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NONCOLLIDING BROWNIAN MOTIONS AND HARISH-CHANDRA FORMULA

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Abstract

We consider a system of noncolliding Brownian motions introduced in our previous paper, in which the noncolliding condition is imposed in a finite time interval $(0,T]$. This is a temporally inhomogeneous diffusion process whose transition probability density depends on a value of T, and in the limit $T\to\infty$ it converges to a temporally homogeneous diffusion process called Dyson's model of Brownian motions. It is known that the distribution of particle positions in Dyson's model coincides with that of eigenvalues of a Hermitian matrix-valued process, whose entries are independent Brownian motions. In the present paper we construct such a Hermitian matrix-valued process, whose entries are sums of Brownian motions and Brownian bridges given independently of each other, that its eigenvalues are identically distributed with the particle positions of our temporally inhomogeneous system of noncolliding Brownian motions. As a corollary of this identification we derive the Harish-Chandra formula for an integral over the unitary group.

1 Introduction

Dyson introduced a Hermitian matrix-valued process whose ij-entry equals to $B_{ij}(t)/\sqrt{2}+\sqrt{-1}\widehat{B}_{ij}(t)/\sqrt{2}$, if $1\le i<j\le N$, and equals to $B_{ii}(t)$, if $i=j$, where $B_{ij}(t),\widehat{B}_{ij}(t)$, $1\le i\le j\le N$, are independent Brownian motions [5]. He found that its eigenvalues perform the Brownian motions with the drift terms acting as repulsive two-body forces proportional to the inverse of distances between them, which is now called Dyson's model of Brownian motions. A number of processes of eigenvalues have been studied for random matrices by

Bru [2, 3], Grabiner [7], König and O'Connell [18], and others, but all of them are temporally homogeneous diffusion processes. In the present paper we introduce a Hermitian matrix-valued process, whose eigenvalues give a temporally inhomogeneous diffusion process.

Let $\mathbf{Y}(t) = (Y_1(t), Y_2(t), \ldots, Y_N(t))$ be the system of N independent Brownian motions in $\mathbf{R}$ conditioned never to collide to each other. It is constructed by the h-transform, in the sense of Doob [4], of the absorbing Brownian motion in a Weyl chamber of type A_{N-1},

$$\mathbf{R}^N_< = \{\mathbf{x} \in \mathbf{R}^N : x_1 < x_2 < \cdots < x_N\} \tag{1.1}$$

with its harmonic function

$$h_N(\mathbf{x}) = \prod_{1 \le i < j \le N} (x_j - x_i), \tag{1.2}$$

$\mathbf{x} \in \mathbf{R}^N_<$. We can prove that it is identically distributed with Dyson's model of Brownian motion. In our previous papers [14, 15], we introduce another system of noncolliding Brownian motions $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_N(t))$, in which Brownian motions do not collide with each other in a finite time interval $(0, T]$. This is a temporally inhomogeneous diffusion process, whose transition probability density depends on the value of T. It is easy to see that it converges to the process $\mathbf{Y}(t)$ in the limit $T \to \infty$. Moreover, it was shown that $P(\mathbf{X}(\cdot) \in dw)$ is absolutely continuous with respect to $P(\mathbf{Y}(\cdot) \in dw)$ and that, in the case $\mathbf{X}(0) = \mathbf{Y}(0) = \mathbf{0}$, the Radon-Nikodym density is given by a constant multiple of $1/h_N(w(T))$. Since this fact can be regard as an N-dimensional generalization of the relation proved by Imhof [9] between a Brownian meander, which is temporally inhomogeneous, and a three-dimensional Bessel process, we called it *generalized Imhof's relation* [15].

The problem we consider in the present paper is to determine a matrix-valued process that realizes $\mathbf{X}(t)$ as the process of its eigenvalues. We found a hint in Yor [24] to solve this problem : equivalence in distribution between the square of the Brownian meander and the sum of the squares of a two-dimensional Bessel process and of an independent Brownian bridge. We prepare independent Brownian bridges $\beta_{ij}(t)$, $1 \le i \le j \le N$ of duration T, which are independent of the Brownian motions $B_{ij}(t)$, $1 \le i \le j \le N$, and set a Hermitian matrix-valued process $\Xi^T(t)$, $t \in [0, T]$, such that its ij-entry equals to $B_{ij}(t)/\sqrt{2} + \sqrt{-1}\beta_{ij}(t)/\sqrt{2}$, if $1 \le i < j \le N$, and it equals to $B_{ii}(t)$, if $i = j$. Then we can prove that the eigenvalues of the matrix $\Xi^T(t)$ realize $\mathbf{X}(t), t \in [0, T]$ (Theorem 2.2). This result demonstrates the fact that a temporally inhomogeneous diffusion process $\mathbf{X}(t)$ in the N dimensional space can be represented as a projection of a combination of $N(N+1)/2$ independent Brownian motions and $N(N-1)/2$ independent Brownian bridges.

