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# SPECTRAL GAP FOR THE INTERCHANGE PROCESS IN A BOX

BEN MORRIS[1]
*Department of Mathematics, University of California, Davis CA 95616.*
email: morris@math.ucdavis.edu

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*Abstract*
We show that the spectral gap for the interchange process (and the symmetric exclusion process) in a $d$-dimensional box of side length $L$ is asymptotic to $\pi^2/L^2$. This gives more evidence in favor of Aldous's conjecture that in any graph the spectral gap for the interchange process is the same as the spectral gap for a corresponding continuous-time random walk. Our proof uses a technique that is similar to that used by Handjani and Jungreis, who proved that Aldous's conjecture holds when the graph is a tree.

# 1 Introduction

## 1.1 Aldous's conjecture

This subsection is taken (with minor alterations) from David Aldous's web page. Consider an $n$-vertex graph $G$ which is connected and undirected. Take $n$ particles labeled $1, 2, ..., n$. In a configuration, there is one particle at each vertex. The *interchange process* is the following continuous-time Markov chain on configurations. For each edge $(i, j)$, at rate 1 the particles at vertex $i$ and vertex $j$ are interchanged.
The interchange process is reversible, and its stationary distribution is uniform on all $n!$ configurations. There is a spectral gap $\lambda_{\mathrm{IP}}(G) > 0$, which is the absolute value of the largest non-zero eigenvalue of the transition rate matrix. If instead we just watch a single particle, it performs a continuous-time random walk on $G$ (hereafter referred to simply as "the continuous-time random walk on $G$"), which is also reversible and hence has a spectral gap $\lambda_{\mathrm{RW}}(G) > 0$. Simple arguments (the contraction principle) show $\lambda_{\mathrm{IP}}(G) \leq \lambda_{\mathrm{RW}}(G)$.

**Problem**. Prove $\lambda_{\mathrm{IP}}(G) = \lambda_{\mathrm{RW}}(G)$ for all $G$.

**Discussion.** Fix $m$ and color particles $1, 2, ...., m$ red. Then the red particles in the interchange process behave as the usual exclusion process (i.e., $m$ particles performing the continuous-time

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random walk on $G$, but with moves that take two particles to the same vertex suppressed). But in the finite setting, the interchange process seems more natural.

### 1.2 Results

Aldous's conjecture has been proved in the case where $G$ is a tree [7] and in the case where $G$ is the complete graph [5]; see also [12]. In this note we prove an asymptotic version of Aldous's conjecture for $G$ a box in $\mathbf{Z}^d$. We show that if $B_L$ denotes a box of side length $L$ in $\mathbf{Z}^d$ then

$$\frac{\lambda_{\mathrm{IP}}(B_L)}{\lambda_{\mathrm{RW}}(B_L)} \to 1,$$

as $L \to \infty$.

**Remark:** After completing a draft of this paper, I learned that Starr and Conomos had recently obtained the same result (see [14]). Their proof uses a similar approach, although the present paper is somewhat shorter. □

**Connection to simple exclusion.** Our result gives a bound on the spectral gap for the exclusion process. The exclusion process is a widely studied Markov chain, with connections to card shuffling [16, 1], statistical mechanics [8, 13, 2, 15], and a variety of other processes (see e.g., [10, 6]); it has been one of the major examples behind the study of convergence rates for Markov chains (see, e.g., [6, 3, 16, 1]). Our result implies that the spectral gap for the symmetric exclusion process in $B_L$ is asymptotic to $\pi^2/L^2$. The problem of bounding the spectral gap for simple exclusion was studied in Quastel [13] and a subsequent independent paper of Diaconis and Saloff-Coste [3]. Both of these papers used a comparison to Bernoulli-Laplace diffusion (i.e., the exclusion process in the complete graph) to obtain a bound of order $1/dL^2$. Diaconis and Saloff-Coste explicitly wondered whether the factor $d$ in the denominator is necessary; in the present paper we show that it is not.

