Elect. Comm. in Probab. **11** (2006), 182–192

ELECTRONIC COMMUNICATIONS in PROBABILITY

GLOBAL GEOMETRY UNDER ISOTROPIC BROWNIAN FLOWS

SREEKAR VADLAMANI[1]
Faculty of Industrial Engineering and Management, Technion - Israel Institute of Technology, Haifa, Israel
http://tx.technion.ac.il/∼ sreekar/
email: `sreekar@ieadler.technion.ac.il`

ROBERT J. ADLER[2]
Faculty of Industrial Engineering and Management, Technion - Israel Institute of Technology, Haifa, Israel
http://ie.technion.ac.il/Adler.phtml
email: `robert@ieadler.technion.ac.il`

Submitted July 9 2006, *accepted in final form* July 31 2006

AMS 2000 Subject classification: Primary 60H10, 60J60; Secondary 52A39, 28A75.
Keywords: Stochastic flows, Brownian flows, manifolds, Lipschitz-Killing curvatures, evolution equations, Lyapunov exponents

Abstract
We consider global geometric properties of a codimension one manifold embedded in Euclidean space, as it evolves under an isotropic and volume preserving Brownian flow of diffeomorphisms. In particular, we obtain expressions describing the expected rate of growth of the Lipschitz-Killing curvatures, or intrinsic volumes, of the manifold under the flow.
These results shed new light on some of the intriguing growth properties of flows from a global perspective, rather than the local perspective, on which there is a much larger literature.

1 Introduction

We are interested in Brownian flows Φ_{st}, $0 \leq s \leq t < \infty$ from $\mathbb{R}^n \to \mathbb{R}^n$, obtained by solving the collection of stochastic differential equations

$$x_t \;=\; \Phi_t(x) \;=\; x + \int_0^t \partial U_s(\Phi_s(x)), \tag{1}$$

where we write Φ_t for Φ_{0t} when there is no danger of confusion. Here, ∂ denotes the Stratonovich stochastic differential and $U_t(x)$ is a vector field valued Brownian motion with

[1]RESEARCH SUPPORTED IN PART BY THE LOUIS AND SAMUEL SEIDEN TECHNION ACADEMIC CHAIR
[2]RESEARCH SUPPORTED IN PART BY US-ISRAEL BINATIONAL SCIENCE FOUNDATION, GRANT 2004064, AND BY TECHNION VPR FUNDS.

smooth spatial covariance structure, on which we shall have more to say in the subsequent section. However, we note already that we shall assume U is such that, with probability one, for each $s \leq t$,

(i) Φ_{st} is a C^2 diffeomorphism.

(ii) Φ_{st} is volume preserving; i.e. for any compact $D \subset \mathbb{R}^n$, $\lambda_n(\Phi_t(D)) = \lambda_n(D)$, where λ_n is Lebesgue measure in $\mathbb{R}^n$.

(iii) Φ_{st} is isotropic in the sense of (8) below.

It is standard fare, following from (1) and our three assumptions, that $\Phi_{ut} \circ \Phi_{su} = \Phi_{st}$, that Φ_{tt} is the identity map on $\mathbb{R}^n$, that $\Phi_{st}(x)$ and $\Phi_{st}^{-1}(x)$ are jointly continuous in x, s, t as are the spatial derivatives

$$D\Phi_{st}(x) \;\triangleq\; \left(\frac{\partial \Phi^i_{st}(x)}{\partial x^j}\right)_{i.j=1}^n \tag{2}$$

and $D\Phi_{st}^{-1}(x)$, and that the 'increments' $\Phi_{s_1t_1}, \Phi_{s_2t_2}, \ldots, \Phi_{s_nt_n}$ are independent for all $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$. For a full study of isotropic Brownian flows, with history and references, we refer the reader to Kunita's monograph, [6].

The study of the evolution of curvature under such flows was pioneered by LeJan in [8], where he established the positive recurrence of the curvature of a curve moving under an isotropic Brownian flow. Quite recently, Cranston and LeJan [2] followed this with a striking analysis of the growth of local curvature. Working with isotropic and volume preserving flows in $\mathbb{R}^n$, they took a codimension one manifold M embedded in $\mathbb{R}^n$ and considered its image under the flow, which we denote by $M_t \;\triangleq\; \Phi_t(M)$.

Taking a point $x \in M$, they developed an Itô formula for the symmetric polynomials of the principal curvatures of $\Phi_t(M)$ at the points $\Phi_t(x)$, including the mean and Gaussian curvatures. In addition, and this will be more important for us, they showed that these polynomials grow exponentially in time, with mean rates that are related to the Lyapunov exponents of the flow. In simple terms, this means that the manifold M_t, while it may begin at time zero as something as simple as the unit sphere S^{n-1} in $\mathbb{R}^n$ (which has unit Gaussian curvature everywhere) it tends to develop sharply rounded 'corners' as time progresses.

