Elect. Comm. in Probab. **8** (2003) 51–62

ELECTRONIC COMMUNICATIONS in PROBABILITY

COMPUTATION OF GREEKS FOR BARRIER AND LOOK-BACK OPTIONS USING MALLIAVIN CALCULUS

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Submitted 13 November 2002, *accepted in final form* 31 March 2003

AMS 2000 Subject classification: 60H07, 60J60, 65C05
Keywords: barrier and lookback options, option sensitivities, Malliavin calculus

Abstract
In this article, we consider the numerical computations associated to the Greeks of barrier and lookback options, using Malliavin calculus. For this, we derive some integration by parts formulae involving the maximum and minimum of a one dimensional diffusion. Numerical tests illustrate the gain of accuracy compared to classical methods.

Introduction

In a frictionless market, let us consider a one-dimensional risky asset $(S_t)_{t\geq 0}$, whose dynamic is given, under the risk neutral probability $\mathbf{P}$, by:

$$S_t = S_0 + \int_0^t rS_s ds + \int_0^t \sigma(S_s)S_s\, dW_s,$$

where r is the interest rate. We focus our attention on barrier and lookback European style options with payoff functions Φ of the type

$$\Phi\left(\max_{s\in I} S_s, \min_{s\in I} S_s, S_T\right),$$

for some set $I \subset [0,T]$. In the following, we will sometimes omit the arguments of the function $\Phi = \Phi(\max_{s\in I} S_s, \min_{s\in I} S_s, S_T)$ as they will be clear from the context. The form of the payoff function Φ shall remain quite general (up to the technical condition **(S)** below): in particular, this includes usual single and double barrier options, backward start lookback options, and in

[1]RESEARCH SUPPORTED BY GRANTS BFM 2000-807 AND BFM 2000-0598

each case, the risky asset can be monitored in continuous time ($I = [0, T]$) or discrete time ($I = \{t_i : 0 \leq i \leq N\}$); we will recall later some standard examples that fit our framework.
The price at time 0 of the option is equal to $\mathbf{E}(e^{-rT}\Phi)$ and through the paper, we are more specifically interested in computing some option derivatives (the so-called Greeks), in particular Delta $\Delta = \partial_{S_0}\mathbf{E}(e^{-rT}\Phi)$ and Gamma $\Gamma = \partial^2_{S_0}\mathbf{E}(e^{-rT}\Phi)$, quantities related to the hedging strategy of the option. Actually, our purpose is first to derive, using an integration by parts formula of Malliavin calculus, some weights H to rewrite each Greek above as $\mathbf{E}(e^{-rT}\Phi\, H)$; and second, to apply this representation to numerically compute the Greeks and test the gain of efficiency, compared to alternative approaches such as the finite difference method (FD in short, see Glasserman and Yao [GY92], L'Ecuyer and Perron [LP94]).
In a series of recent articles (see Fournié, Lasry, Lebuchoux, Lions and Touzi [FLLLT99]; Fournié, Lasry, Lebuchoux and Lions [FLLL01]; Benhamou [Ben00]) the interest for applications of the integration by parts formula of Malliavin Calculus has been increased due to the possibility of performing efficient Monte Carlo simulations to estimate the Greeks. The trick of using an integration by parts formula is natural and not very recent: Broadie and Glasserman [BG96] have introduced this idea (the *likelihood ratio* method) when the density of the random variables involved is explicitly known. The real advantage of using Malliavin Calculus comes into play when one starts to deal with random variables whose density is not explicitly known as the case of Asian options treated in [FLLLT99] and [Ben00].
This approach using simulations of additional weights compared to FD method has proven to be efficient: the number of parameters to be chosen is smaller and the estimation is unbiased; moreover, when the payoff function is irregular, the variance of simulations is in general smaller (see the discussion in [FLLLT99]). Since for barrier options, the payoff Φ involves indicator functions, one may expect much of this method (see numerical results in section 3).
The results presented below for the computations of Greeks in the case of barrier and lookback options are new: the case of vanilla options (or depending on a finite number of dates) has been handled in [FLLLT99], whereas the case of Asian options has been systematically treated in [Ben00]. In comparison with the previous articles, here the technical difficulty comes from the lack of differentiability of the minimum and maximum processes: these random variables are only once differentiable and this is not enough to obtain in a direct way an integration by parts formula. Nevertheless, the problem of obtaining such a formula has already been considered by Nualart and Vives [NV88] where the absolute continuity of the maximum of a differentiable process is proven. More specifically, for barrier options, Cattiaux [Cat91] has performed some Malliavin calculus computations: actually, he has obtained a quasi integration by parts formula, on the time reversed process; unfortunately, although these formulae are useful for theoretical purposes, they are difficult to use in practice for the Greeks. Later in Nualart [Nua95], the smoothness of the density of the supremum of the Wiener sheet is obtained: for this, he uses a localization procedure. We will adapt this idea in our situation, using a dominating process, which controls the extrema of $(S_t)_{0\leq t\leq T}$. Another technical issue is what are the elements that determine a good dominating process. Here, we propose two. One is the extrema of the process and another is obtained through Garsia-Rodemich-Rumsey's Lemma. The choice of the dominating process has effects on the size of the program to compute the Greeks. For this reason, we have studied numerically the behavior of these two dominating processes.
The paper is organized as follows: we first give some preliminaries. In section 1, we state our main results, giving some weights for the Delta and Gamma: then, we give their proofs. The weights cited above depend on some dominating process, and we give some examples of such process in Section 2. Finally, we present some numerical results in Section 3.

