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IDENTIFIABILITY OF EXCHANGEABLE SEQUENCES WITH IDENTICALLY DISTRIBUTED PARTIAL SUMS

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Abstract

Consider two exchangeable sequences $(X_k)_{k\in\mathbb{N}}$ and $(\hat{X}_k)_{k\in\mathbb{N}}$ with the property that $S_n \equiv \sum_{k=1}^n X_k$ and $\hat{S}_n \equiv \sum_{k=1}^n \hat{X}_k$ have the same distribution for all $n \in \mathbb{N}$. David Aldous posed the following question. Does this imply that the two exchangeable sequences have the same joint distributions? We give an example that shows the answer to Aldous' question is, in general, in the negative. On the other hand, we show that the joint distributions of an exchangeable sequence can be recovered from the distributions of its partial sums if the sequence is a countable mixture of i.i.d. sequences that are either nonnegative or have finite moment generating functions in some common neighbourhood of zero.

1 Introduction

In his survey [1] of exchangeability, David Aldous posed the following question. Consider $S_n \equiv \sum_{k=1}^n X_k$ and $\hat{S}_n \equiv \sum_{k=1}^n \hat{X}_k$, where $(X_k)_{k\in\mathbb{N}}$ and $(\hat{X}_k)_{k\in\mathbb{N}}$ are exchangeable sequences of real–valued random variables. Suppose that S_n and $\hat{S_n}$ have the same distribution for all $n \in \mathbb{N}$. Does this imply that the sequences $(X_k)_{k\in\mathbb{N}}$ and $(\hat{X}_k)_{k\in\mathbb{N}}$ have the same joint distributions?

By de Finetti's theorem, exchangeable sequences are just mixtures of i.i.d. sequences, and so Aldous' question becomes one of whether or not we can identify the mixing probability measure associated with an exchangeable sequence given the distribution of each of its partial sums. In this paper, we first construct an example showing such an identification is not possible in

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general. We then prove that it is possible to identify the mixing measure if we know that it is purely atomic and concentrated either on the set of nonnegative i.i.d.sequences or the set of i.i.d. sequences with finite moment generating functions in some common neighbourhood of the origin.

2 A counterexample

For $i = 1, 2$, consider the even functions ϕ_i and $\hat{\phi}_i$ defined by

$$\phi_1(t) := \begin{cases} -\frac{3}{2}t + 1 & \text{for } 0 \le t < \frac{1}{2}, \\ -\frac{1}{4}t + \frac{3}{8} & \text{for } \frac{1}{2} \le t < \frac{3}{2}, \\ 0 & \text{for } \frac{3}{2} \le t, \end{cases} \tag{1}$$

$$\phi_2(t) := \begin{cases} -2t + 1 & \text{for } 0 \le t < \frac{1}{4}, \\ -t + \frac{3}{4} & \text{for } \frac{1}{4} \le t < \frac{3}{4}, \\ 0 & \text{for } \frac{3}{4} \le t, \end{cases} \tag{2}$$

$$\hat{\phi}_1(t) := \begin{cases} -\frac{3}{2}t + 1 & \text{for } 0 \le t < \frac{1}{2}, \\ -t + \frac{3}{4} & \text{for } \frac{1}{2} \le t < \frac{3}{4}, \\ 0 & \text{for } \frac{3}{4} \le t, \end{cases} \tag{3}$$

and

$$\hat{\phi}_2(t) := \begin{cases} -2t + 1 & \text{for } 0 \le t < \frac{1}{4}, \\ -t + \frac{3}{4} & \text{for } \frac{1}{4} \le t < \frac{1}{2}, \\ -\frac{1}{4}t + \frac{3}{8} & \text{for } \frac{1}{2} \le t < \frac{3}{2}, \\ 0 & \text{for } \frac{3}{2} \le t. \end{cases} \tag{4}$$