A somewhat different set of results can be found in a series of papers [3, 4] authored by Cranston, Scheutzow and Steinsaltz. In particular, the combined results of [3, 4, 11] show that, for an isotropic Brownian flow with $n \geq 2$, there are positive constants c and C such that for each compact and connected set $D \subset \mathbb{R}^n$ with at least two points,

$$c \;\leq\; \inf_{u \in S^{n-1}} \sup_{x \in D} \liminf_{t \to \infty} \frac{1}{t} \langle \Phi_t(x), u \rangle \;\leq\; \limsup_{t \to \infty} \frac{1}{t} \sup_{x \in D} \|\Phi_t(x)\| \leq C, \tag{3}$$

almost surely if the top Lyapunov exponent is strictly positive, and with strictly positive probability, otherwise.

One implication of this result is that while $\Phi_t(M)$ is homotopically equivalent to M, for large t it will 'look' quite different. One way to measure this difference, at a global level, is via their Lipschitz-Killing curvatures. Since M has dimension $(n-1)$ there are n such curvatures, $\mathcal{L}_0(M_t), \mathcal{L}_1(M_t), \ldots, \mathcal{L}_{n-1}(M_t)$. The first of these, $\mathcal{L}_0(M_t)$, is the Euler-Poincaré characteristic of M_t, which, because M and M_t are homotopically equivalent, is the same as that of M and so independent of t. The last of these, $\mathcal{L}_{n-1}(M_t)$, gives the $(n-1)$ dimensional surface measure

of M_t and most definitely does change with time, as do all the remaining $\mathcal{L}_j(M_t)$. Further information on the geometric rôles of the Lipschitz-Killing curvatures is given in the following section.
In view of the results of Cranston, Scheutzow and Steinsaltz described above, one would expect that the $\mathcal{L}_j(M_t)$, $1 \le j \le n-1$, would grow rapidly in time, as parts of the set M_t begin to stretch in various directions at rate t. That this is indeed the case is a consequence of the following theorem, one of the two main results of this paper.

Theorem 1.1 *Let M be a smooth codimension one manifold embedded in $\mathbb{R}^n$ and M_t its image under Φ_t, where Φ_{st} is an isotropic and volume preserving Brownian flow of C^2 diffeomorphsims of $\mathbb{R}^n$. Then, for $0 \le k \le n-1$, the expected rate of growth of the Lipschitz-Killing curvatures is given by*

$$\mathbb{E}\left\{\mathcal{L}_{n-k-1}(M_t)\right\} \;=\; \mathcal{L}_{n-k-1}(M)\exp\left(\frac{(n-k-1)(n+1)(k+1)\mu_2 t}{2n(n+2)}\right), \tag{4}$$

where μ_2 is the second moment of the spectral measure F of (5).

The proof of Theorem 1.1 relies on the fact that, loosely speaking, for $1 \le k \le (n-1)$, the $(n-k-1)$-th Lipschitz-Killing curvature of a manifold can be obtained as an average, over the manifold, of the k-th order symmetric polynomial of the principal curvatures. However, as we have already noted above, these have been studied in detail by Cranston and LeJan [2]. Consequently, our proof relies very heavily on their paper, to the extent that one could consider this paper as an addendum to theirs. Nevertheless, we believe that the results are of independent interest, in that they lift the local approach of [2] to a global scenario. The case $k = 0$, which corresponds to the $(n-1)$ dimensional surface measure of the manifold or equivalently the $(n-1)$-th Lipschitz-Killing curvature, is somewhat easier, and is a simple consequence of Lemma 4.1.
The remainder of this paper is organized as follows. In Section 3 we provide the precise definition of Lipschitz-Killing curvatures and the required geometric background, followed by the proofs of the main results of the paper in Section 4.

2 Brownian flows

This section is not so much about Brownian flows *per se*, for which we refer you back to the references of the Introduction, but rather about setting up notation. Since we plan to use the main result of [2] to prove Theorem 1.1, we shall adopt the notation of that paper without much explanation. You can find missing explanations in [2].
The first step is to define the vector field valued Brownian motion U driving the flow in (1). We take this to be a zero mean Gaussian process from $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}^n$ with covariance structure given by

$$\mathbb{E}\left\{U_t^k(x)U_s^l(y)\right\} \;=\; (t\wedge s)C^{kl}(x-y), \quad 1 \le k,l \le n,$$

where each C^{kl} can be written in the form

$$C^{kl}(z) \;=\; \int_0^\infty \int_{S^{n-1}} e^{i\rho\langle z,t\rangle}(\delta_l^k - t^k t^l)\,\sigma_{n-1}(dt)F(d\rho), \tag{5}$$

for a normalized Lebesgue measure σ_{n-1} on S^{n-1} and a non-negative measure F on $\mathbb{R}^+$, with m-th moment, μ_m, given by $\mu_m = \int_0^\infty \rho^m\, F(d\rho)$.