Concerning the Malliavin calculus, we use the notations and definitions that are standard in this area and that can be found in e.g. [Nua95].

Preliminaries

For the sake of simplicity, we are going to consider a transformation of S which avoids some degeneracy problems on the diffusion coefficient, i.e. $X_t = A(S_t)$ where A is the strictly increasing function $A(y) = \int_1^y \frac{du}{u\sigma(u)}$, so that

$$X_t = x + \int_0^t h(X_s)ds + W_t. \tag{1}$$

where $x = A(S_0)$ and $h(u) = (r/\sigma(y) - (y\sigma(y))'/2)|_{y=A^{-1}(u)}$. This corresponds to the usual logarithm change in Black & Scholes model. In the following, we will assume that h is a C_b^∞ function. If we denote $M_t = \max_{s\leq t, s\in I} X_s$ and $m_t = \min_{s\leq t, s\in I} X_s$, the main issue consists in expliciting Malliavin integration by parts formula for Δ and Γ, which are related to (up to the discounted factor and the change of variables)

$$\partial_x \mathbf{E}(\Phi(M_T, m_T, X_T)) = \mathbf{E}(\Phi(M_T, m_T, X_T)H_1)$$
$$\partial^2_{x,x} \mathbf{E}(\Phi(M_T, m_T, X_T)) = \mathbf{E}(\Phi(M_T, m_T, X_T)H_2)$$

for some random variables H_1 and H_2. In the following, the payoff $\Phi(M_T, m_T, X_T)$ is supposed to be squared integrable. Furthermore, we impose a support type condition on Φ

(S) There exists $a > 0$ such that the function $\Phi(M, m, z)$ does not depend on the variables (M, m) for any (M, m, z) such that $0 \leq M - x < a$ or $0 \leq x - m < a$.

If only the maximum M (resp. the minimum m) is involved in the Pay-Off, the above assumption shall be implicitly rewritten omitting m (resp. M): in the first case, **(S)** is stated as

(S) For some $a > 0$, the function $\Phi(M, z)$ does not depend on M if $0 \leq M - x < a$.

and in the second one,

(S) For some $a > 0$, the function $\Phi(m, z)$ does not depend on m if $0 \leq x - m < a$.

These technical conditions are actually not so restrictive: it includes all usual barrier options and most of lookback ones. Let us give some examples, where the parameter $a > 0$ is given.

- Single barrier options.
  - Up & out barrier : $\Phi(M, m, z) = \mathbb{1}_{M<U} f(z)$ with $x < U$. Take $a = U - x$.
  - Down & in barrier : $\Phi(M, m, z) = \mathbb{1}_{m\leq D} f(z)$ with $D < x$. Take $a = x - D$.
- Double barrier options ($D < x < U$).
  - Double in barrier: $\Phi(M, m, z) = \mathbb{1}_{M\geq U} \mathbb{1}_{m\leq D} f(z)$. Take $a = \min(U - x, x - D)$.
  - Mixed In/out barrier: $\Phi(M, m, z) = \mathbb{1}_{M<U} \mathbb{1}_{m\leq D} f(z)$. The function Φ does not satisfy directly (**S**), but since it is a linear combination of Pay-Off verifying **(S)** ($\Phi(M, m, z) = \mathbb{1}_{m\leq D} f(z) - \mathbb{1}_{M\geq U} \mathbb{1}_{m\leq D} f(z)$), the next results also apply for Φ, taking $a = \min(U - x, x - D)$.

- Double out barrier: $\Phi(M,m,z) = \mathbb{1}_{M<U} \mathbb{1}_{m>D} f(z)$. For the same arguments as before, results will apply with $a = \min(U-x, x-D)$.