## 2 Background

Consider a continuous-time Markov chain on a finite state space $W$ with a symmetric transition rate matrix $Q(x, y)$. The spectral gap is the minimum value of $\alpha > 0$ such that

$$Qf = -\alpha f, \tag{1}$$

for some $f : W \to \mathbf{R}$. The spectral gap governs the asymptotic rate of convergence to the stationary distribution. Define

$$\mathcal{E}(f, f) = \frac{1}{2|W|} \sum_{x,y \in W} (f(x) - f(y))^2 Q(x, y),$$

and define

$$\mathrm{var}(f) = \frac{1}{|W|} \sum_{x \in W} (f(x) - \mathbf{E}(f))^2,$$

where

$$\mathbf{E}(f) = \frac{1}{|W|} \sum_{x \in W} f(x).$$

If $f$ is a function that satisfies $Qf = -\lambda f$ for some $\lambda > 0$, then

$$\lambda = \frac{\mathcal{E}(f,f)}{\mathrm{var}(f)}. \tag{2}$$

Furthermore, if $\alpha$ is the spectral gap then for any non-constant $f : W \to \mathbf{R}$ we have

$$\frac{\mathcal{E}(f,f)}{\mathrm{var}(f)} \geq \alpha. \tag{3}$$

Thus the spectral gap can be obtained by minimizing the left hand side of (3) over all non-constant functions $f : W \to \mathbf{R}$.

## 3 Main result

Before specializing to the interchange process, we first prove a general proposition relating the eigenvalues of a certain function of a Markov chain to the eigenvalues of the Markov chain itself. Let $X_t$ be a continuous-time Markov chain on a finite state space $W$ with a symmetric transition rate matrix $Q(x, y)$. Let $T$ be another space and let $g : W \to T$ be a function on $W$ such that if $g(x) = g(y)$ and $U = g^{-1}(u)$ for some $u$, then $\sum_{u' \in U} Q(x, u') = \sum_{u' \in U} Q(y, u')$. Note that $g(X_t)$ is a Markov chain. Let $W'$ denote the collection of subsets of $W$ of the form $g^{-1}(u)$ for some $u \in T$. We can identify the states of $g(X_n)$ with elements of $W'$. Let $Q'$ denote the transition rate matrix for $g(X_n)$. Note that if $U, U' \in W'$, with $U = g^{-1}(u)$ for some $u \in T$ and $U \neq U'$, then $Q'(U, U') = \sum_{y \in U'} Q(u, y)$.
We shall need the following proposition, which generalizes Lemma 2 of [7].

**Proposition 1.** *Let $X_t$, $g$ and $Q'$ be as defined above. Suppose $f : W \to \mathbf{R}$ is an eigenvector of $Q$ with corresponding eigenvalue $-\lambda$ and define $h : W' \to \mathbf{R}$ by $h(U) = \sum_{x \in U} f(x)$. Then $Q'h = -\lambda h$. That is, either $h$ is an eigenvector of $Q'$ with corresponding eigenvalue $-\lambda$, or $h$ is identically zero.*

**Proof:** Note that for all $U' \in W'$ we have

$$\begin{aligned}
(Q'h)(U') &= \sum_{U \in W'} h(U) Q'(U, U') \\
&= \sum_{U \in W'} \sum_{x \in U} f(x) \sum_{y \in U'} Q(x, y) \\
&= \sum_{y \in U'} (Qf)(y) \\
&= -\lambda \sum_{y \in U'} f(y) \\
&= -\lambda h(U'),
\end{aligned}$$

so $Q'h = -\lambda h$. $\square$

The following Lemma is a weaker version of Aldous's conjecture. The proof is similar to the proof of Theorem 1 in [7].

**Lemma 2.** *Let $G$ be a connected, undirected graph with vertices labeled $1, \ldots, n$. For $2 \leq k \leq n$ let $G_k$ be the subgraph of $G$ induced by the vertices $1, 2, \ldots, k$. Let $\lambda_{\mathrm{RW}}(G_k)$ be the spectral gap for the continuous-time random walk on $G_k$, and define $\alpha_k = \min_{2 \leq j \leq k} \lambda_{\mathrm{RW}}(G_j)$. Then*