The values of $\phi_i(t)$ and $\hat{\phi}_i(t)$, $i = 1, 2$, $t \ge 0$ are plotted in Figure 1. By Pólya's criterion, these four functions are characteristic functions of symmetric probability distributions. For $i = 1, 2$, let $(X_{i,k})_{k \in \mathbb{N}}$ (resp. $(\hat{X}_{i,k})_{k \in \mathbb{N}}$) be an i.i.d. sequence with $X_{i,k}$ (resp. $\hat{X}_{i,k}$) having characteristic function ϕ_i (resp. $\hat{\phi}_i$).
Now let Θ be a random variable independent of the four sequences $(X_{i,k})_{k \in \mathbb{N}}$ and $(\hat{X}_{i,k})_{k \in \mathbb{N}}$, $i = 1, 2$, such that $\mathbb{P}\{\Theta = 1\} = \mathbb{P}\{\Theta = 2\} = \frac{1}{2}$. Put $X_k = X_{\Theta,k}$ and $\hat{X}_k = \hat{X}_{\Theta,k}$ and, as above, set $S_n = \sum_{k=1}^n X_k$ and $\hat{S}_n = \sum_{k=1}^n \hat{X}_k$.
It is easy to check that

$$\mathbb{E}[\exp(itS_n)] = \frac{1}{2}\phi_1^n(t) + \frac{1}{2}\phi_2^n(t) = \frac{1}{2}\hat{\phi}_1^n(t) + \frac{1}{2}\hat{\phi}_2^n(t) = \mathbb{E}[\exp(it\hat{S}_n)] \tag{5}$$

for all $t \in \mathbb{R}$ and integers $n \in \mathbb{N}$, so that S_n and $\hat{S}_n$ have the same distribution for all n. However, $\mathbb{E}[\exp(it_1X_1 + it_2X_2)] \neq \mathbb{E}[\exp(it_1\hat{X}_1 + it_2\hat{X}_2)]$ for some (t_1, t_2), say $(1, 2)$. Therefore the sequences $(X_k)_{k \in \mathbb{N}}$ and $(\hat{X}_k)_{k \in \mathbb{N}}$ do not have the same joint distributions.

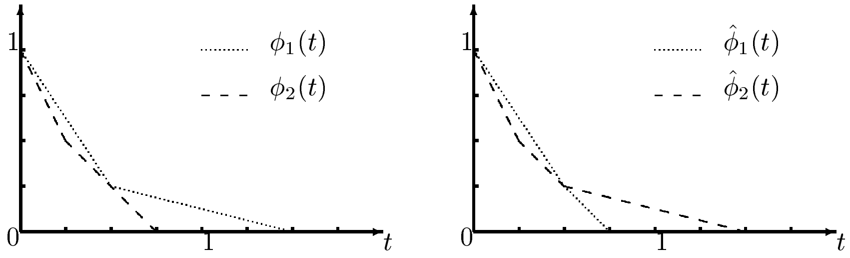

Figure 1

3 Countable mixtures of nonnegative sequences

Throughout this section, let $(X_{i,k})_{k\in\mathbb{N}}$, $i \in \mathbb{N}$, be nonnegative i.i.d sequences with distinct distributions and hence distinct Laplace transforms ϕ_i, $i \in \mathbb{N}$. Let Θ be an $\mathbb{N}$–valued random variable independent of the sequences $(X_{i,k})_{k\in\mathbb{N}}$. Put $X_k = X_{\Theta,k}$ and $S_n = \sum_{k=1}^n X_{\Theta,k}$. Write F_Y for the distribution of a random variable Y.
Note that for each $t \geq 0$, $F_{\phi_\Theta(t)}$ (the push-forward of the distribution of Θ by the map $i \mapsto \phi_i(t)$ from $\mathbb{N}$ into $[0,1]$) is determined by its sequence of moments

$$\int x^n dF_{\phi_\Theta(t)}(x) = \sum_{i\in\mathbb{N}} \phi_i^n(t)\mathbb{P}\{\Theta = i\} = \mathbb{E}[\exp(-tS_n)],\ n \in \mathbb{N}. \tag{6}$$

We therefore have the following result.

Lemma 3.1 *The collection $(F_{\phi_\Theta}(t), t \geq 0)$ of distributions on $[0,1]$ is uniquely determined by the sequence of distributions $(F_{S_n})_{n\in\mathbb{N}}$.*

The next result, which is essentially due to Müntz [2] , plays a central role in the succeeding proofs.