- Backward start lookback Put: $\Phi(M,m,z) = \max(M_0, M) - z$, with $M_0 > x$. Take $a = M_0 - x$.
- Out of money Put on minimum: $\Phi(M,m,z) = (K-m)_+$, with $K < x$. Take $a = x - K$.

1 Computation of the delta and gamma

1.1 Assumptions and notations

Our general approach consists in assuming that there exists an non-decreasing adapted right-continuous process $(Y_t)_{0\leq t\leq T}$ such that:

$$\forall\, t \in I \qquad |X_t - x| \leq Y_t. \tag{2}$$

We shall call it *a process dominating* X or an X*-dominating process.*

Remark 1.1. *If the maximum and the minimum were computed on different time sets I and J, our results stated below would be true, still under* **(S)**, *taking a dominating process Y relatively to the bigger time set $I \cup J$.*

We assume that the following estimate holds true.

(H) There exists a positive function $\alpha : \mathbf{N} \mapsto \mathbf{R}^+$, with $\lim_{q\to\infty} \alpha(q) = \infty$, such that, for any $q \geq 1$, one has: $\forall\, t \in [0,T] \quad \mathbf{E}(Y_t^q) \leq C_q t^{\alpha(q)}$.

In particular, one has $Y_0 = 0$. Furthermore, we shall expect Y to be somewhat smooth in the Malliavin sense. For this, choose a C_b^∞ function $\Psi : [0,\infty) \mapsto [0,1]$, with $\Psi(x) = 1$ if $x \leq a/2$, and 0 if $x \geq a$: the real positive number a is the one appearing in the support condition **(S)**. For $q \in \mathbf{N}^*$, consider the following regularity assumption:

(R(q)) The random variable $\Psi(Y_t)$ belongs to $\mathbf{D}^{q,\infty}$ for each t. Moreover, for $j = 1, \cdots, q$, one has

$$\forall\, p \geq 1 \qquad \sup_{r_1,\cdots,r_j \in [0,T]} \mathbf{E}\left(\sup_{r_1\vee\cdots\vee r_j \leq t \leq T} |\mathcal{D}_{r_1,\cdots,r_j}\Psi(Y_t)|^p \right) \leq C_p.$$

Moreover, for $q \geq 2$, the $q-1$ first derivatives of $\Psi(Y_t)$ w.r.t. x, i.e. $\partial_x(\Psi(Y_t)), \cdots, \partial_x^{q-1}(\Psi(Y_t))$, exist and satisfy the same estimates as above.

The construction of the process $(Y_t)_{0\leq t\leq T}$ will be done later in section 2. Actually, it can depend on the type of subset I.

We will carry out the Malliavin calculus computations on the extrema of $(X_t)_{t\in I}$ if X is a Brownian motion without drift (or with a deterministic one): up to a change of probability measure, we can reduce our problem to this. Consider

$$Z_T = \left.\frac{d\mathbf{P}}{d\mathbf{Q}}\right|_{\mathcal{F}_T} = \exp\Big(\int_0^T h(X_s)dV_s - \frac{1}{2}\int_0^T h^2(X_s)ds\Big)$$

which defines the measure $\mathbf{Q}$, under which $(V_t)_{t\geq 0} = (X_t - x)_{t\geq 0}$ is a standard Brownian motion. Thus, one has:

$$\mathbf{E_P}(\Phi(M_T, m_T, X_T)) = \mathbf{E_Q}(\Phi(\max_{s\leq T, s\in I} V_s + x, \min_{s\leq T, s\in I} V_s + x, V_T + x)\ Z_T). \tag{3}$$

Hence, the derivation of formulae for the Greeks can be performed under $\mathbf{Q}$ and then, rewritten under $\mathbf{P}$: for simplicity of notation, we leave final formulae expressed under $\mathbf{Q}$.
If for any $t \in I \cap [0,T]$ the random variable U_t belongs to $\mathbf{D}^{1,2}$, it is well known that, under some additional mild conditions (see Nualart and Vives [NV88] for a precise statement), the random variables $\min_{s\leq T, s\in I} U_s$ and $\max_{s\leq T, s\in I} U_s$ also belong to $\mathbf{D}^{1,2}$. The next lemma develops this result when U is a standard Brownian motion with a non random drift.

Lemma 1.1. *Let V be a standard Brownian motion and consider $V_t^f = V_t + \int_0^t f(s)\, ds$, for a deterministic function f. Then, the random variables $\min_{s\leq T, s\in I} V_s^f$ and $\max_{s\leq T, s\in I} V_s^f$ belongs to $\mathbf{D}^{1,\infty}$ and their first weak derivatives are defined as follows: for $t \in [0,T]$, one has*

$$\mathcal{D}_t \left(\min_{s\leq T, s\in I} V_s^f\right) = \mathbb{1}_{\, t\leq \tau^m}, \qquad \mathcal{D}_t \left(\max_{s\leq T, s\in I} V_s^f\right) = \mathbb{1}_{\, t\leq \tau^M},$$

where τ^m and τ^M are the random times in $I \cap [0,T]$ for which by $V^f_{\tau^m} = \min_{s\leq T, s\in I} V_s^f$ and $V^f_{\tau^M} = \max_{s\leq T, s\in I} V_s^f$.