The various spatial derivatives of U, implicitly assumed to exist, are denoted by

$$W^i_j \;=\; \frac{\partial U^i}{\partial x^j}, \quad B^i_{jk} \;=\; \frac{\partial^2 U^i}{\partial x^j \partial x^k}.$$

Writing $\langle \cdot, \cdot \rangle$ for quadratic covariation, it is not hard to check that

$$\langle dW^i_j(t,y), dW^k_l(t,y) \rangle \;=\; \frac{\mu_2}{n(n+2)} \left[(n+1)\delta^i_k \delta^j_l - \delta^i_j \delta^k_l - \delta^i_l \delta^k_j \right] dt, \tag{6}$$

$$\langle dB^i_{jk}(t,y), dW^p_q(t,y) \rangle \;=\; 0, \tag{7}$$

for any $1 \le i, j, k, l, p, q \le n$. Furthermore,

$$\langle \langle dB(u,u), v \rangle, \langle dB(u,u), v \rangle \rangle = \frac{3\mu_4}{n(n+2)(n+4)} \left[(n+3)\|u\|^4 \|v\|^2 - 4\langle u, v\rangle^2 \|u\|^2 \right] dt,$$

for all vectors $u, v \in \mathbb{R}^n$.

A simple calculation, together with (6), yields $E(\sum W^i_i)^2 = E(\text{div } U)^2 = 0$, and subsequently makes Φ, defined in (1), a volume preserving flow. Moreover, this particular choice of the covariance function makes the flow isotropic in the sense that the spatial covariance matrices $C(x) = (C^{kl}(x))^n_{l,k=1}$ satisfy

$$C(x) \;=\; G^* C(Gx) G \tag{8}$$

for any real orthonormal matrix G, as well as making the flow volume preserving.

With the flow defined, we now turn to setting up the notation required for studying its (differential) geometry. Our basic references for this are Lee [9, 10] and Part II of [1].

We start with a codimension one Riemannian manifold M embedded in $\mathbb{R}^n$ and, for $x \in M$, let $u = \{u_i\}^{(n-1)}_{i=1}$ be an orthonormal basis of T_xM, the tangent space at $x \in M$. (Note that u actually depends on x, but we shall not write this explicitly.) Then by a simple *push-forward* argument, $D\Phi_t(x)u_i = u_i(t) \in T_{x_t}M_t$, where $x_t = \Phi_t(x)$ (cf. (1)) and $D\Phi_t(x)$ is as defined in (2). Furthermore,

$$du_i(t) \;= \partial W u_i(t) = dW u_i(t).$$

Writing $\Pi_t : T_{x_t}\mathbb{R}^n \to T_{x_t}M_t$ as the orthogonal projection onto $T_{x_t}M_t$ and $\widetilde{\nabla}$ for the canonical connection on the ambient Euclidean space $\mathbb{R}^n$, the second fundamental form at $x_t \in M_t$ is given by

$$S_t(u(t), v(t)) \;=\; (I - \Pi_t)\, \widetilde{\nabla}_{u(t)} v(t),$$

for any $u(t), v(t) \in T_{x_t}M_t$. Subsequently, *scalar* second fundamental form is defined as

$$S_{\nu_t}(u(t), v(t)) \;=\; \langle S_t(u(t), v(t)), \nu_t \rangle,$$

where ν_t denotes the unit normal vector field and $\langle \cdot, \cdot \rangle$ now denotes the usual Euclidean inner product rather than quadratic covariation.

It is standard fare that the scalar second fundamental form can be used to induce a linear operator on the exterior algebra $\Lambda^k(T_{x_t}M_t)$ of alternating covariant tensors on $T_{x_t}M_t$, built over the usual wedge product, for each $1 \le k \le (n-1)$. First we define $S^{(k)}_{\nu_t}$ as

$$S^{(k)}_{\nu_t}(u_{l_1}(t) \wedge \ldots \wedge u_{l_k}(t),\; u_{m_1}(t) \wedge \ldots \wedge u_{m_k}(t)) \;=\; \sum_{\sigma \in S_k} (-1)^{\eta_\sigma} \prod_{j=1}^{k} S_{\nu_t}(u_{l_{\sigma(j)}}(t), u_{m_j}(t)),$$

where $\{u_{l_p}(t)\}, \{u_{m_q}(t)\} \subset \{u_i(t)\}_{i=1}^{(n-1)}$, S_k is the collection of all permutations of $\{1, \ldots, k\}$ and η_σ denotes the sign of the permutation σ. This gives rise to the linear operator

$$u_{l_1}(t) \wedge \ldots \wedge u_{l_k}(t) \;\mapsto\; S_{\nu_t}^{(k)}(u_{l_1}(t) \wedge \ldots \wedge u_{l_k}(t), \cdot),$$

where $\{u_i(t)\}_{i=1}^{(n-1)} \subset T_{x_t}M_t$ is a basis of $T_{x_t}M_t$.

