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LOOP-ERASED WALKS INTERSECT INFINITELY OFTEN IN FOUR DIMENSIONS

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Abstract
In this short note we show that the paths two independent loop-erased random walks in four dimensions intersect infinitely often. We actually prove the stronger result that the cut-points of the two walks intersect infinitely often.

Let $S(t)$ be a transient Markov chain with integer time t on a countable state space. Associated to S, is the loop-erased process $\hat{S}$ obtained by erasing loops in chronological order defined as follows. Let

$$s_0 = \sup\{t : S(t) = S(0)\},$$

and for $n > 0$,

$$s_n = \sup\{t : S(t) = S(s_{n-1} + 1)\}.$$

Then the loop-erased process $\hat{S}(n)$ is defined by

$$\hat{S}(n) = S(s_n).$$

This is well-defined with probability one by transience. (In many cases the process is well-defined for recurrent chains with a slightly modified definition, but we are only interested in transient chains here.) Note that the path of the loop-erased process is contained in the path of the original process. The loop-erased process was first studied when S is simple random walk on the integer lattice (see [2] and references therein), but recent results relating loop-erased processes to uniform spanning trees has caused an interest in the loop-erased process for arbitrary Markov chains.

Lyons, Peres, and Schramm [6] have recently shown that if S^1, S^2 are two independent realizations of the Markov chain starting at the same point then the probability that $S^1[0,\infty) \cap$

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$\hat{S}^2(0,\infty)$ is infinite is one if and only if the probability that $S^1[0,\infty)\cap S^2[0,\infty)$ is infinite is one. In the case of simple random walk, this says that the loop-erased walk and the simple random walk intersect infinitely often in dimensions three and four (this was already known, see [2]). One can naturally ask the same question for two loop-erased processes: what is the probability that the loop erasures of two processes intersect infinitely often? The purpose of this note is to show that loop-erased paths intersect infinitely often in four dimensions, i.e., we prove that if S^1, S^2 are simple random walks starting at the origin in $\mathbb{Z}^4$, then

$$\mathbf{P}\{\#[\hat{S}^1[0,\infty)\cap\hat{S}^2[0,\infty)]=\infty\}=1. \tag{1}$$

It is unknown rigorously whether or not this is true for $d=3$. Numerical simulations [1] suggest that the paths of loop-erased walks in three dimensions have dimension greater than 1.5, and hence we would expect them to intersect infinitely often.

We will prove a stronger result than (1). If $x\in\mathbb{Z}^4$, let

$$\tau_x^i=\inf\{t: S^i(t)=x\}.$$

We will say that x is a cut-point for S^i if $\tau_x^i<\infty$ and

$$S^i[0,\tau_x^i]\cap S^i[\tau_x^i+1,\infty)=\emptyset.$$

It might be more natural to say that x is a cut-point if there exists an n with $S[0,n]\cap S[n+1,\infty)=\emptyset$ and $S(n)=x$, but it will be more convenient for us to use the stronger condition above. Note that if x is a cut-point for S^i then

$$x\in\hat{S}^i[0,\infty).$$

We call x a double cut-point if x is a cut-point for both S^1 and S^2. Let H denote the set of double cut-points; let

$$C^n=\{x\in\mathbb{Z}^4:|x|<2^n\},$$

and

$$H_n=H\cap(C^{n+1}\setminus C^n).$$

We will show that

$$\mathbf{P}\{H_n\neq\emptyset \text{ i. o.}\}=1, \tag{2}$$

from which we can conclude (1). We use a second moment argument in the proof. At the end of this note we show that (2) is not true in three dimensions, i.e., with probability one, the cut points of two independent three dimensional simple random walks intersect only finitely often. Hence, although we expect the loop-erased walks to intersect infinitely often in three dimensions, proving this will require looking at more than cut-points.