Actually, τ^m and τ^M are uniquely defined *a.s.* (see Karatzas and Shreve [KS91]). Note that the random variables $\min_{s\leq T, s\in I} V_s$ and $\max_{s\leq T, s\in I} V_s$ do not belong to $\mathbf{D}^{2,p}$, so that a classical integration by parts formula can not apply for them. Nevertheless, a localization procedure using the dominating process Y will give some results (some analogous situations are handled in Nualart [Nua95], Proposition 2.1.5).

1.2 Integration by parts formulae

We now state the main results of the paper.

Theorem 1.1. *Assume* **(S)** *and* **(H)**.

1) *If Y satisfies* **(R(1))**, *set $H_1 = \delta\left(\frac{Z_T}{\int_0^T \Psi(Y_t)dt}\Psi(Y_\cdot)\right) + \partial_x Z_T$ (the explicit expression for $\partial_x Z_T$ is given in the proof below). Then, one has*

$$\partial_x \mathbf{E_P}(\Phi(M_T, m_T, X_T)) = \mathbf{E_Q}(\Phi(M_T, m_T, X_T) H_1). \tag{4}$$

2) *If Y satisfies* **(R(2))**, *set $H_2 = \delta\left(\frac{H_1}{\int_0^T \Psi(Y_t)dt}\Psi(Y_\cdot)\right) + \partial_x H_1$. Then, one has*

$$\partial^2_{x,x} \mathbf{E_P}(\Phi(M_T, m_T, X_T)) = \mathbf{E_Q}(\Phi(M_T, m_T, X_T) H_2). \tag{5}$$

Proof. 1) Formula for Δ. We can assume that the function Φ is smooth with bounded derivatives: the general case follows by a density argument (see [FLLLT99]). Denote by $\text{div}\ \Phi = \Phi'_M + \Phi'_m + \Phi'_z$ the usual divergence of the function Φ. Starting from (3), one has

$$\begin{aligned}\partial_x \mathbf{E_P}(\Phi(M_T, m_T, X_T)) =& \mathbf{E_Q}\Big((\text{div}\ \Phi)(\max_{s\leq T, s\in I} V_s + x, \min_{s\leq T, s\in I} V_s + x, V_T + x)\ Z_T \\ &+ \Phi(\max_{s\leq T, s\in I} V_s + x, \min_{s\leq T, s\in I} V_s + x, V_T + x)\ \partial_x Z_T\Big),\end{aligned} \tag{6}$$

where $\partial_x Z_T = Z_T \int_0^T h'(x+V_s)(dV_s - h(x+V_s)\,ds)$. The weak derivative of Φ is equal to

$$\mathcal{D}_t\Phi = \Phi'_M \mathbf{1}_{t\le\tau^M} + \Phi'_m \mathbf{1}_{t\le\tau^m} + \Phi'_z,$$

using the notation from Lemma (1.1). The key point is to remark that one has

$$\mathcal{D}_t\Phi\ \Psi(Y_t) = (\mathrm{div}\ \Phi)(\max_{s\le T, s\in I} V_s + x, \min_{s\le T, s\in I} V_s + x, V_T + x)\Psi(Y_t). \tag{7}$$

Indeed, due to the condition **(S)**, both sides of the above expression equal $\Phi'_z\Psi(Y_t)$ on the event $A = \{\max_{s\le T, s\in I} V_s \le a\} \cup \{\min_{s\le T, s\in I} V_s \ge -a\}$. On A^c and for t such that $\Psi(Y_t)\neq 0$, one has $Y_t < a$; hence using (2), one has $\max_{s\le t, s\in I} V_s < a < \max_{s\le T, s\in I} V_s$ and $\min_{s\le t, s\in I} V_s > -a > \min_{s\le T, s\in I} V_s$, that is $t\le\tau^M$ and $t\le\tau^m$. This proves that, on A^c, one has

$$\mathbf{1}_{t\le\tau^M}\Psi(Y_t) = \Psi(Y_t) \qquad \text{and} \qquad \mathbf{1}_{t\le\tau^m}\Psi(Y_t) = \Psi(Y_t).$$