Theorem 3.2 *Let ψ and ρ be Laplace transforms of probability measures μ and ν on $\mathbb{R}_+$. Then $\mu = \nu$ if and only if there exist $0 < t_1 < t_2 < \dots$ such that $t_n \uparrow \infty$, $\sum_n t_n^{-1} = \infty$ and $\psi(t_n) = \rho(t_n)$ for all $n \in \mathbb{N}$.*

Lemma 3.3 and Lemma 3.4 are corollaries of Theorem 3.2.

Lemma 3.3 *Let ψ and ρ be Laplace transforms of distinct probability measures on $\mathbb{R}_+$. Then for T sufficiently large, $\{t \geq T : \psi(t) = \rho(t)\}$ is a discrete set.*

Proof: If the assertion is not true, we could find sequence $0 < a_1 < a_1 + 1 < a_2 < a_2 + 1 < \dots$ such that $\#\{t \in [a_n, a_n + 1] : \psi(t) = \rho(t)\} = \infty$ for all n. Then choose $t_{n,i} \in [a_n, a_n + 1], i = 1, \dots, \lfloor a_n + 1\rfloor + 1$ such that $\psi(t_{n,i}) = \rho(t_{n,i})$. Since $\sum_{n=1}^\infty \sum_{i=1}^{\lfloor a_n+1\rfloor+1} t_{n,i}^{-1} = \infty$, we get a contradiction to Theorem 3.2. ∎

Lemma 3.4 *Suppose that $\psi_1, \dots, \psi_m$ are Laplace transforms of distinct probability measures. Then for any $t_0 \geq 0$ the set of functions $\{\psi_1, \dots, \psi_m\}$ is uniquely determined by the union $\bigcup_{t\geq t_0} \bigcup_{i=1}^m \{(t, \psi_i(t))\}$.*

Proof: Suppose that $\rho_1, \ldots, \rho_n$ are distinct Laplace transforms such that the set $\bigcup_{1 \le i \le m} \{\psi_i(t)\}$ coincides with the set $\bigcup_{1 \le j \le n} \{\rho_j(t)\}$ for all $t \ge t_0$ but $\{\psi_1, \ldots, \psi_m\} \ne \{\rho_1, \ldots, \rho_n\}$.
By interchanging the roles of the two sets of functions and renumbering, we may suppose without loss of generality that $\rho_1 \notin \{\psi_1, \ldots, \psi_m\}$. By assumption, $\rho_1(t) \in \bigcup_{1 \le i \le m} \{\psi_i(t)\}$ for all $t \ge t_0$ so that $\bigcup_{1 \le i \le m} \{t \ge t_0 : \rho_1(t) = \psi_i(t)\} = [t_0, \infty)$, contradicting Lemma 3.3. ∎

The next theorem shows that if $(\hat{X}_i)_{i \in \mathbb{N}}$ is another countable mixture of nonnegative i.i.d. sequences with partial sums $\hat{S}_n$ such that S_n and $\hat{S}_n$ have the same distribution for each $n \in \mathbb{N}$, then $(X_i)_{i \in \mathbb{N}}$ and $(\hat{X}_i)_{i \in \mathbb{N}}$ have identical joint distributions.

Theorem 3.5 *The set of pairs $\{(\phi_i, \mathbb{P}\{\Theta = i\})\}$ is uniquely determined by the sequence of distributions $(F_{S_n})_{n \in \mathbb{N}}$.*

Proof: For $t \ge 0$ put $\Gamma(t) \equiv \bigcup_{i=1}^\infty \{(t, \phi_i(t))\} \subseteq \{t\} \times [0,1]$, so that $\{x : (t,x) \in \Gamma(t)\}$ is the set of atoms of $F_{\phi_\Theta(t)}$. Recall from Lemma 3.1 that $F_{\phi_\Theta(t)}$ is determined by $(F_{S_n})_{n \in \mathbb{N}}$.
Given $a > 0$, define $\Gamma_a(t) \equiv \{(t,x) \in \Gamma(t) : F_{\phi_\Theta(t)}(\{x\}) \ge a\}$. Write $\Gamma_a \equiv \bigcup_{t \ge 0} \Gamma_a(t)$. Construct a set $\Gamma_a^*(t)$ by removing from $\Gamma_a(t)$ all the points (t,x) for which there is no continuous function $f : \mathbb{R}_+ \to [0,1]$ such that $x = f(t)$ and $\{(s, f(s)) : s \ge 0\} \subseteq \Gamma_a$.
Write $p_i \equiv \mathbb{P}\{\Theta = i\}$, $i \in \mathbb{N}$. We can suppose that $p_1 \ge p_2 \ge \ldots$. It is clear that for all $t \ge 0$, $\bigcup_{p_i \ge a} \{(t, \phi_i(t))\} \subset \Gamma_a^*(t)$. We claim that, in fact,

$$\bigcup_{p_i \ge a} \{(t, \phi_i(t))\} = \Gamma_a^*(t) \text{ for all } t \text{ sufficiently large.} \tag{7}$$