Before proceeding with the proof of (2), we will review results about cut-points. In this paper we write

$$f(n)\sim g(n),\quad n\to\infty$$

if

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=1;$$

we write

$$f(n)\asymp g(n),$$

if there exists positive constants c_1, c_2 with

$$c_1 g(n) \leq f(n) \leq c_2 g(n);$$

and we write

$$f(n) \approx g(n),$$

if $\log f(n) \asymp \log g(n)$. Let

$$T_n^i = \inf\{t : |S^i(t)| \geq 2^n\}.$$

The following was proved in [3]. (Note that (4) follows almost immediately from (3) and vice versa since $T_n^i \approx 2^{2n}$.)

Proposition 1 *There exists a constant c_3 such that if S^1, S^2 are independent simple random walks starting at the origin in $\mathbb{Z}^4$,*

$$\mathbf{P}\{S^1[0,n] \cap S^2[1,n] = \emptyset\} \sim c_3 (\log n)^{-1/2}, \tag{3}$$

$$\mathbf{P}\{S^1[0,T_n^1] \cap S^2[1,T_n^2] = \emptyset\} \sim \frac{c_3}{\sqrt{2\log 2}} n^{-1/2}. \tag{4}$$

Moreover there exists a constant c such that, if S^1, S^2 start at x, y respectively, with $|x|, |y| \leq k$,

$$\mathbf{P}\{S^1[0,n^2] \cap S^2[1,n^2] = \emptyset\} \leq c[\frac{\log k}{\log n}]^{1/2}.$$

One can see from the Proposition 1, that in four dimensions being a cut-point is in some sense a local property, e.g.,

$$\lim_{n\to\infty} \mathbf{P}\{S^1[0,T_n^1] \cap S^2[1,T_n^2] \neq \emptyset \mid S^1[0,T_{n-\log n}^1] \cap S^2[1,T_{n-\log n}^2] = \emptyset\} = 0. \tag{5}$$

Suppose we are given a positive integer k and a sequence of events $\{V_m\}$ such that the event V_m depends only on

$$\{S^i(t) : T_{m-k}^i \leq t \leq T_m^i,\ i = 1,2\}.$$

Suppose also that $\mathbf{P}(V_m) \geq \alpha$ for some positive constant α independent of m. By the discrete Harnack inequality there is a constant c_4 (independent of k, m, V_m) such that for $m > k+1$,

$$\mathbf{P}[V_m \mid \mathcal{F}_{m-k-1}] \geq c_4 \mathbf{P}[V_m] \geq c_4 \alpha,$$

where $\mathcal{F}_j$ denotes the σ-algebra generated by

$$\{S^i(t) : 0 \leq t \leq T_j^i,\ i = 1,2\}.$$

In particular,

$$\mathbf{P}(V_m \mid S^1[0,T_{m-k-1}^1] \cap S^2[1,T_{m-k-1}^2] = \emptyset) \geq c_4\alpha.$$

From (5) we can see that for all m sufficiently large (sufficiently large depending only on α),

$$\mathbf{P}\{V_m; S^1[0,T_m^1] \cap S^2[1,T_m^2] = \emptyset \mid S^1[0,T_{m-k-1}^1] \cap S^2[1,T_{m-k-1}^2] = \emptyset\} \geq \frac{1}{2} c_4 \alpha. \tag{6}$$

Lemma 2 *There exist positive constants c_1, c_2 such that for any $x \neq 0$,*

$$\frac{c_1}{|x|^2\sqrt{\log|x|}} \leq \mathbf{P}\{x \text{ cut-point for } S^i\} \leq \frac{c_2}{|x|^2\sqrt{\log|x|}}.$$