From (7), it readily follows, using the integration by parts of Malliavin Calculus (formula (1.41) from Nualart [Nua95]) since $\Psi(Y_t)$ is smooth and satisfies **(R(1))**, that

$$\begin{aligned}\mathbf{E_Q}\Big((\mathrm{div}\ \Phi)(\max_{s\le T, s\in I} V_s + x, \min_{s\le T, s\in I} V_s + x, V_T + x)\ Z_T\Big) &= \mathbf{E_Q}\Big(\Big\langle \mathcal{D}\Phi, \frac{Z_T}{\int_0^T \Psi(Y_t)dt}\Psi(Y_.)\Big\rangle\Big)\\ &= \mathbf{E_Q}\Big(\Phi\ \delta\Big(\frac{Z_T}{\int_0^T \Psi(Y_t)dt}\Psi(Y_.)\Big)\Big).\end{aligned} \tag{8}$$

The above computations are valid up to verifying that $(\int_0^T \Psi(Y_t)dt)^{-1}$ belongs to any $L_p, p\ge 1$. For this, it is enough to prove that $\mathbf{P}\left(\int_0^T \Psi(Y_t)dt < \epsilon\right) = O_{\epsilon\to 0}(\epsilon^p)$ for any $p\ge 1$ (see Lemma 2.3.1 in [Nua95]). Due to the non-decreasing property of Y, this probability is upper bounded by $\mathbf{P}\left(a/2 < Y_\epsilon\right) \le \frac{\mathbf{E_P}(Y_\epsilon^q)}{(a/2)^q}$, and the conclusion follows from assumption **(H)**.
The combination of (6) and (8) leads to the required expression for H_1.

2) Formula for Γ. We remark that the smooth random variable Z_T in (3) for the computation of Δ has been replaced by the smooth random variable H_1 in (4). Hence, the derivation of H_2 is straightforward. □

It is worth noticing that formulae from Theorem (1.1) can be simplified when the drift function $h(x)$ in (1) is constant equal to μ (this includes the Black & Scholes model): in that situation, X is, under $\mathbf{P}$, a Brownian motion with deterministic drift, so that Lemma (1.1) directly applies and no change of measure is needed. Hence, one obtains the

Theorem 1.2. *Assume* **(S)** *and* **(H)** *and suppose that* $X_t = x + W_t + \mu\, t$.

1) If Y *satisfies* **(R(1))**, *set* $H_1 = \delta\left(\frac{1}{\int_0^T \Psi(Y_t)dt}\Psi(Y_.)\right)$. *Then, one has*

$$\partial_x \mathbf{E_P}(\Phi(M_T, m_T, X_T)) = \mathbf{E_P}(\Phi(M_T, m_T, X_T)H_1). \tag{9}$$

2) If Y *satisfies* **(R(2))**,*set* $H_2 = \delta\left(\frac{H_1}{\int_0^T \Psi(Y_t)dt}\Psi(Y_.)\right) + \partial_x H_1$. *Then, one has*

$$\partial^2_{x,x}\mathbf{E_P}(\Phi(M_T, m_T, X_T)) = \mathbf{E_P}(\Phi(M_T, m_T, X_T)H_2). \tag{10}$$

The next sections provide two examples of dominating processes $(Y_t)_{0\le t\le T}$, which do not depend on x in the Black & Scholes model: thus, in that case, one has $\partial_x H_1 = 0$.

2 Construction of dominating processes Y

Until now, we have assumed the existence of a dominating process Y satisfying some regularity assumptions and some estimates. In this section, we explicit these processes in two cases.

1) I is the full interval $[0, T]$.

2) I consists in a finite number of times.

The first case corresponds to continuous time monitored barrier and lookback options, and the second, to discrete time ones. These two cases include most standard situations that appear in practice.
Note that any dominating process for X in the case $I = [0, T]$ is also a dominating process for any case $I \subset [0, T]$; however, its numerical efficiency may not be optimal as shown in the section of numerical simulations. Throughout this section, X is the solution of the SDE (1), although the results in this section are also valid if X were one component of a general multidimensional diffusion process.

2.1 The case of continuous time maximum/minimum: $I = [0, T]$

2.1.1 Extrema process

The simplest dominating process is probably the extrema process

$$Y_t = \max_{s \le t}(X_s - x) - \min_{s \le t}(X_s - x).$$

It is easy to check that it satisfies (2), assumptions **(H)** and **(R(1))**. Hence, the computation of Δ can be performed using formula (4) or (9) with such a dominating process.
However, since Y_t does not belong to $\mathbf{D}^{2,p}$, one needs smoother dominating process to compute Γ and higher sensibilities. That is what we develop now.