It is known that Brownian motions $B_{ij}(t), 1 \le i, j \le N$ are decomposed orthogonally into the Brownian bridges $B_{ij}(t) - (t/T)B_{ij}(T)$ and the processes $(t/T)B_{ij}(T)$ (see, for instance, [23, 24]). Then the process $\Xi^T(t)$ can be decomposed into two independent matrix-valued processes $\Theta^{(1)}(t)$ and $\Theta^{(2)}(t)$ such that, for each t, the former realizes the distribution of Gaussian unitary ensemble (GUE) of complex Hermitian random matrices and the latter does of the Gaussian orthogonal ensemble (GOE) of real symmetric random matrices, respectively. This implies that the process $\Xi^T(t)$ is identified with a two-matrix model studied by Pandey and Mehta [20, 22], which is a one-parameter interpolation of GUE and GOE, if the parameter of the model is appropriately related with time t. In [14] we showed this equivalence by using the Harish-Chandra formula for an integral over the unitary group [8]. The proof of Theorem 2.2 makes effective use of our generalized version of Imhof's relation and this equivalence is established. The Harish-Chandra formula is then derived as a corollary of our theorem (Corollary 2.3).

As clarified by this paper, the Harish-Chandra integral formula implies the equivalence between temporally inhomogeneous systems of Brownian particles and multi-matrix models. This equivalence is very useful to calculate time-correlation functions of the particle systems. By using the method of orthogonal polynomials developed in the random matrix theory [19], determinantal expressions are derived for the correlations and by studying their asymptotic behaviors, infinite particle limits can be determined as reported in [21, 13].
Extensions of the present results for the systematic study of relations between noncolliding Brownian motions with geometrical restrictions (*e.g.* with an absorbing wall at the origin [15, 17]) and other random matrix ensembles than GUE and GOE (see [19, 25, 1], for instance), will be reported elsewhere [16].

2 Preliminaries and Statement of Results

2.1 Noncolliding Brownian motions

We consider the Weyl chamber of type A_{N-1} as (1.1) [6, 7]. By virtue of the Karlin-McGregor formula [11, 12], the transition density function $f_N(t,\mathbf{y}|\mathbf{x})$ of the absorbing Brownian motion in $\mathbf{R}^N_<$ and the probability $\mathcal{N}_N(t,\mathbf{x})$ that the Brownian motion started at $\mathbf{x}\in\mathbf{R}^N_<$ does not hit the boundary of $\mathbf{R}^N_<$ up to time $t>0$ are given by

$$f_N(t,\mathbf{y}|\mathbf{x}) = \det_{1\le i,j\le N}\Big[G_t(x_j,y_i)\Big], \ \mathbf{x},\mathbf{y}\in\mathbf{R}^N_<, \tag{2.1}$$

and

$$\mathcal{N}_N(t,\mathbf{x}) = \int_{\mathbf{R}^N_<} d\mathbf{y}\, f_N(t,\mathbf{y}|\mathbf{x}),$$

respectively, where $G_t(x,y)=(2\pi t)^{-1/2}\ e^{-(y-x)^2/2t}$. The function $h_N(\mathbf{x})$ given by (1.2) is a strictly positive harmonic function for absorbing Brownian motion in the Weyl chamber. We denote by $\mathbf{Y}(t)=(Y_1(t),Y_2(t),\dots,Y_N(t)), t\in[0,\infty)$ the corresponding Doob h-transform [4], that is the temporally homogeneous diffusion process with transition probability density $p_N(s,\mathbf{x},t,\mathbf{y})$;

$$p_N(0,\mathbf{0},t,\mathbf{y}) = \frac{t^{-N^2/2}}{C_1(N)}\exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\}h_N(\mathbf{y})^2, \tag{2.2}$$

$$p_N(s,\mathbf{x},t,\mathbf{y}) = \frac{1}{h_N(\mathbf{x})}f_N(t-s,\mathbf{y}|\mathbf{x})h_N(\mathbf{y}), \tag{2.3}$$

for $0<s<t<\infty$, $\mathbf{x},\mathbf{y}\in\mathbf{R}^N_<$, where $C_1(N)=(2\pi)^{N/2}\prod_{j=1}^N\Gamma(j)$. The process $\mathbf{Y}(t)$ represents the system of N Brownian motions conditioned never to collide. The diffusion process $\mathbf{Y}(t)$ solves the equation of Dyson's Brownian motion model [5] :