The last and the most important remaining definition is that of the trace of $S_{\nu_t}^{(k)}$. For this, however, we need some more notation. For $1 \le k \le (n-1)$ define the index set I_k by

$$I_k \;=\; \left\{\vec{m} \in \{1, \ldots, n-1\}^k : \; m_1 < m_2 < \cdots < m_k\right\}.$$

Then, for $\vec{l} \in I_k$, define

$$\begin{aligned}
|\vec{l}| &= l_1 + \cdots + l_k, \\
\alpha_{\vec{l}}(t) &= u_{l_1}(t) \wedge \cdots \wedge u_{l_k}(t), \\
\alpha^{\vec{l}}(t) &= (-1)^{|\vec{l}|+k} u_1(t) \wedge \cdots \wedge \widehat{u}_{l_1}(t) \wedge \cdots \wedge \widehat{u}_{l_k}(t) \wedge \cdots \wedge u_{n-1}(t), \\
\alpha(t) &= u_1(t) \wedge \cdots \wedge u_{n-1}(t),
\end{aligned}$$

where the hatted vectors are understood to be omitted from the wedge product. Now, for $\vec{l}, \vec{m} \in I_k$, define

$$\langle \alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t) \rangle \;=\; \det\left(\langle u_{l_i}(t), u_{m_j}(t) \rangle\right), \tag{9}$$

and naturally $\|\alpha(t)\|^2 \;=\; \det\left(\langle u_i(t), u_j(t) \rangle\right)$.

We now have all that we need to define the all important trace, $\mathrm{Tr} S_{\nu_t}^{(k)}$, as

$$\mathrm{Tr} S_{\nu_t}^{(k)} \;=\; \frac{\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t) \rangle}{\|\alpha(t)\|^2} \, S_{\nu_t}^{(k)}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t)), \tag{10}$$

where the Einstein summation convention is carried over the indices $\vec{l}, \vec{m} \in I_k$. Now for the case $k = 0$, which has thus far remained untouched, we define $TrS_{\nu_t}^{(0)} = 1$.

The temporal development of this trace was studied in detail in [2], where Cranston and LeJan proved that $\{\mathrm{Tr} S_{\nu_t}^{(k)}\}_{k=1}^{(n-1)}$ is a $(n-1)$-dimensional diffusion. This is the precise formulation of the result we were referring to in the Introduction when we spoke of the Itô formula of symmetric polynomials of principal curvatures, which are essentially equivalent to the traces $\{\mathrm{Tr} S_{\nu_t}^{(k)}\}_{k=1}^{(n-1)}$.

3 Lipschitz-Killing curvatures

There are a number of ways to define Lipschitz-Killing curvatures, but perhaps the easiest is via their appearance in the so-called tube formulae, which, in their original form, are due to Weyl [12]. (For more details and applications see either the monograph of Gray [5] or Chapter 10 of [1].)

To state the tube formula, let M be a C^2, $(n-1)$-dimensional manifold embedded in $\mathbb{R}^n$ and endowed with the canonical Riemannian structure on $\mathbb{R}^n$. The tube of radius ρ around M is defined as

$$\mathrm{Tube}(M, \rho) \;=\; \{x \in \mathbb{R}^n : \; d(x, M) \le \rho\},$$

where

$$d(x,M) \;=\; \inf_{y\in M} \|x-y\|.$$

Weyl's tube formula states that there exists a $\rho_c > 0$, known as the critical radius of M, such that, for $\rho \le \rho_c$, the volume of the tube is given by

$$\lambda_n(\mathrm{Tube}(M,\rho)) \;=\; \sum_{j=0}^{n-1} \rho^{n-j}\omega_{n-j}\mathcal{L}_j(M), \tag{11}$$

where ω_j is the volume of the j-dimensional unit ball and $\mathcal{L}_j(M)$ is the j^{th}-Lipschitz-Killing curvature of M.
Writing $\mathcal{H}_j$ for j-dimensional Hausdorff measure, it is easy to check from (11) that $\mathcal{L}_{n-1}(M) = \mathcal{H}_{n-1}(M)$. That is, it is the surface 'area' of M. $\mathcal{L}_0(M)$ is the Euler-Poincaré characteristic of M and while the remaining Lipschitz-Killing curvatures have less transparent interpretations, it is easy to see that they satisfy simple scaling relationships, in that $\mathcal{L}_j(\alpha M) = \alpha^j \mathcal{L}_j(M)$ for all $1 \le j \le n-1$, where $\alpha M = \{x \in \mathbb{R}^n \; x = \alpha y \text{ for some } y \in M\}$. Furthermore, despite the fact that defining the $\mathcal{L}_j$ via (11) involves the embedding of M in $\mathbb{R}^n$, the $\mathcal{L}_j(M)$ are actually intrinsic and so independent of the embedding space.
While (11) characterises the $\mathcal{L}_j(M)$ it does not generally help one compute them. There are a number of ways in which to do this, but we choose the following, which is most appropriate for our purposes. (cf. [1, 5] for further details and examples)

$$\mathcal{L}_{n-k-1}(M) \;=\; K_{n,k}\int_M \int_{S(\mathbb{R})} \mathrm{Tr} S_\nu^{(k)} 1_{N_xM}(-\nu)\,\mathcal{H}_0(d\nu)\mathcal{H}_{n-1}(dx), \tag{12}$$

where $K_{n,k} \;=\; \frac{1}{2(\pi)^{(k+1)/2}k!}\Gamma\left(\frac{k+1}{2}\right)$ and N_xM is the normal cone to M at the point x. Since M has codimension 1 in $\mathbb{R}^n$, each N_xM contains only the outward normals to T_xM and is of unit dimension.
In general, we shall write $S(\mathbb{R}^n)$ for the unit sphere in $\mathbb{R}^n$. Thus the $S(\mathbb{R})$ appearing in (12) contains only the two vectors $+1$ and -1 in $\mathbb{R}$ and $\mathcal{H}_0$ is counting measure, which makes the integral over $S(\mathbb{R})$ a rather pretentious way of writing things, as, indeed, was the introduction of the normal cone. Nevertheless, both will be of use to us later on, when we discuss possible generalisations of our results.
We, of course, are interested in the temporal evolution of the $\mathcal{L}_j(M_t)$ and it follows directly from (12) that, with the notation of the previous section,

