(M',O)-d_H(M,M')\right) .$$

Let us check that R_H is positive definite:

$$\sum_{i,j=1}^{p}\lambda_i\overline{\lambda_j}R_H(M_i,M_j) =$$

$$-\frac{1}{2C_{2H}}\int_0^{+\infty}\frac{\sum_{i,j=1}^{p}\lambda_i\overline{\lambda_j}\left[\mathbb{E}\left(e^{i\sqrt{2x}B(M_i)}\right)+\mathbb{E}\left(e^{-i\sqrt{2x}B(M_j)}\right)-\mathbb{E}\left(e^{i\sqrt{2x}(B(M_i)-B(M_j))}\right)-1\right]}{x^{1+2H}}dx$$

$$= \frac{1}{2C_{2H}}\int_0^{+\infty}\frac{\mathbb{E}\left|\sum_{i=1}^{p}\lambda_i\left(1-e^{i\sqrt{2x}B(M_i)}\right)\right|^2}{x^{1+2H}}dx . \qquad (9)$$

(9) is clearly positive and Lemma 3.1 is proved.

We now prove by contradiction that the Fractional Brownian Motion indexed by a CROSS does not exist for $H > 1/2$. The geodesic of a CROSS are periodic. Let $\mathbf{G}$ be such a geodesic containing 0. Therefore, the process $B_H(M),\ M \in \mathbf{G}$ is a PFBM. We know from Theorem 2.1 that PFBM exits if, and only if, $0 < H \leq 1/2$.

Proof of Corollary 3.1

Let ϕ be the isometric mapping between $\mathcal{M}$ and the CROSS and let $\widetilde{d}$ (resp. d) be the metric of $\mathcal{M}$ (resp. the CROSS). Then, for all $M,M' \in \mathcal{M}$, one has:

$$\widetilde{d}(M,M') = d(\phi(M),\phi(M')) .$$

Let $\widetilde{O}$ be a given point of $\mathcal{M}$ and $O = \phi(\widetilde{O})$. Denote by $\widetilde{R}_H$ (resp. R_H) the covariance function of the Fractional Brownian Motion indexed by $\mathcal{M}$ (resp. the CROSS).

$$\begin{aligned}\widetilde{R}_H(M,M') &\overset{def}{=} \frac{1}{2}(\widetilde{d}^{2H}(\widetilde{O},M)+\widetilde{d}^{2H}(\widetilde{O},M')-\widetilde{d}^{2H}(M,M'))\\ &= \frac{1}{2}(d^{2H}(O,\phi(M))+d^{2H}(O,\phi(M'))-d^{2H}(\phi(M),\phi(M'))\\ &= R_H(\phi(M),\phi(M')) .\end{aligned}$$

It follows that $\widetilde{R}_H$ is positive definite if and only if, R_H is positive definite. Corollary 3.1 is proved.

4 Hyperbolic Fractional Brownian Motion

Let us consider real hyperbolic spaces $\mathbb{H}_d$:

$$\mathbb{H}_d = \{(x_1,\ldots,x_d),\ x_1>0,\ x_1^2-\sum_2^d x_i^2=1\}\ ,$$

with geodesic distance:

$$d(M,M') = Arccosh[M,M']\ ,$$

where

$$[M,M'] = x_1x_1'-\sum_2^d x_ix_i'\ .$$

The HFBM is the Gaussian centered field such that:

$$\begin{aligned} B_H(O) &= 0 \ \text{(a.s.)}\ , \\ \mathbb{E}|B_H(M)-B_H(M')|^2 &= d^{2H}(M,M')\ \ M,M'\in\mathbb{H}_d\ , \end{aligned}$$

where O is a given point of $\mathbb{H}_d$.

Theorem 4.1
The HFBM exists if, and only if, $0<H\le 1/2$.

Since the Brownian Motion indexed by a real hyperbolic space can be defined [4, 5], Lemma 3.1 implies the existence of HFBM when $0<H\le 1/2$.
Let H be a real function. [5, Prop. 7.6] prove that if $H(d(M,M'))$ is negative definite, then $H(x)=O(x)$ as $x\to+\infty$. It follows that $d^{2H}(M,M')$, when $H>1/2$, is not negative definite: the HFBM, when $H>1/2$ does not exist.

Corollary 4.1
Let $(\mathcal{M},\tilde{d})$ be a complete Riemannian manifold such that $\mathcal{M}$ and $\mathbb{H}_d$ are isometric. Then the Fractional Brownian Motion indexed by $\mathcal{M}$ and defined by:

$$\begin{aligned} B_H(\widetilde{O}) &= 0 \ \textit{(a.s.)}\ , \\ \mathbb{E}|B_H(M)-B_H(M')|^2 &= \tilde{d}^{2H}(M,M')\ \ M,M'\in\mathcal{M}\ , \end{aligned}$$

exists if, and only, if $0<H\le 1/2$.

The proof of Corollary 4.1 is identical to the proof of Corollary 3.1.

5 Real trees

A metric space (X,d) is a real tree (e.g. [3]) if the following two properties hold for every $x,y\in X$.

- There is a unique isometric map $f_{x,y}$ from $[0,d(x,y)]$ into X such that $f_{x,y}(0)=x$ and $f_{x,y}(d(x,y))=y$.

- If ϕ is a continuous injective map from $[0,1]$ into X, such that $\phi(0) = x$ and $\phi(1) = y$, we have
$$\phi([0,1]) = f_{x,y}([0, d(x,y)]).$$

Theorem 5.1
The Fractional Brownian Motion indexed by a real tree (X,d) exists for $0 < H \leq 1/2$.

[16] proves that the distance d is negative definite. It follows from [15] that function $R(x,y) = \frac{1}{2}(d(0,x) + d(0,y) - d(x,y))$ is positive definite. Lemma 3.1 then implies theorem 5.1.

Acknowledgements. I would like the anonymous referee for his valuable suggestions.

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