$$dY_i(t) = dB_i(t) + \sum_{1\le j\le N, j\ne i}\frac{1}{Y_i(t)-Y_j(t)}dt, \quad t\in[0,\infty), \quad i=1,2,\dots,N, \tag{2.4}$$

where $B_i(t)$, $i=1,2,\dots,N$, are independent one dimensional Brownian motions.
For a given $T>0$, we define

$$g_N^T(s,\mathbf{x},t,\mathbf{y}) = \frac{f_N(t-s,\mathbf{y}|\mathbf{x})\mathcal{N}_N(T-t,\mathbf{y})}{\mathcal{N}_N(T-s,\mathbf{x})}, \tag{2.5}$$

for $0 < s < t \le T,\ \mathbf{x},\mathbf{y} \in \mathbf{R}^N_<$. The function $g_N^T(s,\mathbf{x},t,\mathbf{y})$ can be regarded as the transition probability density from the state $\mathbf{x} \in \mathbf{R}^N_<$ at time s to the state $\mathbf{y} \in \mathbf{R}^N_<$ at time t, and associated with the temporally inhomogeneous diffusion process, $\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_N(t))$, $t \in [0,T]$, which represents the system of N Brownian motions conditioned not to collide with each other in a finite time interval $[0,T]$. It was shown in [15] that as $|\mathbf{x}| \to 0$, $g_N^T(0,\mathbf{x},t,\mathbf{y})$ converges to

$$g_N^T(0,\mathbf{0},t,\mathbf{y}) = \frac{T^{N(N-1)/4} t^{-N^2/2}}{C_2(N)} \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} h_N(\mathbf{y}) \mathcal{N}_N(T-t,\mathbf{y}), \tag{2.6}$$

where $C_2(N) = 2^{N/2}\prod_{j=1}^N \Gamma(j/2)$. Then the diffusion process $\mathbf{X}(t)$ solves the following equation:

$$dX_i(t) = dB_i(t) + b_i^T(t,\mathbf{X}(t))dt, \quad t \in [0,T], \quad i = 1,2,\dots,N, \tag{2.7}$$

where

$$b_i^T(t,\mathbf{x}) = \frac{\partial}{\partial x_i} \ln \mathcal{N}_N(T-t,\mathbf{x}), \qquad i = 1,2,\dots,N.$$

From the transition probability densities (2.2), (2.3) and (2.6), (2.5) of the processes, we have the following relation between the processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ in the case $\mathbf{X}(0) = \mathbf{Y}(0) = \mathbf{0}$ [14, 15]:

$$P(\mathbf{X}(\cdot) \in dw) = \frac{C_1(N) T^{N(N-1)/4}}{C_2(N) h_N(w(T))} P(\mathbf{Y}(\cdot) \in dw). \tag{2.8}$$

This is the generalized form of the relation obtained by Imhof [9] for the Brownian meander and the three-dimensional Bessel process. Then, we call it *generalized Imhof's relation.*

2.2 Hermitian matrix-valued processes

We denote by $\mathcal{H}(N)$ the set of $N \times N$ complex Hermitian matrices and by $\mathcal{S}(N)$ the set of $N \times N$ real symmetric matrices. We consider complex-valued processes $x_{ij}(t)$, $1 \le i,j \le M$ with $x_{ij}(t) = x_{ji}(t)^\dagger$, and Hermitian matrix-valued processes $\Xi(t) = (x_{ij}(t))_{1\le i,j\le N}$.
Here we give two examples of Hermitian matrix-valued process. Let $B_{ij}^{\mathrm{R}}(t)$, $B_{ij}^{\mathrm{I}}(t)$, $1 \le i \le j \le N$ be independent one dimensional Brownian motions. Put

$$x_{ij}^{\mathrm{R}}(t) = \begin{cases} \dfrac{1}{\sqrt{2}} B_{ij}^{\mathrm{R}}(t), & \text{if } i<j, \\ B_{ii}^{\mathrm{R}}(t), & \text{if } i=j, \end{cases} \quad \text{and} \quad x_{ij}^{\mathrm{I}}(t) = \begin{cases} \dfrac{1}{\sqrt{2}} B_{ij}^{\mathrm{I}}(t), & \text{if } i<j, \\ 0, & \text{if } i=j, \end{cases}$$

with $x_{ij}^{\mathrm{R}}(t) = x_{ji}^{\mathrm{R}}(t)$ and $x_{ij}^{\mathrm{I}}(t) = -x_{ji}^{\mathrm{I}}(t)$ for $i > j$.