Otherwise, there exist two sequences $(t_j)_{j \in \mathbb{N}}$ and $(x_j)_{j \in \mathbb{N}}$ such that $t_1 < t_2 < \ldots$, $t_j \uparrow \infty$, and $(t_j, x_j) \in \Gamma_a^*(t_j) \setminus \bigcup_{p_i \ge a} \{(t_j, \phi_i(t_j))\}$ for all $j \in \mathbb{N}$. It follows from the definition of $\Gamma_a^*(t)$ that there are disjoint intervals (t_j', t_j'') and continuous functions $f_j : \mathbb{R}_+ \to [0,1]$, $j \in \mathbb{N}$, such that for all j we have $t_j \in (t_j', t_j'')$, $(t, f_j(t)) \in \Gamma_a(t)$ for all $t \ge 0$, and $\{(t, f_j(t)) : t \in (t_j', t_j'')\} \cap \bigcup_{t \ge 0} \bigcup_{p_i \ge a} \{(t, \phi_i(t))\} = \emptyset$. Set $I_{a+} \equiv \max\{i : p_i \ge a\}$ and $I_{a-} \equiv \min\{n : \sum_{i=n}^\infty p_i < a - \max\{p_i : p_i < a\}\}$. Then for any $j \in \mathbb{N}$ and any $t \in (t_j', t_j'')$, there must exist two indices $I_{a+} < i_1 < i_2 < I_{a-}$ such that $\phi_{i_1}(t) = \phi_{i_2}(t) = f_j(t)$ and hence $\bigcup_{I_{a+} < i_1 < i_2 < I_{a-}} \{t : \phi_{i_1}(t) = \phi_{i_2}(t)\} = \bigcup_j (t_j', t_j'')$, contradicting Lemma 3.3.
Thus the claim (7) holds. This combined with Lemma 3.4 implies that the set of functions $\{\phi_i : p_i \ge a\}$ can be identified for any $a > 0$ and hence the set $\{\phi_i\}$ can also be identified. It follows from Lemma 3.3 that $p_i = \inf_{t \ge 0} F_{\phi_\Theta(t)}(\{\phi_i(t)\})$. ∎

4 Countable mixtures of sequences with m.g.f.'s

We can use ideas similar to those in the previous section to show that two countable mixtures $(X_k)_{k \in \mathbb{N}}$ and $(\hat{X}_k)_{k \in \mathbb{N}}$ of real–valued i.i.d. sequences have the same joint distributions if their partial sums S_n and $\hat{S}_n$ have the same distribution for all $n \in \mathbb{N}$ and also for some $\epsilon > 0$ we have $\mathbb{E}[\exp(tX_1)] = \mathbb{E}[\exp(t\hat{X}_1)] < \infty$ for all $t \in (-\epsilon, \epsilon)$. The key ingredient is the following counterpart of Lemma 3.3. This result follows from the uniqueness of characteristic functions, the fact that a moment generating function which is finite in the interval $(-\epsilon, \epsilon)$ can be uniquely extended to an analytic function in the strip $\{z \in \mathbb{C} : \Re z \in (-\epsilon, \epsilon)\}$, and the fact that the zeroes of analytic functions defined in some region have no points of accumulation in the region.

Lemma 4.1 *Let μ and ν be distinct probability measures on $\mathbb{R}$. Suppose for some $\epsilon > 0$ that $\alpha(t) \equiv \int \exp(tx) d\mu(x) < \infty$ and $\beta(t) \equiv \int \exp(tx) d\nu(x) < \infty$ for all $t \in (-\epsilon, \epsilon)$. Then the set $\{t \in (-\epsilon', \epsilon') : \alpha(t) = \beta(t)\}$ is finite for all $\epsilon' < \epsilon$.*

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