$$\begin{aligned}
\mathcal{L}_{n-k-1}(M_t) &= K_{n,k}\int_{M_t}\int_{S(\mathbb{R})} \mathrm{Tr} S_{\nu_t}^{(k)} 1_{N_{x_t}M_t}(-\nu_t)\,\mathcal{H}_0(d\nu_t)\mathcal{H}_{n-1}(dx_t) \\
&= K_{n,k}\int_{M}\int_{S(\mathbb{R})} \mathrm{Tr} S_{\nu_t}^{(k)} \sqrt{\det(\langle u_i(t), u_j(t)\rangle)}\, 1_{N_{x_t}M_t}(-\nu_t)\,\mathcal{H}_0(d\nu_t)\mathcal{H}_{n-1}(dx) \\
&= K_{n,k}\int_{M}\int_{S(\mathbb{R})} \mathrm{Tr} S_{\nu_t}^{(k)} \|\alpha_t\| 1_{N_{x_t}M_t}(-\nu_t)\,\mathcal{H}_0(d\nu_t)\mathcal{H}_{n-1}(dx).
\end{aligned} \tag{13}$$

4 An Itô formula for $\mathcal{L}_{n-k-1}(M_t)$

Before commencing a serious stochastic analysis of (13) we recall some of the results and further notation from LeJan [7].

Let $\xi(t) = \xi_1(t) \wedge \ldots \wedge \xi_k(t)$ and $\psi(t) = \psi_1(t) \wedge \ldots \wedge \psi_k(t)$, where $\{\xi_i(t)\}, \{\psi_i(t)\} \subset T_{x_t}M_t$. Then, by Lemma 3 of [7],

$$d\langle \xi(t), \psi(t)\rangle = \sum_{l,\, j} (\langle \tau_i^j \xi(t), \psi(t)\rangle + \langle \xi(t), \tau_i^j \psi(t)\rangle)\, dW_j^i(t) \;+\; \frac{k(n-k)\mu_2}{n} \langle \xi(t), \psi(t)\rangle\, dt,$$

where

$$\tau_l^j \xi(t) \;=\; e^j \wedge \left\{ \sum_{i=1}^{k} (-1)^{i+1} \langle \xi_i(t), e^l\rangle \xi_1(t) \wedge \cdots \wedge \hat{\xi}_i(t) \wedge \cdots \wedge \xi_k(t) \right\},$$

with $\{e^k\}_{k=1}^n$ being the standard basis of $\mathbb{R}^n$ and $\tau_l^j \psi(t)$ being defined similarly.
It follows immediately from the above that if $\xi(t) = \psi(t) = \alpha(t) = u_1(t) \wedge \cdots \wedge u_{n-1}(t)$, where $u_i(t) = D\Phi_t(x) u_i$ and $(u_1, \ldots, u_{n-1})$ is an orthonormal basis of $T_x M$, then

$$d\|\alpha(t)\|^2 \;=\; \|\alpha(t)\|^2 \Big(2 \sum_{i=1}^{n-1} dW_i^i(t) \;+\; \frac{(n-1)\mu_2}{n} dt \Big).$$

Now we derive an Itô formula for $\|\alpha(t)\|$ and use it together with Theorem A.2 of [2] to obtain an expression for the Itô derivative of the Lipschitz-Killing curvatures.

Lemma 4.1 *Let M be a smooth $(n-1)$-dimensional manifold embedded in $\mathbb{R}^n$ and M_t its image at time t under the stochastic, isotropic, and volume preserving flow Φ_t described in Section 2. Then, in the notation of Section 2,*

$$d\|\alpha(t)\| \;=\; \|\alpha(t)\| \Big(\sum_{i=1}^{n-1} dW_i^i(t) \;+\; \frac{(n-1)(n+1)\mu_2}{2n(n+2)} dt \Big).$$

Proof: Using the standard Itô formula and (6) we obtain

$$\begin{aligned} d\|\alpha(t)\| &= d(\|\alpha(t)\|^2)^{\frac{1}{2}} \\ &= \frac{1}{2} \|\alpha(t)\| \Big(2 \sum_{i=1}^{n-1} dW_i^i(t) + \frac{(n-1)\mu_2}{n} dt \Big) - \frac{(n-1)\mu_2}{2n(n+2)} \|\alpha(t)\| dt \\ &= \|\alpha(t)\| \Big(\sum_{i=1}^{n-1} dW_i^i(t) + \frac{(n-1)(n+1)\mu_2}{2n(n+2)} dt \Big). \end{aligned}$$