(i) **GUE type matrix-valued process.** Let $\Xi^{\mathrm{GUE}}(t) = (x_{ij}^{\mathrm{R}}(t) + \sqrt{-1}x_{ij}^{\mathrm{I}}(t))_{1\le i,j\le N}$. For fixed $t \in [0,\infty)$, $\Xi^{\mathrm{GUE}}(t)$ is in the Gaussian unitary ensemble (GUE), that is, its probability density function with respect to the volume element $\mathcal{U}(dH)$ of $\mathcal{H}(N)$ is given by

$$\mu^{\mathrm{GUE}}(H,t) = \frac{t^{-N^2/2}}{C_3(N)} \exp\left\{-\frac{1}{2t}\mathrm{Tr}H^2\right\}, \quad H \in \mathcal{H}(N),$$

where $C_3(N) = 2^{N/2}\pi^{N^2/2}$. Let $\mathbf{U}(N)$ be the space of all $N \times N$ unitary matrices. For any $U \in \mathbf{U}(N)$, the probability $\mu^{\rm GUE}(H,t)\mathcal{U}(dH)$ is invariant under the automorphism $H \to U^\dagger HU$. It is known that the distribution of eigenvalues $\mathbf{x} \in \mathbf{R}^N_<$ of the matrix ensembles is given as

$$g^{\rm GUE}(\mathbf{x},t) = \frac{t^{-N^2/2}}{C_1(N)} \exp\left\{-\frac{|\mathbf{x}|^2}{2t}\right\} h_N(\mathbf{x})^2,$$

[19], and so $p_N(0,\mathbf{0},t,\mathbf{x}) = g^{\rm GUE}(\mathbf{x},t)$ from (2.2).

(ii) **GOE type matrix-valued process.** Let $\Xi^{\rm GOE}(t) = (x^{\rm R}_{ij}(t))_{1\le i,j\le N}$. For fixed $t \in [0,\infty)$, $\Xi^{\rm GOE}(t)$ is in the Gaussian orthogonal ensemble (GOE), that is, its probability density function with respect to the volume element $\mathcal{V}(dA)$ of $\mathcal{S}(N)$ is given by

$$\mu^{\rm GOE}(A,t) = \frac{t^{-N(N+1)/4}}{C_4(N)} \exp\left\{-\frac{1}{2t}{\rm Tr}A^2\right\}, \quad A \in \mathcal{S}(N),$$

where $C_4(N) = 2^{N/2}\pi^{N(N+1)/4}$. Let $\mathbf{O}(N)$ be the space of all $N \times N$ real orthogonal matrices. For any $V \in \mathbf{O}(N)$, the probability $\mu^{\rm GOE}(H,t)\mathcal{V}(dA)$ is invariant under the automorphism $A \to V^TAV$. It is known that the probability density of eigenvalues $\mathbf{x} \in \mathbf{R}^N_<$ of the matrix ensemble is given as

$$g^{\rm GOE}(\mathbf{x},t) = \frac{t^{-N(N+1)/4}}{C_2(N)} \exp\left\{-\frac{|\mathbf{x}|^2}{2t}\right\} h_N(\mathbf{x}),$$

[19], and so $g^t_N(0,\mathbf{0},t,\mathbf{x}) = g^{\rm GOE}(\mathbf{x},t)$ from (2.6).

We denote by $U(t) = (u_{ij}(t))_{1\le i,j\le N}$ the family of unitary matrices which diagonalize $\Xi(t)$:

$$U(t)^\dagger \Xi(t) U(t) = \Lambda(t) = {\rm diag}\{\lambda_i(t)\},$$

where $\{\lambda_i(t)\}$ are eigenvalues of $\Xi(t)$ such that $\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_N(t)$. By a slight modification of Theorem 1 in Bru [2] we have the following.

Proposition 2.1 *Let $x_{ij}(t)$, $1 \le i,j \le N$ be continuous semimartingales. The process $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t))$ satisfies*

$$d\lambda_i(t) = dM_i(t) + dJ_i(t), \quad i = 1,2,\ldots,N, \tag{2.9}$$

where $M_i(t)$ is the martingale with quadratic variation $\langle M_i\rangle_t = \int_0^t \Gamma_{ii}(s)ds$, and $J_i(t)$ is the process with finite variation given by

$$\begin{aligned} dJ_i(t) &= \sum_{j=1}^N \frac{1}{\lambda_i(t) - \lambda_j(t)} 1(\lambda_i \ne \lambda_j)\Gamma_{ij}(t)dt \\ &+ \text{the finite variation part of } (U(t)^\dagger d\Xi(t) U(t))_{ii} \end{aligned}$$

with

$$\Gamma_{ij}(t)dt = (U^\dagger(t)d\Xi(t)U(t))_{ij}(U^\dagger(t)d\Xi(t)U(t))_{ji}. \tag{2.10}$$