2.1.2 Averaged modulus continuity process

For an even integer γ, define

$$Y_t := 8\left(4\int_0^t \int_0^t \frac{|X_s - X_u|^\gamma}{|s-u|^{m+2}}\, ds\, du\right)^{1/\gamma} \frac{m+2}{m}\, t^{m/\gamma}. \tag{11}$$

Using the classical estimates $\mathbf{E}(|X_s - X_u|^\gamma) \le C_q |s-u|^{\gamma/2}$, one remarks that the condition $0 < m < \frac{\gamma}{2} - 2$ implies that $\mathbf{E}\left(\int_0^t \int_0^t \frac{|X_s - X_u|^\gamma}{|s-u|^{m+2}}\, ds\, du\right) < \infty$, so that $(Y_t)_{0 \le t \le T}$ is *a.s.* well defined. This process is clearly non-decreasing, adapted and continuous; the next lemma justifies that Y is a good candidate for our procedure.

Lemma 2.1. *Let γ be a even integer and set m such that $0 < m < \frac{\gamma}{2} - 2$. Then, one has:*

i) For any $t \in [0, T]$, one has $|X_t - x| \le Y_t$.

ii) The assumption **(H)** *is satisfied.*

iii) For q such that $1 \le q \le \gamma - 2(m+2)$, assumption **(R(q))** *is satisfied.*

Proof. Assertion i) is a consequence of Garsia-Rodemich-Rumsey's Lemma [GRR70]. Indeed, following their notation, take $\Psi(x) = x^\gamma$, $p(x) = x^{\frac{m+2}{\gamma}}$ and put $B_t = \int_0^t \int_0^t \frac{|X_s - X_u|^\gamma}{|s-u|^{m+2}} \, ds \, du$; then, for any $s \in [0, t]$, one has

$$|X_s - x| \leq 8 \int_0^s \left(\frac{4B_t}{u^2}\right)^{1/\gamma} \left(\frac{m+2}{\gamma}\right) u^{\frac{m+2-\gamma}{\gamma}} du = 8 \, (4B_t)^{1/\gamma} \, \frac{m+2}{m} \, s^{m/\gamma} \leq Y_t.$$

Assertion ii) is easy to prove using classical estimates on the modulus of continuity of SDEs: we omit the details.
Proof of Assertion iii). We prove the estimates for $\Psi(Y_t)$, those for its derivatives w.r.t. x being similar. Standard computations (see Chapter 2.2, Nualart [Nua95]) prove that, for any $t \in [0, T]$, $B_t \in \mathbf{D}^{q,\infty}$ if $q \leq \gamma - 2(m+2)$. Moreover, for $j = 1, \cdots, q$, one has

$$\forall \, p \geq 1 \qquad \sup_{r_1, \cdots, r_j \in [0,T]} \mathbf{E}\left(\sup_{r_1 \vee \cdots \vee r_j \leq t \leq T} |\mathcal{D}_{r_1, \cdots, r_j} B_t|^p \right) \leq C_p.$$

Since the function $x \in \mathbf{R}^+ \mapsto \Psi\left(8 \, (4x)^{1/\gamma} \frac{m+2}{m} \, t^{m/\gamma}\right)$ is of class C_b^∞, we are finished. □

2.2 The case of discrete time maximum/minimum: $I = \{0 \leq t_0 < \cdots < t_i < \cdots < t_N \leq T\}$

In that situation, to find dominating processes with good properties is much easier than for the continuous case. As before, the extrema process $Y_t = \max_{0 \leq i \leq N : t_i \leq t}(X_{t_i} - x) - \min_{0 \leq i \leq N : t_i \leq t}(X_{t_i} - x)$ is still valid, only for the computation of Δ: it clearly satisfies (2), assumptions **(H)** and **(R(1))**.
For higher sensibilities, following the idea of using the averaged modulus continuity process from Garsia-Rodemich-Rumsey's Lemma, let us consider the non-decreasing, adapted and right-continuous process

$$Y_t = \sqrt{N \sum_{1 \leq i \leq N : t_i \leq t} (X_{t_i} - X_{t_{i-1}})^2}. \tag{12}$$

Lemma 2.2. *One has:*

i) For any $t \in \{t_i : 0 \leq i \leq N\}$, one has $|X_t - x| \leq Y_t$.

ii) The assumption **(H)** *is satisfied.*

iii) For any $q \geq 1$, assumption **(R(q))** *is satisfied.*

Proof. For $t = t_j$, one has $|X_t - x| \leq \sum_{i=1}^j |X_{t_i} - X_{t_{i-1}}| \leq Y_t$, using Jensen's inequality: this proves Assertion i). Others assertions are also easy to justify, we omit the details. □