Proof. Let
$$p_n(x) = \mathbf{P}\{x \text{ cut-point for } S^i; \tau_x^i = n\}.$$
Note that by translation invariance and symmetry of the random walk,
$$p_n(x) = \mathbf{P}\{S^1[0,n] \cap S^2[1,\infty) = \emptyset; S^1(n) = x; 0 \notin S^1[1,n]\}.$$
Also note that
$$\mathbf{P}\{x \text{ cut-point for } S^i\} = \sum_{n=0}^{\infty} p_n(x).$$
To get the upper bound, let integer m satisfy $2^m \leq |x| < 2^{m+1}$. Then
$$\sum_{n=0}^{\infty} p_n(x) \leq \mathbf{P}\{S^1[0,T^1_{m-1}] \cap S^2[0,T^2_{m-1}] = \emptyset\}$$
$$\cdot \mathbf{E}[\,[\sum_{n=0}^{\infty} I\{S^1(n) = x\}] \mid S^1[0,T^1_{m-1}] \cap S^2[0,T^2_{m-1}] = \emptyset\}.$$
By Proposition 1,
$$\mathbf{P}\{S^1[0,T^1_{m-1}] \cap S^2[0,T^2_{m-1}] = \emptyset\} \leq c(\log|x|)^{-1/2},$$
and the standard estimate for the Green's function of random walk in four dimensions tells us
$$\mathbf{E}[\,[\sum_{n=0}^{\infty} I\{S^1(n) = x\}] \mid S^1[0,T^1_{m-1}] \cap S^2[0,T^2_{m-1}] = \emptyset\} \leq c|x|^{-2}.$$
For the lower bound, let m be as above and let $\mathcal{B}(r) = \mathcal{B}_x(r)$ be the ball of radius $r|x|$ with center x and $-\mathcal{B}(r)$ the corresponding ball centered at $-x$. Consider the event
$$U_x = \{S^1[0,T^1_m] \cap S^2[1,T^2_m] \neq \emptyset; 0 \notin S^1[1,T^1_m];$$
$$S^1(T^1_m) \in \mathcal{B}(2/3); S^1[0,T^1_m] \cap [-\mathcal{B}(4/5)] = \emptyset; S^2(T^2_m) \in -\mathcal{B}(2/3); S^2[0,T^2_m] \cap \mathcal{B}(4/5) = \emptyset\}.$$
From the invariance principle and (6) we can see that
$$\mathbf{P}[U_x] \geq c(\log|x|)^{-1/2}.$$
An easy estimate gives that
$$\mathbf{P}\{S^2[T_m,\infty) \subset -\mathcal{B}(4/5) \cup [\mathbb{Z}^4 \setminus (C_m \cup \mathcal{B}(4/5))] \mid U_x\} \geq c.$$
Also, standard estimates for the Green's function in a ball tell us that starting at $y \in \mathcal{B}(2/3)$, the expected number of visits to x before leaving the ball of radius $\mathcal{B}(3/4)$ is bounded below by $c|x|^{-2}$. This gives the lower bound.

The following slight improvement of the above result can be proved similarly; we omit the proof.

Lemma 3 *There exists a $c > 0$ such that the following is true. Suppose $x \in \mathbb{Z}^4$ with $2^m \leq |x| < 2^{m+1}$. Then*

$$\mathbf{P}\{x \text{ cut-point for } S^i; \quad S^i[T^i_m, \infty) \cap C_{m-1} = \emptyset;$$
$$S^i[T^i_{m+3}, \infty) \cap C_{m+2} = \emptyset\} \geq \frac{c}{|x|^2 \sqrt{\log |x|}}.$$

Using the discrete Harnack inequality we can see that this lemma implies that if $2^m \leq |x| < 2^{m+1}$, $|y| \leq 2^{m-1}$, then

$$\mathbf{P}^y\{x \text{ cut-point for } S^i;$$
$$S^i[T_{m+3}, \infty) \cap C_{m+2} = \emptyset \mid S^i[T^i_m, \infty) \cap C_{m-1} = \emptyset\} \geq \frac{c}{|x|^2 \sqrt{\log |x|}}. \qquad (7)$$

From Lemma 2 we see that

$$\mathbf{P}\{x \text{ double cut-point}\} \asymp |x|^{-4} (\log |x|)^{-1}. \qquad (8)$$

By summing over x, we see that

$$\mathbf{E}[\#(H)] = \infty,$$

which suggests (2) is true.