For the process $\Xi^{\rm GUE}(t)$, $d\Xi^{\rm GUE}_{ij}(t)d\Xi^{\rm GUE}_{k\ell}(t)=\delta_{i\ell}\delta_{jk}dt$ and $\Gamma_{ij}(t)=1$. The equation (2.9) is given as

$$d\lambda_i(t)=dB_i(t)+\sum_{j:j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}dt,\quad 1\leq i\leq N.$$

Hence, the process $\lambda(t)$ is the homogeneous diffusion that coincides with the system of noncolliding Brownian motions $\mathbf{Y}(t)$ with $\mathbf{Y}(0)=\mathbf{0}$.
For the process $\Xi^{\rm GOE}(t)$, $d\Xi^{\rm GOE}_{ij}(t)d\Xi^{\rm GOE}_{k\ell}(t)=\frac{1}{2}\Big(\delta_{i\ell}\delta_{jk}+\delta_{ik}\delta_{j\ell}\Big)dt$ and $\Gamma_{ij}(t)=\frac{1}{2}(1+\delta_{ij})$. The equation (2.9) is given as

$$d\lambda_i(t)=dB_i(t)+\frac{1}{2}\sum_{j:j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}dt,\quad 1\leq i\leq N.$$

2.3 Results

Let $\beta_{ij}(t)$, $1\leq i<j\leq N$ be independent one dimensional Brownian bridges of duration T, which are the solutions of the following equation:

$$\beta_{ij}(t)=B^{\rm I}_{ij}(t)-\int_0^t\frac{\beta_{ij}(s)}{T-s}ds,\quad 0\leq t\leq T.$$

For $t\in[0,T]$, we put

$$\xi_{ij}(t)=\begin{cases}\dfrac{1}{\sqrt{2}}\beta_{ij}(t), & \text{if } i<j,\\[2ex] 0, & \text{if } i=j,\end{cases}$$

with $\xi_{ij}(t)=-\xi_{ji}(t)$ for $i>j$. We introduce the $\mathcal{H}(N)$-valued process $\Xi^T(t)=(x^{\rm R}_{ij}(t)+\sqrt{-1}\xi_{ij}(t))_{1\leq i,j\leq N}$. Then, the main result of this paper is the following theorem.

Theorem 2.2 *Let $\lambda_i(t)$, $i=1,2,\dots,N$ be the eigenvalues of $\Xi^T(t)$ with $\lambda_1(t)\leq\lambda_2(t)\leq\cdots\leq\lambda_N(t)$. The process $\boldsymbol{\lambda}(t)=(\lambda_1(t),\lambda_2(t),\dots,\lambda_N(t))$ is the temporally inhomogeneous diffusion that coincides with the noncolliding Brownian motions $\mathbf{X}(t)$ with $\mathbf{X}(0)=\mathbf{0}$.*

As a corollary of the above result, we have the following formula, which is called the Harish-Chandra integral formula [8] (see also [10, 19]). Let dU be the Haar measure of the space $\mathbf{U}(N)$ normalized as $\int_{\mathbf{U}(N)}dU=1$.

Corollary 2.3 *Let $\mathbf{x}=(x_1,x_2,\dots,x_N),\mathbf{y}=(y_1,y_2,\dots,y_N)\in\mathbf{R}^N_<$. Then*

$$\int_{\mathbf{U}(N)}dU\,\exp\left\{-\frac{1}{2\sigma^2}{\rm Tr}(\Lambda_{\mathbf{x}}-U^\dagger\Lambda_{\mathbf{y}}U)^2\right\}=\frac{C_1(N)\sigma^{N^2}}{h_N(\mathbf{x})h_N(\mathbf{y})}\det_{1\leq i,j\leq N}\Big[G_{\sigma^2}(x_i,y_j)\Big],$$

where $\Lambda_{\mathbf{x}}={\rm diag}\{x_1,\dots,x_N\}$ and $\Lambda_{\mathbf{y}}={\rm diag}\{y_1,\dots,y_N\}$.

Remark Applying Proposition 2.1 we derive the following equation:

$$d\lambda_i(t)=dB_i(t)+\sum_{j:j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}dt-\frac{\lambda_i(t)-\int_{\mathcal{S}(N)}\mu^{\rm GOE}(dA)(U(t)^\dagger AU(t))_{ii}}{T-t}dt,\qquad(2.11)$$

$i = 1, 2, \ldots, N$, where $U(t)$ is one of the families of unitary matrices which diagonalize $\Xi^T(t)$. From the equations (2.7) and (2.11) we have

$$\int_{\mathcal{S}(N)} \mu^{\rm GOE}(dA)(U(t)^\dagger A U(t))_{ii}$$
$$= \lambda_i(t) + (T-t)\left\{ \frac{\frac{\partial}{\partial \lambda_i}\mathcal{N}_N(T-t, \boldsymbol{\lambda}(t))}{\mathcal{N}_N(T-t, \boldsymbol{\lambda}(t))} - \sum_{j:j\neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} \right\}. \tag{2.12}$$

The function $\mathcal{N}_N(t, \mathbf{x})$ is expressed by a Pfaffian of the matrix whose ij-entry is $\Psi((x_j - x_i)/2\sqrt{t})$ with $\Psi(u) = \int_0^u e^{-v^2} dv$. (See Lemma 2.1 in [15].) Then the right hand side of (2.12) can be written explicitly.

