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# LINEAR SPEED LARGE DEVIATIONS FOR PERCOLATION CLUSTERS

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*Abstract*
Let $C_n$ be the origin-containing cluster in subcritical percolation on the lattice $\frac{1}{n}\mathbb{Z}^d$, viewed as a random variable in the space $\Omega$ of compact, connected, origin-containing subsets of $\mathbb{R}^d$, endowed with the Hausdorff metric $\delta$. When $d \geq 2$, and $\Gamma$ is any open subset of $\Omega$, we prove that

$$\lim_{n\to\infty} \frac{1}{n} \log P(C_n \in \Gamma) = -\inf_{S\in\Gamma} \lambda(S)$$

where $\lambda(S)$ is the one-dimensional Hausdorff measure of $S$ defined using the *correlation norm*:

$$||u|| := \lim_{n\to\infty} -\frac{1}{n} \log P(u_n \in C_n)$$

where $u_n$ is $u$ rounded to the nearest element of $\frac{1}{n}\mathbb{Z}^d$. Given points $a^1, \ldots, a^k \in \mathbb{R}^d$, there are finitely many correlation-norm Steiner trees spanning these points and the origin. We show that if the $C_n$ are each conditioned to contain the points $a^1_n, \ldots, a^k_n$, then the probability that $C_n$ fails to approximate one of these trees tends to zero exponentially in $n$.

## 1 Introduction

Let $C_n$ be the origin-containing cluster in subcritical Bernoulli bond-percolation with parameter $p$ on the lattice $\frac{1}{n}\mathbb{Z}^d$; we view $C_n$ as a random variable in the space $\Omega$ of compact, connected,

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origin-containing subsets of $\mathbb{R}^d$. When the probability measure involved is clear from context, we use $P(A)$ to denote the probability of an event $A$. When $u \in \mathbb{R}^d$, let $u_n$ be the vector $u$ rounded to the nearest element in $\frac{1}{n}Z^d$. We define the "correlation norm" by

$$||u|| := \lim_{n\to\infty} -\frac{1}{n} \log P(u_n \in C_n).$$

This limit exists for all $u \in \mathbb{R}^d$ (with $||u|| \in (0, \infty)$ for $u \neq 0$) and $||\cdot||$ is a strictly convex norm (i.e., if $u$ and $v$ are not on the same line through the origin, then $||u + v|| < ||u|| + ||v||$) that is real-analytic on the Euclidean unit sphere $S^{d-1}$ [3]. Denote by $\lambda(S)$ the *one-dimensional Hausdorff measure* of the set $S$ defined with the above norm; in particular, if $S \in \Omega$ is a finite union of rectifiable arcs in $\mathbb{R}^d$, then $\lambda(S)$ is the sum of the correlation-norm lengths of those arcs.

Given a set $X \subset \mathbb{R}^d$, denote by $B_\epsilon(X)$ the set of all points of distance less than $\epsilon$ from some point in $X$. Given sets $X, Y \in \Omega$, let $\delta(X, Y)$ be their *Hausdorff distance*, i.e.,

$$\delta(X, Y) = \inf\{\epsilon : X \subset B_\epsilon(Y), Y \subset B_\epsilon(X)\}.$$

Many authors, including [1], [2], [6], [3], and [12], have investigated the shapes of "typical" large finite clusters in supercritical percolation on $\mathbb{Z}^d$ by proving surface order large deviation principles for clusters conditioned to contain at least $m$ vertices. They have shown that as $m$ gets large, the shapes of typical clusters are approximately minimizers of surfaces tension integrals, called Wulff crystals. Moreover, the surface tension integral is a rate function for a large deviation principle—with *surface order* speed $m^{d-1/d}$—on cluster shapes. These results are one way of precisely answering the questions, "What does the typical 'large' cluster look like? How unlikely are large deviations from this typical shape?"