$$\lambda_{\mathrm{IP}}(G) \geq \alpha_n.$$

**Proof:** Our prooof will be by induction on the number of vertices $n$. The base case $n = 2$ is trivial, so assume $n > 2$. Let $W$ and $Q$ be the state space and transition rate matrix, respectively, for the interchange process on $G$. Let $f : W \to \mathbf{R}$ be a function that satisfies $Qf = -\lambda f$. We shall show that $\lambda \geq \alpha_n$. Note that a configuration of the interchange process can be identified with a permutation $\pi$ in $S_n$, where if particle $i$ is in vertex $j$, then $\pi(i) = j$. For positive integers $m$ and $k$ with $m, k \leq n$, we write $f(\pi(m) = k)$ for

$$\sum_{\pi : \pi(m) = k} f(\pi).$$

We consider two cases.

**Case 1:** For some $m$ and $k$ we have $f(\pi(m) = k) \neq 0$. Define $h : V \to \mathbf{R}$ by $h(j) = f(\pi(m) = j)$. Then $h$ is not identically zero, and using Proposition 1 with $g$ defined by $g(\pi) = \pi(m)$ gives that if $Q'$ is the transition rate matrix for continuous time random walk on $G$, then $Q'h = -\lambda h$. It follows that $\lambda$ is an eigenvalue of $Q'$ and hence $\lambda \geq \lambda_{\mathrm{RW}}(G) = \alpha_n$.

**Case 2:** For all $m$ and $k$ we have $f(\pi(m) = k) = 0$. Define the *suppressed process* as the interchange process with moves involving vertex $n$ suppressed. That is, the Markov chain with the following transition rule:

For every edge $e$ not incident to $n$, at rate 1 switch the particles at the endpoints of $e$.

For $1 \leq k \leq n$, let $W_k = \{\pi \in W : \pi^{-1}(n) = k\}$. Note that the $W_k$ are the irreducible classes of the suppressed process, and that for each $k$ the restriction of the suppressed process to $W_k$ can be identified with the interchange process on $G_{n-1}$. For $k$ with $1 \leq k \leq n$, define

$$\mathcal{E}_k(f, f) = \frac{1}{2(n-1)!} \sum_{\pi_1, \pi_2 \in W_k} (f(\pi_1) - f(\pi_2))^2 Q(\pi_1, \pi_2),$$

and define

$$\mathrm{var}_k(f) = \frac{1}{(n-1)!} \sum_{\pi \in W_k} f(\pi)^2.$$

(Note that for every $k$ we have $\sum_{\pi \in W_k} f(x) = 0$.)
By the induction hypothesis, the spectral gap for the interchange process on $G_{n-1}$ is at least $\alpha_{n-1}$. Hence for every $k$ with $1 \leq k \leq n$ we have

$$\mathcal{E}_k(f, f) \geq \alpha_{n-1} \mathrm{var}_k(f) \geq \alpha_n \mathrm{var}_k(f).$$

It follows that

$$\begin{aligned} n!\mathcal{E}(f,f) &\geq \tfrac{1}{2}\sum_{k=1}^{n}\sum_{\pi_1,\pi_2\in W_k}(f(\pi_1)-f(\pi_2))^2Q(\pi_1,\pi_2) && (4)\\ &= \sum_{k=1}^{n}(n-1)!\mathcal{E}_k(f,f) && (5)\\ &\geq \sum_{k=1}^{n}\alpha_n(n-1)!\mathrm{var}_k(f) && (6)\\ &= \alpha_n\sum_{k=1}^{n}\sum_{\pi\in W_k}f(\pi)^2 && (7)\\ &= \alpha_n n!\mathrm{var}(f). && (8)\end{aligned}$$

Combining this with equation (2) gives $\lambda \geq \alpha_n$. □

**Remark:** Theorem 2 is optimal if the vertices are labeled in such a way that $\lambda_{\mathrm{RW}}(G_k)$ is nonincreasing in $k$, in which case it gives $\lambda_{\mathrm{IP}}(G) = \lambda_{\mathrm{RW}}(G)$. Since any tree can be built up from smaller trees (with larger spectral gaps), we recover the result proved in [7] that $\lambda_{\mathrm{IP}}(T) = \lambda_{\mathrm{RW}}(T)$ if $T$ is a tree. □

Our main application of Lemma 2 is the following asymptotic version of Aldous's conjecture in the special case where $G$ is a box in $\mathbf{Z}^d$.