3 Proofs

3.1 Proof of Theorem 2.2

For $y \in \mathbf{R}$ and $1 \le i, j \le N$, let $\beta_{ij}^\sharp(t) = \beta_{ij}^\sharp(t:y)$, $t \in [0,T]$, $\sharp = \mathrm{R}, \mathrm{I}$, be diffusion processes which satisfy the following stochastic differential equations:

$$\beta_{ij}^\sharp(t:y) = B_{ij}^\sharp(t) - \int_0^t \frac{\beta_{ij}^\sharp(s:y) - y}{T-s} ds, \quad t \in [0,T]. \tag{3.1}$$

These processes are Brownian bridges of duration T starting form 0 and ending at y. For $H = (y_{ij}^{\rm R} + \sqrt{-1} y_{ij}^{\rm I})_{1 \le i,j \le N} \in \mathcal{H}(N)$ we put

$$\xi_{ij}^{\rm R}(t : y_{ij}^{\rm R}) = \begin{cases} \dfrac{1}{\sqrt{2}} \beta_{ij}^{\rm R}(t : \sqrt{2} y_{ij}^{\rm R}), & \text{if } i < j, \\ \beta_{ii}^{R}(t : y_{ii}^{\rm R}), & \text{if } i = j, \end{cases}$$
$$\xi_{ij}^{\rm I}(t : y_{ij}^{\rm I}) = \begin{cases} \dfrac{1}{\sqrt{2}} \beta_{ij}^{\rm I}(t : \sqrt{2} y_{ij}^{\rm I}), & \text{if } i < j, \\ 0, & \text{if } i = j, \end{cases}$$

with $\xi_{ij}^{\rm R}(t : y_{ij}^{\rm R}) = \xi_{ji}^{\rm R}(t : y_{ji}^{\rm R})$ and $\xi_{ij}^{\rm I}(t : y_{ij}^{\rm I}) = -\xi_{ji}^{\rm I}(t : y_{ji}^{\rm I})$ for $i > j$. We introduce the $\mathcal{H}(N)$-valued process $\Xi^T(t : H) = (\xi_{ij}^{\rm R}(t : y_{ij}^{\rm R}) + \sqrt{-1}\xi_{ij}^{\rm I}(t : y_{ij}^{\rm I}))_{1 \le i,j \le N}$, $t \in [0,T]$. From the equation (3.1) we have the equality

$$\Xi^T(t : H) = \Xi^{\rm GUE}(t) - \int_0^t \frac{\Xi^T(s : H) - H}{T-s} ds, \ t \in [0,T]. \tag{3.2}$$

Let H_U be a random matrix with distribution $\mu^{\rm GUE}(\cdot, T)$, and A_O be a random matrix with distribution $\mu^{\rm GOE}(\cdot, T)$. Note that $\beta_{ij}^\sharp(t : Y)$, $t \in [0,T]$ is a Brownian motion when Y is a Gaussian random variable with variance T, which is independent of $B_{ij}^\sharp(t), t \in [0,T]$. Then when H_U and A_O are independent of $\Xi^{\rm GUE}(t), t \in [0,T]$,

$$\Xi^T(t : H_U) = \Xi^{\rm GUE}(t), \quad t \in [0,T], \tag{3.3}$$
$$\Xi^T(t : A_O) = \Xi^T(t), \quad t \in [0,T], \tag{3.4}$$

in the sense of distribution. Since the distribution of the process $\Xi^{\rm GUE}(t)$ is invariant under any unitary transformation, we obtain the following lemma from (3.2).

Lemma 3.1 *For any $U \in \mathbf{U}(N)$ we have*

$$U^\dagger \Xi^T(t:H)U = \Xi^T(t:U^\dagger HU), \quad t \in [0,T],$$

in distribution.