If instead of number of vertices we define "large" in terms of, say, diameter or volume of the convex hull, then these questions can be answered for subcritical percolation using the following linear speed large deviation principle:

**Theorem 1.1.** *Let $d \geq 2$, $p < p_c$, and $\Gamma \subset \Omega$ be Borel-measurable. Then*

$$-\inf_{S\in\Gamma^o} \lambda(S) \leq \liminf_{n\to\infty} \frac{1}{n} \log P(C_n \in \Gamma) \leq \limsup_{n\to\infty} \frac{1}{n} \log P(C_n \in \Gamma) \leq -\inf_{S\in\overline{\Gamma}} \lambda(S)$$

*where $\Gamma^o$ and $\overline{\Gamma}$ are the interior and closure of $\Gamma$ with respect to the Hausdorff topology.*

In the language of [5], this says that the random variables $C_n$ satisfy a large deviation principle with respect to the Hausdorff metric topology on $\Omega$ and with speed $n$ and rate function $I(S) = \lambda(S)$. Note that since $\lambda : \Omega \to \mathbb{R}$ is continuous, this implies that

$$\lim \frac{1}{n} \log P(C_n \in \Gamma) = -\inf_{S\in\Gamma} \lambda(S)$$

whenever $\Gamma$ is an open subset of $\Omega$.

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# 2 Proof of large deviation principle

## 2.1 Exponential tightness and an equivalent formulation

We now prove Theorem 1.1. The sets $\{S|\delta(S,\{0\}) \leq \alpha\}$ are compact in the Hausdorff metric topology, and $P(\delta(C_n,\{0\}) > \alpha)$ decays exponentially in $n$ and $\alpha$. [11] This implies that the laws of the $C_n$ are *exponentially tight* (in the sense of [5], Sec. 1.2). Given this exponential tightness, Theorem 1.1 is equivalent to the statement that the following bounds hold for $S \in \Omega$:

$$\lim_{\epsilon \to 0} \mathcal{A}(S,\epsilon) \leq \lambda(S)$$

$$\lim_{\epsilon \to 0} \mathcal{B}(S,\epsilon) \geq \lambda(S)$$

where

$$\mathcal{A}(S,\epsilon) = \limsup_{n\to\infty} \frac{-1}{n} \log P(\delta(S,C_n) < \epsilon)$$

$$\mathcal{B}(S,\epsilon) = \liminf_{n\to\infty} \frac{-1}{n} \log P(\delta(S,C_n) < \epsilon)$$

This equivalence is well-known in the large deviations literature ([5], Lemma 1.2.18 and Theorem 4.1.11), and is also not hard to prove directly. We now prove the first of the two bounds above, which involves giving a lower bound on the probabilities $P(\delta(S,C_n) < \epsilon)$.

## 2.2 Lower bound on probabilities

Fix $\epsilon$ and choose $S'$ to be a connected union of finitely many line segments of the form $(a^i,b^i)$, for $1 \leq i \leq k$—intersecting one another only at endpoints—such that $\delta(S,S') < \epsilon/2$ and at least one of the segments includes the origin as an endpoint. No matter how small $\epsilon$ gets, we can always choose such an $S'$ of total length less than or equal to $\lambda(S)$. Thus, it is enough to show that

$$\liminf \frac{-1}{n} \log P(\delta(S',C_n) < \epsilon/2) \leq \lambda(S')$$

for sets $S'$ of this form.

Now, let $A_n^i$ (respectively, $A_{n,c}^i$ be the event that $a_n^i$ and $b_n^i$ are connected by *some* open path whose Hausdorff distance from the line segment $(a^i,b^i)$ is at most $\epsilon/4$ (respectively $c/n$). For any fixed $n$, $P(A_{n,c}^i)$ tends to $P(a_n^i - b_n^i \in C_n)$ as $c$ tends to $\infty$. Subadditivity arguments imply that $\liminf \frac{-1}{n} \log P(A_{n,c}^i)$ tends to $||a^i - b^i||$ as $c$ tends to infinity. It follows that $\liminf \frac{-1}{n} \log P(A_n^i) \leq ||a^i - b^i||$. The FKG inequality then implies that $\liminf \frac{-1}{n} P(\cup A_n^i) \leq \lambda(S')$.