3 Numerical results

In this section, we give illustrations of the efficiency of these integration by parts formulae from Theorem 1.1. For this, let us consider a Black & Scholes model, with interest rate $r = 5\%$, volatility $\sigma = 20\%$ and options with maturity $T = 1$ year; the initial stock price is $S_0 = 100$. Since the law of $(S_T, \min_{0 \leq t \leq T} S_t)$ and $(S_T, \max_{0 \leq t \leq T} S_t)$ are explicit, one can derive in some

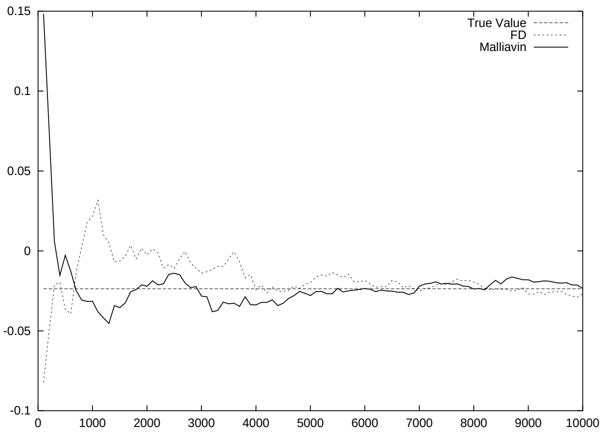

Figure 1: Delta of an up & out Call, using the averaged modulus continuity process.

cases closed formulae for the Greeks: these values are taken as references for some of our numerical tests.
On figure 1, we represent some results of the numerical computation of the Delta for a continuous time monitored up & out Call, with upper barrier $U = 120$ and Strike $K = 100$ (the x-range correspond to the number of simulations). We compare the standard Finite Difference method (*FD* plot) (see [LP94]), and our Malliavin calculus approach (*Malliavin* plot) (i.e. formula (9), up to the logarithm change and to the discounting factor). For the second method, we here use the averaged modulus continuity process (11) as the X-dominating process: the involved parameters have been taken here equal to $\gamma = 20$ and $m = 3$, but other values merely lead to same qualitative results.

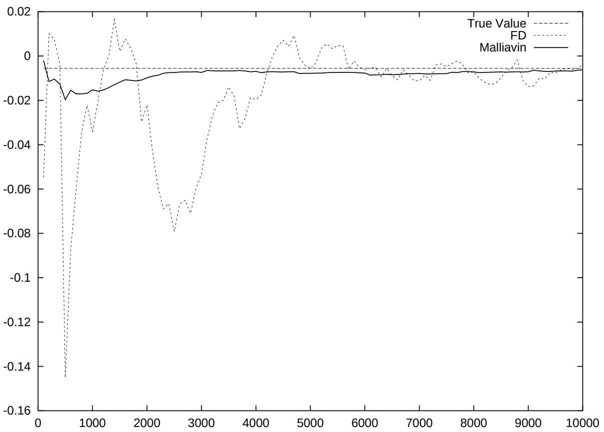

Figure 2: Gamma of an up & out Call.

Figure 2 corresponds to the computation of the Gamma, with the same parameters.
On figure 3, we use the extrema process $Y_t = \max_{s \leq t} X_s - x$ as dominating process, the upper barrier being $U = 110$. Results are good in both cases. The array below sums up the estimated

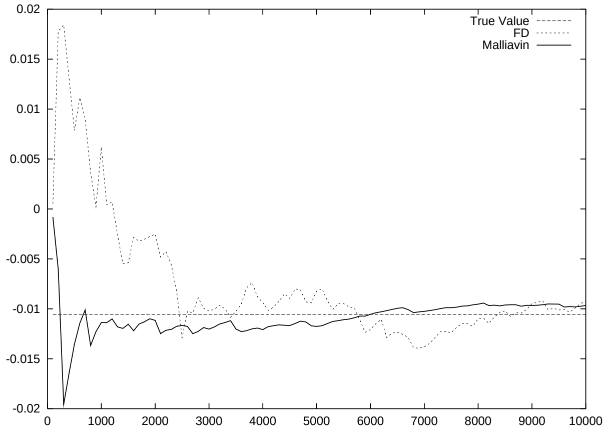

Figure 3: Delta of an up & out Call, using the extrema process.

values for the mean and the standard deviation (in bracket) of the procedures for $N = 10000$ paths.