From the above lemma it is obvious that if $H^{(1)}$ and $H^{(2)}$ are $N \times N$ Hermitian matrices having the same eigenvalues, the processes of eigenvalues of $\Xi^T(t:H^{(1)}), t \in [0,T]$ and $\Xi(t:H^{(2)}), t \in [0,T]$ are identical in distribution. For an $N \times N$ Hermitian matrix H with eigenvalues $\{a_i\}_{1\le i\le N}$, we denote the probability distribution of the process of the eigenvalues of $\Xi^T(t:H)$ by $Q^T_{0,\mathbf{a}}(\cdot), t \in [0,T]$. We also denote by $Q^{\rm GUE}(\cdot)$ the distribution of the process of eigenvalues of $\Xi^{\rm GUE}(t), t \in [0,T]$, and by $Q^T(\cdot)$ that of $\Xi^T(t), t \in [0,T]$. From the equalities (3.3) and (3.4) we have

$$Q^{\rm GUE}(\cdot) = \int_{\mathbf{R}^{\mathbf{N}}_<} Q^T_{0,\mathbf{a}}(\cdot) g^{\rm GUE}(\mathbf{a},T)d\mathbf{a},$$
$$Q^T(\cdot) = \int_{\mathbf{R}^{\mathbf{N}}_<} Q^T_{0,\mathbf{a}}(\cdot) g^{\rm GOE}(\mathbf{a},T)d\mathbf{a}.$$

Since $Q^{\rm GUE}(\cdot)$ is the distribution of the temporally homogeneous diffusion process $\mathbf{Y}(t)$ which describes noncolliding Brownian motions, by our generalized Imhof's relation (2.8) we can conclude that $Q^T(\cdot)$ is the distribution of the temporally inhomogeneous diffusion process $\mathbf{X}(t)$ which describes our noncolliding Brownian motions. ∎

3.2 Proof of Corollary 2.3

By (2.6) we have

$$\begin{aligned}
g^T_N(0,\mathbf{0},t,\mathbf{y}) &= \frac{1}{C_2(N)} T^{N(N-1)/4} t^{-N^2/2} \exp\Big\{-\frac{|\mathbf{y}|^2}{2t}\Big\} h_N(\mathbf{y}) \\
&\quad \times \int_{\mathbf{R}^N_<} d\mathbf{z} \det_{1\le i,j\le N}\left[\frac{1}{\sqrt{2\pi(T-t)}} \exp\left\{-\frac{(y_j-z_i)^2}{2(T-t)}\right\}\right] \\
&= \frac{1}{C_2(N)} T^{N(N-1)/4} t^{-N^2/2} (2\pi(T-t))^{-N/2} h_N(\mathbf{y}) \\
&\quad \times \int_{\mathbf{R}^N_<} d\mathbf{z} \det_{1\le i,j\le N}\left[\exp\left\{-\frac{y_j^2}{2t} - \frac{(y_j-z_i)^2}{2(T-t)}\right\}\right] \\
&= \frac{1}{C_2(N)} T^{N(N-1)/4} t^{-N^2/2} (2\pi(T-t))^{-N/2} h_N(\mathbf{y}) \\
&\quad \times \int_{\mathbf{R}^N_<} d\mathbf{z} \exp\left\{-\frac{|\mathbf{z}|^2}{2T}\right\} \det_{1\le i,j\le N}\left[\exp\left\{-\frac{T}{2t(T-t)}\left(y_j - \frac{t}{T}z_i\right)^2\right\}\right].
\end{aligned}$$

Setting $(t/T)z_i = a_i$, $i = 1, 2, \dots, N$, $t(T-t)/T = \sigma^2$ and $T/t^2 = \alpha$, we have

$$g_N^T(0, \mathbf{0}, t, \mathbf{y}) = \frac{(2\pi)^{-N/2}}{C_2(N)} \sigma^{-N} \alpha^{N(N+1)/4} h_N(\mathbf{y}) \times \int_{\mathbf{R}_<^N} d\mathbf{a} \exp\left\{-\frac{\alpha}{2}|\mathbf{a}|^2\right\} \det_{1\le i,j\le N}\left[\exp\left\{-\frac{1}{2\sigma^2}(y_j - a_i)^2\right\}\right]. \tag{3.5}$$

We write the transition probability density of the process $\Xi^T(t)$ by $q_N^T(s, H_1, t, H_2)$, $0 \le s < t \le T$, for $H_1, H_2 \in \mathcal{H}(N)$. Then by Theorem 2.2 and the fact that $\mathcal{U}(dH) = C_U(N) h_N(\mathbf{y})^2 dU d\mathbf{y}$, with $C_U(N) = C_3(N)/C_1(N)$, we have

$$g_N^T(0, \mathbf{0}, t, \mathbf{y}) = C_U(N) h_N(\mathbf{y})^2 \int_{\mathbf{U}(N)} dU \; q_N^T(0, O, t, U^\dagger \Lambda_{\mathbf{y}} U), \tag{3.6}$$