□

We need just a little more preparation before we can turn to our main result.
Let $\alpha_{\vec{l}}(t) = u_{l_1} \wedge \cdots \wedge u_{l_k}(t)$, be a k-form for $\vec{l} \in I_k$, then define

$$\alpha_{\vec{l_p}}(t) = (-1)^{p+1} u_{l_1}(t) \wedge \cdots \wedge \hat{u}_{l_p}(t) \wedge \cdots \wedge u_{l_k}(t), \tag{14}$$

for $1 \le k \le (n-1)$, where $\vec{l} \in I_k$, $\vec{l_p} \in I_{k-1}$ and $1 \le p \le k$.
Rewriting the above expression as

$$\alpha_{\vec{l_p}}(t) \;=\; (-1)^{p+1} u_{l_1}^{(p)}(t) \wedge \cdots \wedge u_{l_{k-1}}^{(p)}(t), \tag{15}$$

defines $u_l^{(p)}$.
Then according to Theorem A.2 of [2], for $1 \leq k \leq n-1$

$$
\begin{aligned}
d\mathrm{Tr}S_{\nu_t}^{(k)} &= \sum_{i,p} \left[S_{\nu_t}^{(k-1)}(\alpha_{\vec{l_p}}(t), \alpha_{\vec{m}_i}(t)) \langle dB(u_{l_p}(t), u_{m_i}(t)), \nu_t \rangle \right] \frac{\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t) \rangle}{\|\alpha(t)\|^2} \\
&+ \mathrm{Tr}S_{\nu_t}^{(k)} \Big(k dW_n^n(t) - 2 \sum_{i=1}^{n-1} dW_i^i(t) \Big) \\
&+ \sum_{i,j} S_{\nu_t}^{(k)}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t)) \frac{\langle \tau_i^j \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t) \rangle + \langle \alpha^{\vec{l}}(t), \tau_i^j \alpha^{\vec{m}}(t) \rangle}{\|\alpha(t)\|^2} dW_j^i(t) \\
&+ \frac{(n+1)k(n-k)\mu_2}{2n(n+2)} \mathrm{Tr}S_{\nu_t}^{(k)} dt, \qquad (16)
\end{aligned}
$$

where the Einstein summation convention is carried over the indices $\vec{l}$ and $\vec{m}$.
We now have everything we need to present the main result of this paper.

Theorem 4.1 *Retain the assumptions and notation of Lemma 4.1. Let $\mathcal{L}_k$ be the Lipschitz-Killing curvatures defined by* (13)*. Then the Itô derivatives of the Lipschitz-Killing curvatures for $1 \leq k \leq n-1$ are given by*

$$
\begin{aligned}
d\mathcal{L}_{n-k-1}(M_t) &= \Big[K_{n,k} \int_M \int_{S(\mathbb{R})} \Big(\sum_{i,\, p=1}^{k} S_{\nu_t}^{(k-1)}(\alpha_{\vec{l_p}}(t), \alpha_{\vec{m}_i}(t)) \langle dB(u_{l_p}(t), u_{m_i}(t)), \nu_t \rangle \\
&\times \langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t) \rangle \|\alpha(t)\|^{-1} \\
&+ \mathrm{Tr}S_{\nu_t}^{(k)} \|\alpha(t)\| \big(k dW_n^n - \sum_{i=1}^{n-1} dW_i^i \big) \\
&+ \sum_{i,\, j} S^{(k)}(\alpha_{\vec{l}}, \alpha_{\vec{m}}) (\langle \tau_i^j \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t) \rangle + \langle \alpha^{\vec{l}}(t), \tau_i^j \alpha^{\vec{m}}(t) \rangle) dW_j^i \|\alpha(t)\|^{-1} \Big) \\
&\times 1_{N_{x_t}M_t}(-\nu_t) \mathcal{H}_0(d\nu_t) \mathcal{H}_{n-1}(dx) \Big] \\
&+ \frac{(n-k-1)(n+1)(k+1)\mu_2}{2n(n+2)} \mathcal{L}_{n-k-1}(M_t) dt, \qquad (17)
\end{aligned}
$$

where $\mathcal{H}_k$ is k-dimensional Hausdorff measure and $N_{x_t}M_t$ is the normal cone to M_t at $x_t \in M_t$.

Recall that many of the terms in (17) depend on space parameter $x \in M$ through the vector field U, its various spatial derivatives W and B, as well as the various tangent vectors at x. Before giving the proof of Theorem 4.1 we note that (17) simplifies considerably when $k = 0$, in which case, as we have already noted, $\mathcal{L}_{n-1}(M_t) \equiv \mathcal{H}_{n-1}(M_t)$. Then

$$
\begin{aligned}
\mathcal{L}_{n-1}(M_t) &= K(n,0) \int_M \int_{S(\mathbb{R})} \|\alpha(t)\| 1_{N_{x_t}M_t}(-\nu_t) \mathcal{H}_0(d\nu_t) \mathcal{H}_{n-1}(dx) \\
&= \int_M \|\alpha(t)\| \mathcal{H}_{n-1}(dx).
\end{aligned}
$$

Hence, by Theorem 4.1,

$$d\mathcal{L}_{n-1}(M_t) \quad = \quad \int_M \|\alpha(t)\|\Big(\sum_{i=1}^{n-1} dW_i^i(t)\Big)\mathcal{H}_{n-1}(dx) + \frac{(n-1)(n+1)\mu_2}{2n(n+2)}\mathcal{L}_{n-1}(M_t)dt.$$