**Corollary 3.** *Let $B_L = \{0,\ldots,L\}^d$ be a box of side length $L$ in $\mathbf{Z}^d$. Then the spectral gap for the interchange process on $B_L$ is asymptotic to $\pi^2/L^2$.*

**Proof:** In order to use Lemma 2 we need to label the vertices of $B_L$ in some way. Our goal is to label in such a way that for every $k$ the quantity $\lambda_{\mathrm{RW}}(G_k)$ (i.e., the spectral gap corresponding to the subgraph of $B_L$ induced by the vertices $1,\ldots,k$) is not too much smaller than $\lambda_{\mathrm{RW}}(B_L)$. So our task is to build $B_L$, one vertex at a time, in such a way that the spectral gaps of the intermediate graphs don't get too small.
We shall build $B_L$ by inductively building $B_{L-1}$ and then building $B_L$ from $B_{L-1}$. Since $\lambda_{\mathrm{RW}}(B_L) \downarrow 0$, it is enough to show that

$$\frac{\beta_L}{\lambda_{\mathrm{RW}}(B_L)} \to 1,$$

where $\beta_L$ is the minimum spectral gap for any intermediate graph between $B_{L-1}$ and $B_L$.
For a graph $H$, let $V(H)$ denote the set of vertices in $H$. For $j \geq 0$, let $\mathcal{L}_j = \{0,\ldots,j\}$ be the line graph with $j+1$ vertices. Define $\gamma_L = \lambda_{\mathrm{RW}}(\mathcal{L}_L)$. It is well known that $\gamma_L$ is decreasing in $L$ and asymptotic to $\pi^2/L^2$ as $L\to\infty$. It is also well known that if $H$ and $H'$ are graphs and $\times$ denotes Cartesian product, then $\lambda_{\mathrm{RW}}(H\times H') = \min(\lambda_{\mathrm{RW}}(H),\lambda_{\mathrm{RW}}(H'))$. Since $B_L = \mathcal{L}_L^d$, it follows that $\lambda_{\mathrm{RW}}(B_L) = \gamma_L$.
We construct $B_L$ from $B_{L-1}$ using intermediate graphs $H_0,\ldots,H_d$, where for $k$ with $1\leq k\leq d$ we define $H_k = \mathcal{L}_L^k \times \mathcal{L}_{L-1}^{d-k}$. Note that $H_0 = B_{L-1}$ and $H_d = B_L$. We obtain $H_k$ from $H_{k-1}$ by adding vertices to lengthen $H_{k-1}$ by one unit in direction $k$. The order in which the vertices in $V(H_k) - V(H_{k-1})$ are added is arbitrary.

Fix $k$ with $1 \le k \le d$, and define $G' = G'(L,k)$ as follows. Let

$$V' = V(H_k), \qquad E' = \{(u,v) : \text{either } u \text{ or } v \text{ is a vertex in } H_{k-1}\},$$

and let $G' = (V', E')$. It is well known and easily shown that if $H$ is a graph, then adding edges to $H$ cannot decrease $\lambda_{\mathrm{RW}}(H)$, nor can removing pendant edges. Since each intermediate graph $\tilde{G}$ between $H_{k-1}$ and $H_k$ can be obtained from $G'$ by adding edges and removing pendant edges, it follows that for any such graph $\tilde{G}$ we have $\lambda_{\mathrm{RW}}(\tilde{G}) \ge \lambda_{\mathrm{RW}}(G')$. Thus, it is enough to bound $\lambda_{\mathrm{RW}}(G')$ from below. We shall show that for any $\epsilon > 0$ we have $\lambda_{\mathrm{RW}}(G'(L,k)) \ge (1-\epsilon)\gamma_L$ if $L$ is sufficiently large.