where O is the zero matrix. We introduce the $\mathcal{H}(N)$-valued process $\Theta^{(1)}(t) = (\theta_{ij}^{(1)}(t))_{1\le i,j\le N}$ and the $\mathcal{S}(N)$-valued process $\Theta^{(2)}(t) = (\theta_{ij}^{(2)}(t))_{1\le i,j\le N}$ which are defined by

$$\theta_{ij}^{(1)}(t) = \begin{cases} \dfrac{1}{\sqrt{2}}\left\{B_{ij}^{\mathrm{R}}(t) - \dfrac{t}{T}B_{ij}^{\mathrm{R}}(T)\right\} + \dfrac{\sqrt{-1}}{\sqrt{2}}\beta_{ij}(t), & \text{if } i < j, \\[2ex] B_{ii}^{\mathrm{R}}(t) - \dfrac{t}{T}B_{ii}^{\mathrm{R}}(T), & \text{if } i = j, \end{cases}$$

and

$$\theta_{ij}^{(2)}(t) = \begin{cases} \dfrac{t}{\sqrt{2}T}B_{ij}^{\mathrm{R}}(T), & \text{if } i < j, \\[2ex] \dfrac{t}{T}B_{ii}^{\mathrm{R}}(T), & \text{if } i = j, \end{cases}$$

respectively. Then $\Xi^T(t) = \Theta^{(1)}(t) + \Theta^{(2)}(t)$. Note that $B_{ij}^{\mathrm{R}}(t) - (t/T)B_{ij}^{\mathrm{R}}(T)$ are Brownian bridges of duration T which are independent of $(t/T)B_{ij}^{\mathrm{R}}(T)$. Hence $\Theta^{(1)}(t)$ is in the GUE and $\Theta^{(2)}(t)$ is in the GOE independent of $\Theta^{(1)}(t)$. Since $E[\theta_{ii}^{(1)}(t)^2] = \sigma^2$ and $E[\theta_{ii}^{(2)}(t)^2] = 1/\alpha$, the transition probability density $q_N^T(0, O, t, H)$ can be written by

$$\begin{aligned} q_N^T(0, O, t, H) &= \int_{\mathcal{S}(N)} \mathcal{V}(dA)\; \mu^{\mathrm{GOE}}\left(A, \frac{1}{\alpha}\right) \mu^{\mathrm{GUE}}(H - A, \sigma^2) \\ &= \frac{C_O(N)\sigma^{-N^2}\alpha^{N(N+1)/4}}{C_3(N)C_4(N)} \int_{\mathbf{R}_<^N} d\mathbf{a}\; h_N(\mathbf{a}) \exp\left\{-\frac{\alpha}{2}|\mathbf{a}|^2 - \frac{1}{2\sigma^2}\mathrm{Tr}(H - \Lambda_{\mathbf{a}})^2\right\}, \end{aligned} \tag{3.7}$$

where we used the fact $\mathcal{V}(dA) = C_O(N) h_N(\mathbf{a}) dV d\mathbf{a}$ with the Haar measure dV of the space $\mathbf{O}(N)$ normalized as $\int_{\mathbf{O}(N)} dV = 1$, and $C_O(N) = C_4(N)/C_2(N)$. Combining (3.5), (3.6) and (3.7) we have

$$\begin{aligned} &\frac{C_1(N)\sigma^{N^2-N}}{(2\pi)^{N/2}h_N(\mathbf{y})} \int_{\mathbf{R}_<^N} d\mathbf{a} \exp\left\{-\frac{\alpha}{2}|\mathbf{a}|^2\right\} \det_{1\le i,j\le N}\left[\exp\left\{-\frac{1}{2\sigma^2}(y_j - a_i)^2\right\}\right] \\ &= \int_{\mathbf{R}_<^N} d\mathbf{a}\, h_N(\mathbf{a}) \exp\left\{-\frac{\alpha}{2}|\mathbf{a}|^2\right\} \int_{\mathbf{U}(N)} dU \exp\left\{-\frac{1}{2\sigma^2}\mathrm{Tr}(U^\dagger\Lambda_{\mathbf{y}}U - \Lambda_{\mathbf{a}})^2\right\}. \end{aligned} \tag{3.8}$$

For each $\sigma > 0$, (3.8) holds for any $\alpha > 0$ and we have

$$\frac{C_1(N)\sigma^{N^2}}{h_N(\mathbf{y})h_N(\mathbf{a})} \det_{1\le i,j\le N}\left[\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}(y_j-a_i)^2\right\}\right]$$
$$= \int_{\mathbf{U}(N)} dU \exp\left\{-\frac{1}{2\sigma^2}\mathrm{Tr}(U^\dagger\Lambda_{\mathbf{y}}U-\Lambda_{\mathbf{a}})^2\right\}.$$

This completes the proof. ∎

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