Proof of Theorem 4.1: We start with

$$\begin{aligned} d(\mathrm{Tr}S_{\nu_t}^{(k)}\|\alpha(t)\|) &= (d(\mathrm{Tr}S_{\nu_t}^{(k)})\|\alpha(t)\| + \mathrm{Tr}S_{\nu_t}^{(k)}d(\|\alpha(t)\|) + \langle d\mathrm{Tr}S_{\nu_t}^{(k)}, d\|\alpha(t)\|\rangle \\ &\triangleq I + II + III. \end{aligned}$$

We shall obtain a closed form expression for each of these terms. By (16) we have

$$\begin{aligned} I &= (d(\mathrm{Tr}S_{\nu_t}^{(k)})\|\alpha(t)\| \\ &= \Big[\sum_{i,p=1}^{k} S_{\nu_t}^{(k-1)}(\alpha_{\vec{l_p}}(t), \alpha_{\vec{m_i}}(t))\langle dB(u_{l_p}(t), u_{m_i}(t)), \nu_t\rangle\Big]\langle\alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle\|\alpha(t)\|^{-1} \\ &\quad + \mathrm{Tr}S_{\nu_t}^{(k)}\Big[kdW_n^n(t) - 2\sum_{i=1}^{n-1} dW_i^i(t)\Big]\|\alpha(t)\| \\ &\quad + \sum_{i,j} S_{\nu_t}^{(k)}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t))(\langle\tau_i^j\alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle + \langle\alpha^{\vec{l}}(t), \tau_i^j\alpha^{\vec{m}}(t)\rangle)dW_j^i(t)\|\alpha(t)\|^{-1} \\ &\quad + \frac{(n+1)k(n-k)\mu_2}{2n(n+2)}\mathrm{Tr}S_{\nu_t}^{(k)}\|\alpha(t)\|dt. \end{aligned} \tag{18}$$

Using Theorem 4.1 we find

$$\begin{aligned} II &= \mathrm{Tr}S_{\nu_t}^{(k)}d(\|\alpha(t)\|) \\ &= \mathrm{Tr}S_{\nu_t}^{(k)}\|\alpha(t)\|\Big(\sum_{i=1}^{n-1} dW_i^i(t) + \frac{(n-1)(n+1)\mu_2}{2n(n+2)}dt\Big). \end{aligned} \tag{19}$$

Finally using (6), (16) and Theorem 4.1 we have

$$III \tag{20}$$

$$\begin{aligned} &= \Big\langle d\mathrm{Tr}S_{\nu_t}^{(k)}, d\|\alpha(t)\|\Big\rangle \\ &= \mathrm{Tr}S_{\nu_t}^{(k)}\|\alpha(t)\|\Big\langle\Big(kdW_n^n - 2\sum_{i-1}^{n-1} dW_i^i\Big), \sum_{i=1}^{n-1} dW_i^i\Big\rangle_t \\ &+ \sum_{i,j} S_{\nu_t}^{(k)}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t))\big(\langle\tau_i^j\alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle + \langle\alpha^{\vec{l}}(t), \tau_i^j\alpha^{\vec{m}}(t)\rangle\big)\Big\langle dW_j^i, \sum_{i=1}^{n-1} dW_i^i\Big\rangle\|\alpha(t)\|^{-1} \\ &= -\frac{(n-1)(k+2)\mu_2}{n(n+2)}\mathrm{Tr}S_{\nu_t}^{(k)}\|\alpha(t)\|dt + \frac{2(n-k-1)\mu_2}{n(n+2)}\mathrm{Tr}S_{\nu_t}^{(k)}\|\alpha(t)\|dt. \end{aligned} \tag{21}$$

Summing (18), (19) and (20) we have

$$
\begin{aligned}
& d(\mathrm{Tr} S^{(k)}_{\nu_t}\|\alpha(t)\|) \\
&= \Big[\sum_{i,p} S^{(k-1)}_{\nu_t}(\alpha_{\vec{l_p}}(t),\alpha_{\vec{m_i}}(t))\langle dB(u_{l_p}(t),u_{m_i}(t)),\nu_t\rangle\Big]\langle\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle\|\alpha(t)\|^{-1} \\
&+ \mathrm{Tr} S^{(k)}_{\nu_t}\|\alpha(t)\|\Big[k dW^n_n(t) - \sum_{i=1}^{n-1} dW^i_i(t)\Big] \\
&+ \sum_{i,j} S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\big(\langle\tau^j_i\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle + \langle\alpha^{\vec{l}}(t),\tau^j_i\alpha^{\vec{m}}(t)\rangle\big)dW^i_j(t)\|\alpha(t)\|^{-1} \\
&+ \frac{(n-k-1)(n+1)(k+1)\mu_2}{2n(n+2)}\mathrm{Tr} S^{(k)}_{\nu_t}\|\alpha(t)\|dt.
\end{aligned}
$$

Substituting the above in (13) gives (17) and so the theorem. □

We can now easily deduce Theorem 1.1 of the Introduction.