Now, we have to show that *given* $\cup A_n^i$, the probability of the event $C_n \not\subset B_{\epsilon/2}(S')$ decays exponentially. Let $D_n$ be the event that there is a path from *any* point $x$ outside of $B_{\epsilon/2}(S')$ to *any* point $y \in B_{\epsilon/4}(S')$. This event is independent of $\cup A_n^i$. Since $D_n$ contains the event $C_n \not\subset B_{\epsilon/2}(S')$, it is enough for us to show that $P(D_n)$ decays exponentially. To see this, we introduce and sketch a proof of the following lemma. (See [3] for more delicate asymptotics of $P(u_n \in C_n)$.)

**Lemma 2.1.** *There exists a constant $\alpha$ such that $P(u_n \in C_n) \leq \alpha e^{-n||u||}$ for all $n$ and $u$.*

*Proof.* If $u = u_n$, then it is clear that $P(u_n \in C_n) \leq e^{-n||u||}$. (Simply use the FKG inequality to observe that for any integer $m$, we have $P(u_{mn} \in C_{mn}) \geq P(u_n \in C_n)^m$ and apply the standard subadditivity argument to the log limits.) If $u \neq u_n$, then it suffices to observe that $e^{-n||u||}$ and $e^{-n||u_n||}$ differ by at most a constant factor. $\square$

The probability that any particular vertex of $B_\epsilon(S')\backslash B_{\epsilon/2}(S')$ is connected to any particular vertex in $B_{\epsilon/4}(S)$ is bounded above by $\alpha \exp[-n \inf\{||u|| : |u| = \epsilon/4\}]$, where $|u|$ is the Euclidean norm. Since the number of pairs of points of this type grows polynomially in $n$, the result follows.

### 2.3 Upper bound on probabilities

Fix $\gamma > 0$ and choose a finite set of points $a^1, a^2, \ldots a^k$ in $S$ such that every collection $S'$ of line segments that contains the points $a^i$ has total length greater than $\lambda(S) - \gamma$ (or greater than some large value $N$ if $\lambda(S)$ is infinite) and that for some sufficiently small $\epsilon > 0$, this remains true if each $a^i$ is replaced by some $c^i \in B_\epsilon(a^i)$. (The reader may check that such a set of points and such an $\epsilon$ exist for any $\gamma > 0$.) We know that

$$\limsup -\frac{1}{n} \log P(\delta(S, C_n) < \epsilon)$$

is at least as large as

$$\limsup -\frac{1}{n} \log P(\text{some } c_n^i \in B_\epsilon(a^i) \text{ is contained in } C_n).$$

We claim that the latter is at least $\lambda(S) - \gamma$. If $C_n$ does contain all of the $c_n^i$, then it must contain a subgraph that is a tree with the $c_n^i$ as vertices. If we remove all branches of this tree that do not contain a $c_n^i$, then a straightforward induction on $k$ shows that we are left with a tree $T$ in which at most $k-2$ vertices have more than two neighbors. Denote by $b_n^i$ the vertices with this property. The path-connectedness-in-$T$ relation puts a tree structure on the set of $b_n^i$ and $c_n^i$. Each edge of this new tree $T'$ represents a pair of these points joined by a path, and all of these paths are disjoint.
Now, given a specific set of set of points $b_n^i$ and $c_n^i$ and $T'$, we have by the BK inequality and Lemma 2.1 that the probability that these disjoint paths are contained in $C_n$ is at most $\alpha e^{-\lambda(T')n}$, where $\lambda(T')$ is the sum of the correlation lengths of the edges of $T'$, and by assumption this value is at least $\lambda(S) - \gamma$. Since the number of possible choices for the $b_n^i$ and the $c_n^i$ grows polynomially, and since $\gamma$ can be chosen arbitrarily small, this completes the proof.