Lemma 4 *There exists a $c > 0$ such that for all x, y with $2^n \leq |x|, |y| < 2^{n+1}$,*

$$\mathbf{P}\{x, y \text{ cut-points for } S^i\} \leq \frac{c}{2^{2n} |x-y|^2 \sqrt{n \log |x-y|}}. \qquad (9)$$

Proof. Let $k = \lfloor |x-y|/64 \rfloor$ and without loss of generality assume $k \geq 3$. We will show

$$\mathbf{P}\{x, y \text{ cut-points for } S^i; \tau_x < \tau_y\} \leq \frac{c}{2^{2n} k^2 \sqrt{n \log k}},$$

and by interchanging x and y we get the estimate. In order for x and y to be cut-points with $\tau_x < \tau_y$, it is necessary for the random walk to visit x and then visit y. Also, our definition of cut-points requires these visits to be at unique times. We will construct possible random walks by first defining them near x and y and then patching together appropriate pieces.
Let S^3, S^4, S^5, S^6 be independent random walks; S^3, S^4 starting at x, and S^5, S^6 starting at y. Define the following stopping times:

$$\rho_i = \inf\{t : |S^i(t) - x| \geq k\}, \quad i = 3, 4;$$
$$\rho_i = \inf\{t : |S^i(t) - y| \geq k\}, \quad i = 5, 6;$$
$$\eta = \inf\{t : |S^3(t) - x| \geq 2^{n-2}\}.$$

Note that $k \leq 2^{n-3}$, and hence $\rho_3 < \eta$. To get a simple random walk starting at the origin and hitting x, y in order we take any finite length random walk path starting at the origin that does not visit x or y and whose final point is $S^3(\eta)$. We then follow S^3 in reverse time from $S^3(\eta)$ to $S^3(0) = x$. Next we follow S^4 from time 0 to time ρ_4, Next we take any finite length simple random walk path from $S^4(\rho_4)$ to $S^5(\rho_5)$. Next we follow S^5 in reverse time from $S^5(\rho_5)$ to $S^5(0) = y$. Finally we follow the infinite walk S^6 starting at y. The Green's function for the number of finite simple random walk paths starting at the origin and ending at $S^3(\eta)$ (weighted by the appropriate probability) is comparable to $|x|^{-2} \asymp 2^{-2n}$. Similarly the Green's function for walks starting at $S^4(\rho_4)$ ending at $S^5(\rho_5)$ is comparable to $k^{-2} \asymp |x-y|^{-2}$. The probability that

$$S^3[0,\rho_3] \cap S^4[1,\rho_4] = \emptyset, \tag{10}$$

and the probability that

$$S^5[0,\rho_5] \cap S^6[1,\rho_6] = \emptyset, \tag{11}$$

are each comparable to $(\log k)^{-1/2}$. Finally, given (10) and (11), the probability that

$$S^3[\rho_3,\eta] \cap S^6[\rho_6,\infty) = \emptyset$$

is bounded by a constant times

$$[\frac{\log k}{\log 2^n}]^{1/2}.$$

(This last inequality follows from the last inequality in Proposition 1.) Combining all these estimates gives the upper bound.

Let Z_n denote the number of points in $C^{n+1} \setminus C^n$ that are cut-points for both S^1 and S^2. By summing 8 and 9, we see that

$$\mathbf{E}[Z_n] \geq \frac{c}{n},$$

$$\mathbf{E}[Z_n^2] \leq \frac{c \log n}{n^2}.$$

The estimate

$$\mathbf{P}\{Z_n > 0\} \geq \frac{\mathbf{E}[Z_n]^2}{\mathbf{E}[Z_n^2]},$$

for nonnegative random variables gives

$$\mathbf{P}\{Z_n > 0\} \geq \frac{c}{n \log n}.$$

In particular, the expected number of n such that $Z_n > 0$ is infinite.
To finish the argument we will say that x is a special cut-point for S^i if x is a cut-point and

$$\tau_x^i < T_{m+2}^i,$$

$$S^i[T_{m-1}^i,\infty) \cap C_{m-2} = \emptyset,$$

$$S^i[T_{m+3}^i,\infty) \cap C_{m+2} = \emptyset,$$

where integer m is chosen so that

$$2^m \leq |x| < 2^{m+1}.$$

We call x a double special cut-point if it is a special cut-point for both S^1 and S^2. From (7), we can see that if $2^n \leq |x| < 2^{n+1}$,

$$\mathbf{P}\{x \text{ special double cut-point}\} \asymp |x|^{-4}(\log|x|)^{-1} \asymp n^{-4}(\log n)^{-1}.$$