Proof of Theorem 1.1: In (17) we note that, with the single exception of the last term, all terms are zero mean martingales due to the presence of the martingale integrators $dW(t)$ or $dB(t)$. Therefore, taking expectations in (17), after taking the integral over time t, immediately yields

$$
\mathbb{E}\left\{\mathcal{L}_{n-k-1}(M_t)\right\} = \frac{(n-k-1)(n+1)(k+1)\mu_2}{2n(n+2)}\int_0^t \mathbb{E}\left\{\mathcal{L}_{n-k-1}(M_s)\right\}ds.
$$

Solving this linear differential equation gives (4) and we are done. □

We close with one further result and two open problems.

As for the first open problem, we believe that, in the setting of Theorem 1.1,

$$
\lim_{t\to\infty}\frac{1}{t}\log\left(\frac{\mathcal{L}_{n-k-1}(M_t)}{\mathcal{L}_{n-k-1}(M)}\right) = \frac{(n-k-1)(n+1)(k+1)\mu_2}{2n(n+2)},
$$

where the limit here is in L_1. At this point we do not have an air tight proof of this. We would also like to add that the almost sure growth rates may well be different from the ones conjectured above.

As for the additional result, recall that throughout the paper, we have assumed that M was a codimension one manifold in $\mathbb{R}^n$. From a technical point of view, this has a substantial simplifying effect on the definition (12) of Lipschitz-Killing curvatures. If $\dim(M) = m < (n-1)$, then (12) changes in that the normal cones are now of dimension $(n-m)$, $S(\mathbb{R})$ is replaced by $S(\mathbb{R}^{n-m})$ and so is of dimension $(n-m-1)$, and $\mathcal{H}_0$ and $\mathcal{H}_{n-1}$ are replaced by $\mathcal{H}_{n-m-1}$ and $\mathcal{H}_m$, respectively. (The constant also changes, but this is less important. See [1] for details.) All told, we have

$$
\mathcal{L}_{m-k}(M) = K'_{m,k}\int_M\int_{S(\mathbb{R}^{m-m})}\mathrm{Tr} S^{(k)}_\nu 1_{N_xM}(-\nu)\,\mathcal{H}_{n-m-1}(d\nu)\mathcal{H}_m(dx), \tag{22}
$$

for some constants $K'_{m,k}$. For all $k \neq 0$ here, we have found that the complications introduced by increasing the dimension of the normal space are such that computations analogous to those we have carried out are forbiddingly complex.

For $k = 0$ the trace term in (22) disappears and so it is not hard to show that

$$d\mathcal{L}_m(M_t) \;=\; \int_M \|\alpha(t)\| \Big(\sum_{i=1}^{m} dW_i^i(t) \Big) \mathcal{H}_m(dx) + \frac{m(n-m)(n+1)\mu_2}{2n(n+2)} \mathcal{L}_m(M_t)dt. \quad (23)$$

Since $\mathcal{L}_m(M_t)$ is the m-dimensional content of M_t, this is a far from uninteresting result. Of course, as before, this implies that

$$\mathbb{E}\left\{\mathcal{L}_m(M_t)\right\} \;=\; \mathcal{L}_m(M) \exp\left(\frac{m(n-m)(n+1)\mu_2 t}{2n(n+2)} \right). \quad (24)$$

We have not, however, been able to find corresponding results for the more general case.

References

[1] R.J. Adler and J.E. Taylor. *Random Fields and Geometry.* Springer, 2006. In press.

[2] M. Cranston and Y. LeJan. Geometric evolution under isotropic stochastic flow. *Electron. J. Probab.,* 3, paper 4, 36 pp. 1998.

[3] M. Cranston, M. Scheutzow, and D. Steinsaltz. Linear expansion of isotropic Brownian flows. *Electron. Comm. Probab.,* 4, 91–101 (electronic), 1999.

[4] M. Cranston, M. Scheutzow, and D. Steinsaltz. Linear bounds for stochastic dispersion. *Ann. Probab.,* 28, 1852–1869, 2000.

[5] A. Gray. *Tubes.* Addison-Wesley, Redwood City, 1990.

[6] H. Kunita. *Stochastic Flows and Stochastic Differential Equations,* volume 24 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997.

[7] Y. Le Jan. On isotropic Brownian motions. *Z. Wahrsch. Verw. Gebiete,* 70, 609–620, 1985.

[8] Y. Le Jan. Asymptotic properties of isotropic Brownian flows. In *Spatial Stochastic Processes,* volume 19 of Progr. Probab., 219–232. Birkhäuser Boston, Boston, MA, 1991.

[9] J.M. Lee. *Riemannian Manifolds: An Introduction to Curvature,* Springer-Verlag, New York, 1997.

[10] J.M. Lee. *Introduction to Smooth Manifolds*, Springer-Verlag, New York, 2003.

[11] M. Scheutzow and D. Steinsaltz. Chasing balls through martingale fields. *Ann. Probab.,* 30, 2046–2080, 2002.

[12] H. Weyl. On the volume of tubes. *Amer. J. Math.,* 61(2):461–472, 1939.