### 2.4 Steiner trees

Given points $a^1, \ldots, a^k$, a (correlation norm) *Steiner tree spanning* $\{a^i\}$ *and the origin* is an element $T$ of $\Omega$ for which $\lambda(T)$ is minimal among sets containing the $\{a^i\}$. Existence of at least one Steiner tree follows from compactness arguments, and Steiner trees are trees with at most $k-2$ vertices in addition to $a^1, \ldots, a^k$ [8]. Although the Steiner tree spanning a set of points is not always unique, strict convexity of the correlation norm implies that the number of Steiner trees is always finite. See [10] for a general reference on Steiner trees. The proof of Theorem 1.1 now yields the following:

**Theorem 2.2.** *Let $d \geq 2$ and let $\Gamma \subset \Omega$ be Borel-measurable. If $C_n$ is the origin-containing cluster in a subcritical percolation conditioned on $\{a_n^i\} \subset C_n$, then*

$$-\inf_{S\in\Gamma^o} \lambda(S) - \lambda(T) \leq \liminf_{n\to\infty} \frac{1}{n} \log P(C_n \in \Gamma) \leq \limsup_{n\to\infty} \frac{1}{n} \log P(C_n \in \Gamma) \leq -\inf_{S\in\overline{\Gamma}} \lambda(S) - \lambda(T)$$

*where $T$ is any Steiner tree spanning $\{a^i\}$ and the origin.*

In other words, these conditioned $C_n$ satisfy a large deviation principle with rate function given by $I(S) = \lambda(S) - \lambda(T)$. In particular, if $T_j$, for $1 \leq j \leq m$, are the Steiner trees spanning $\{a^i\}$ and the origin, and $B_\epsilon(T_j) = \{S : \delta(S, T_j) < \epsilon\}$, then we have

$$\lim_{n\to\infty} \frac{1}{n} \log P(C_n \notin \cup B_\epsilon(T_j)) = -\inf_{S\notin \cup B_\epsilon(T_j)} \lambda(S) - \lambda(T).$$

That is, the probability that $C_n$ fails to approximate one of these Steiner trees tends to zero exponentially.

# References


[1] K.S.Alexander, J.T.Chayes and L.Chayes. *The Wulff Construction and Asymptotics of The Finite Cluster in Two-Dimensional Percolation.* Commun. Math. Phys. **131** (1990), 1–50.

[2] M.Campanino, J.T.Chayes and L.Chayes. *Gaussian Fluctuations in the Subcritical Regime of Percolation.* Prob. Th. Rel. Fields **88** (1991), 269–341.

[3] M.Campanino and D.Ioffe. *Ornstein-Zernike Theory For The Bernoulli Bond Percolation On $\mathbb{Z}^d$.* Ann.Prob. **30** (2002), no. 2, 652–682.

[4] O. Couronné. *A large deviations result for the subcritical Bernoulli percolation.* [preprint]

[5] A. Dembo, O. Zeitouni. *Large Deviations and Applications* (second edition). Springer, New York (1998).

[6] R.L.Dobrushin, R.Kotecký and S.Shlosman. *Wulff construction. A global shape from local interaction.* Translation of Mathematical Monographs **104** American Mathematical Society, Providence (1992).

[7] R.G.Edwards and A.D.Sokal. *Generalization of the Fortuin-Kesteleyn-Swendsen-Wang representation and Monte Carlo algorithm.* The Physical Review D **38** (1988), 2009–2012.

[8] E.N. Gilbert and H.O. Pollak. *Steiner minimal trees.* SIAM Journal on Applied Mathematics **16** (1968), 1–29.

[9] G.R.Grimmett. *Percolation* (second edition). Springer, New York (1999).

[10] R. Hwang, D. Richards, and P. Winter. *The Steiner Tree Problem.* Vol. **53** of Annals of Discrete Math., North Holland, Amsterdam (1992).

[11] M.V.Menshikov. *Coincidence of critical points in percolation problems.* Soviet Mathematics Doklady **33** (1986), 856–859.

[12] A.Pisztora. *Surface order large deviations for Ising, Potts and percolation models.* Probab. Th. Rel. Fields **104** (1996), 427–466.