Let $\tilde{Z}_n$ denote the number of special double cut-points in $C^{n+1} \setminus C^n$. Then as before,

$$\mathbf{E}[\tilde{Z}_n] \geq \frac{c}{n}, \quad \mathbf{E}[\tilde{Z}_n^2] \leq \frac{c \log n}{n}.$$

If A_n is the event

$$A_n = \{\tilde{Z}_n > 0\},$$

then the second moment method gives

$$P[A_n] \geq \frac{c}{n \log n}.$$

Assume $n < m - 3$. Then it is easy to see that

$$\mathbf{P}[A_m \mid A_n] \leq \sup_{y_1, y_2} \mathbf{P}^{y_1, y_2}\{A_m \mid S^i[0, \infty) \cap C_{n+2} = \emptyset, i = 1, 2\},$$

where the supremum is over all $2^{n+3} \leq |y_1|, |y_2| \leq 2^{n+3} + 1$ and $\mathbf{P}^{y_1, y_2}$ denotes probabilities assuming $S^i(0) = y_i$.
Let

$$r = r_{m,n} = \sup \mathbf{P}^{y_1, y_2}[A_m],$$

where the supremum is as above. By the discrete Harnack principle, for all $2^{n+3} \leq |z_1|, |z_2| < 2^{n+3} + 1$,

$$\mathbf{P}^{z_1, z_2} \geq cr.$$

Note also that the strong Markov property applied to S^1 implies that

$$\mathbf{P}^{z_1, z_2}(A_m \mid T^1_{n+2} < \infty) \leq r,$$

and hence for all such z_1, z_2,

$$\mathbf{P}^{z_1, z_2}(A_m; S^1[0, \infty) \cap C_{n+2} = \emptyset) \geq c_1 r.$$

Repeating this argument we can see that

$$\mathbf{P}^{z_1, z_2}(A_m; S^i[0, \infty) \cap C_{n+2} = \emptyset, i = 1, 2) \geq c_2 r.$$

From this we can use the discrete Harnack principle to conclude for all such z_1, z_2,

$$\mathbf{P}^{z_1, z_2}(A_m \mid S^i[0, \infty) \cap C_{n+2} = \emptyset, i = 1, 2) \asymp \mathbf{P}(A_m).$$

In particular, there exists a constant c_3 such that for all $n < m - 3$,

$$\mathbf{P}(A_n \cap A_m) \leq c_3 \mathbf{P}(A_n) \mathbf{P}(A_m).$$

Since

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty,$$

(2) now follows from the generalized Borel-Cantelli Lemma (see, e.g., [7, P26.3]).

We finish by discussing the situation in three dimensions. In [4] it was shown that if S^1, S^2 are independent simple random walks starting at the origin in $\mathbb{Z}^3$, then

$$\mathbf{P}\{S^1[0, n^2] \cap S^2[1, n^2] = \emptyset\} \asymp n^{-\xi},$$

where $\xi = \xi_3(1,1)$ is the intersection exponent for Brownian motions. It has recently been proved [5], that $\xi > 1/2$ (the bound $\xi \geq 1/2$ was already known, see [2]). Using a proof similar to the upper bound in 2, we see that

$$\mathbf{P}\{x \text{ cut point for } S^i\} \leq c|x|^{-1}|x|^{-\xi},$$

and hence,

$$\mathbf{P}\{x \text{ cut point for } S^1 \text{ and } S^2\} \leq c|x|^{-2-2\xi}.$$

Since $\xi > 1/2$, we can sum over x and see that the expected number of double cut points is finite and hence the probability of an infinite number of double cut points is zero.

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