N=10000	True Value	FD	Malliavin
Figure 1	-0.0237	-0.0269 (0.0116)	-0.0235 (0.0097)
Figure 2	-0.0055	-0.0041 (0.0233)	-0.0063 (0.0014)
Figure 3	-0.0105	-0.0092 (0.0035)	-0.0096 (0.0097)

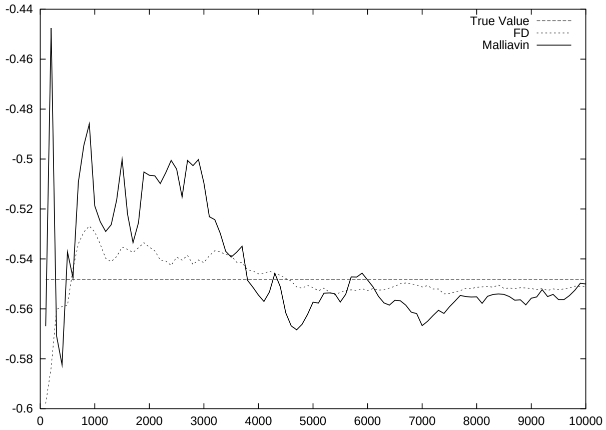

Figure 4: Delta of an lookback Put.

Concerning the influence of the support parameter a, the variance of the weights to simulate $(H_1, H_2, \cdots)$ tends to increase when a goes to 0 (for the above examples, this means U closed to S_0): nevertheless, figure 3 ($U = 110$) shows that the Malliavin calculus approach can remain good. Besides, the use of the extrema process as dominating process provides in general better results than with the averaged modulus continuity process; moreover, the computational time is smaller.

We now consider the example of a backward start lookback Put with $M_0 = 130$. Figure 4 presents the result for the FD and Malliavin methods (the latter being performed with the extrema process). Here, we observe that the variance for the second method is much higher than for the FD one: this may be explained by the fact that the payoff is merely smooth w.r.t. S_0. This phenomena is well-known and occurs e.g. for the Vanilla Call, for which a local integration by parts formula shall be performed (see [FLLLT99]). We do not investigate furthermore in that direction.

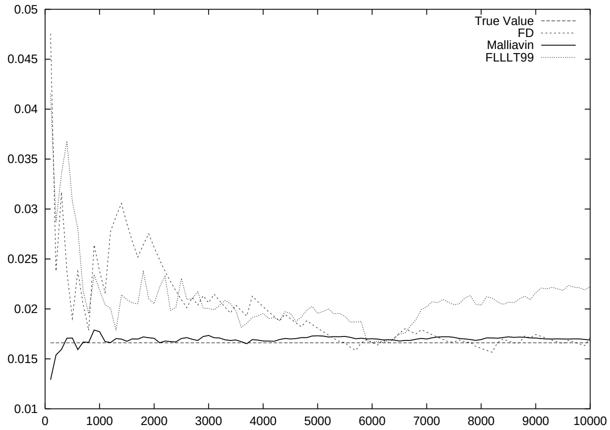

Figure 5: Delta of up in & down out digital Call.

Another interesting example is discrete time monitored barrier options. Let us consider a daily monitored up in & down out digital Call, with upper barrier $U = 130$, lower barrier $D = 70$ and strike $K = 100$. Note that here, no closed formula is available for the price of the option and its Greeks, so the *True Value* in Figure 5 is obtained using FD method with very large number of simulations. Since the payoff function

$$\Phi = \mathbb{1}_{\min_{1\le i\le 250} S_{t_i} > D}\ \mathbb{1}_{\max_{1\le i\le 250} S_{t_i} \ge U}\ \mathbb{1}_{S_T \le K}$$

involves only a finite number of S_{t_i}, results from [FLLLT99] can be applied. We find that

$$\Delta = \mathbf{E}\left(e^{-r\,T}\ \Phi\ \frac{W_{t_1}}{\sigma\ S_0\ t_1}\right).$$

Hence, we can compare the results using this formula (*FLLLT99* plot), FD method and our approach: this is achieved on Figure 5.

N=10000	True Value	FD	Malliavin	FLLLT99
Figure 5	0.0166	0.0171 (0.0028)	0.0168 (0.0004)	0.0222 (0.0033)

Here, the integration by parts involving the maximum and minimum of the process provides much more accurate procedures than the two others. Actually, the FLLLT99 approach worsens as the frequency of monitoring gets higher, i.e. as t_1 gets smaller.

4 Conclusion

In this paper, Malliavin calculus integration by part formulae have been performed for the maximum and minimum of a one dimensional diffusion: the key argument is to introduce an extra dominating process, which localizes Malliavin calculus computations, avoiding some problems with the lack of differentiability of the maximum and minimum processes.
These computations lead to new formulae for the Delta and Gamma of barrier and lookback options, in a continuous or discrete setting.
Numerical tests illustrates that this approach can give more accurate results than with FD method, when the complexity is a bit more important. In particular, in the discrete time case, the method is much more efficient than FD method or that from [FLLLT99].

Acknowledgements. We would like the anonymous referee for his valuable suggestions.

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