Let $e_k$ be the unit vector in direction $k$. Let

$$S = V(H_{k-1}); \qquad \partial S = V' - S.$$

Let $X_t$ be the continuous-time random walk on $G'$, with transition rate matrix $Q$. Fix $f : V' \to \mathbf{R}$ with $Qf = -\lambda f$ for some $\lambda > 0$. For $x \in \mathbf{Z}^d$, let $g_k(x)$ denote the component of $x$ in the $k$th coordinate. Note that $g_k(X_t)$ is the continuous-time random walk on $\mathcal{L}_L$. Let $Q'$ be the transition rate matrix for $g_k(X_t)$. Proposition 1 implies that if $h : \{0,\dots,L\} \to \mathbf{R}$ is defined by $h(j) = \sum_{\substack{x \in V' \\ g_k(x)=j}} f(x)$, then $Q'h = -\lambda h$. Thus if $\lambda < \gamma_L$, then $g$ is identically zero and hence $\sum_{x\in S} f(x) = 0$. Define

$$\mathcal{E}(f,f) = \frac{1}{2|V'|} \sum_{x,y \in V'} (f(x) - f(y))^2 Q(x,y),$$

and let $\mathcal{E}_S(f,f)$ be defined analogously, but with only vertices in $S$ included in the sum. Note that $\mathcal{E}(f,f) \ge \mathcal{E}_S(f,f)$. Since $\sum_{x\in S} f(x) = 0$, we have

$$\frac{\mathcal{E}_S(f,f)}{\sum_{x\in S} f(x)^2} \ge \lambda_{\mathrm{RW}}(H_{k-1}) \ge \gamma_L, \tag{9}$$

where the second inequality follows from the fact that $H_{k-1}$ is a Cartesian product of $d$ graphs, each of which is either $\mathcal{L}_{L-1}$ or $\mathcal{L}_L$.

Fix $\epsilon > 0$ and let $M$ be a positive integer large enough so that $(1-4M^{-1})^{-1} \le (1-\epsilon)^{-1/2}$. For each $x \in \partial S$, say that $x$ is *good* if there is a $y \in S$ such that $x = y + ie_k$ for some $i \le M$ and $|f(y)| \le |f(x)|/2$. Otherwise say that $x$ is *bad*. Let $\mathcal{G}$ and $\mathcal{B}$ denote the set of good and bad vertices, respectively, in $\partial S$. Note that if $x$ is bad and $M \le L$ then $f(x)^2 \le \frac{4}{M}\sum_{j=1}^M f(x - je_k)^2$. Summing this over bad $x$ gives

$$\sum_{x\in\mathcal{B}} f(x)^2 \le \frac{4}{M} \sum_{x \in V'} f(x)^2 \tag{10}$$

Note that if $x$ is good, then there must be an $x' \in S$ of the form $x - ie_k$ such that $|f(x') - f(x'+e_k)| > f(x)/2M$. It follows that

$$\frac{\mathcal{E}(f,f)}{\sum_{x\in\mathcal{G}} f(x)^2} \ge 1/4M^2. \tag{11}$$

Since $V' = S \cup \mathcal{B} \cup \mathcal{G}$, combining equations (11), (9) and (10) gives

$$\sum_{x\in V'} f(x)^2 \le (\gamma_L^{-1} + 4M^2)\mathcal{E}(f,f) + 4M^{-1} \sum_{x\in V'} f(x)^2,$$

and hence

$$\sum_{x\in V'} f(x)^2 \le (1-4M^{-1})^{-1}(\gamma_L^{-1}+4M^2)\mathcal{E}(f,f). \tag{12}$$

Recall that $(1-4M^{-1})^{-1} \le (1-\epsilon)^{-\frac{1}{2}}$, and note that since $\gamma_L \to 0$ as $L \to \infty$, we have $\gamma_L^{-1}+4M^2 \le (1-\epsilon)^{-\frac{1}{2}}\gamma_L^{-1}$ for sufficiently large $L$. Combining this with equation (12) gives

$$\frac{\mathcal{E}(f,f)}{\sum_{x\in V'} f(x)^2} \ge (1-\epsilon)\gamma_L,$$

for sufficiently large $L$. It follows that $\lambda_{\mathrm{RW}}(G') \ge (1-\epsilon)\gamma_L$ for sufficiently large $L$ and so the proof is complete. □

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