*Elect. Comm. in Probab.* **10** (2005), 146–154

**ELECTRONIC COMMUNICATIONS in PROBABILITY**

# ON THE NOVIKOV-SHIRYAEV OPTIMAL STOPPING PROBLEMS IN CONTINUOUS TIME

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*Submitted* 12 November 2004, *accepted in final form* 7 July 2005

AMS 2000 Subject classification: 60G40, 60J75, 91B70, 60G51
Keywords: Lévy processes, optimal stopping problem, Appell polynomials, pasting principles.

*Abstract*
Novikov and Shiryaev (2004) give explicit solutions to a class of optimal stopping problems for random walks based on other similar examples given in Darling et al. (1972). We give the analogue of their results when the random walk is replaced by a Lévy process. Further we show that the solutions show no contradiction with the conjecture given in Alili and Kyprianou (2004) that there is smooth pasting at the optimal boundary if and only if the boundary of the stopping reigion is irregular for the interior of the stopping region.

## 1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions. For $x \in \mathbb{R}$ denote by $\mathbb{P}_x(\cdot)$ the law of $X$ when it is started at $x$ and for simplicity write $\mathbb{P} = \mathbb{P}_0$. We denote its Lévy-Khintchine exponent by $\Psi$. That is to say $\mathbb{E}[e^{i\theta X_1}] = \exp\{-\Psi(\theta)\}$ for $\theta \in \mathbb{R}$ such that

$$\Psi(\theta) = i\theta a + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{|x|<1})\Pi(dx) \tag{1}$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi$ is a measure supported on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}} (1 \wedge x^2)\Pi(dx) < \infty.$$

Consider an optimal stopping problem of the form

$$V(x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}(e^{-q\tau} G(X_\tau)\mathbf{1}_{(\tau<\infty)}) \tag{2}$$

where $q \geq 0$ and $\mathcal{T}_{0,\infty}$ is the family of stopping times with respect to $\{\mathcal{F}_t\}$.
The purpose of this short paper is to characterize the solution to (2) for the choices of gain functions

$$G(x) = (x^+)^n \ \ n = 1, 2, 3...$$

under the hypothesis

**(H)** either $q > 0$ or $q = 0$ and $\limsup_{t\uparrow\infty} X_t < \infty$.

Note that when $q = 0$ and $\limsup_{t\uparrow\infty} X_t = \infty$ it is clear that it is never optimal to stop in (2) for the given choices of $G$.
This short note thus verifies that the results of Novikov and Shiryaev (2004) for random walks carry over into the context of the Lévy process as predicted by the aforementioned authors. Novikov and Shiryaev (2004) write:

*"The results of this paper can be generalized to the case of stochastic processes with continuous time parameter (that is for Lévy processes instead of the random walk). This generalization can be done by passage of limit from the discrete time case (similarly to the technique used in Mordecki (2002) for pricing American options) or by use of the technique of pseudo-differential operators (described e.g. in the monograph Boyarchenko and Levendorskii (2002) in the context of Lévy processes)".*

We appeal to neither of the two methods referred to by Novikov and Shiryaev however. Instead we work with fluctuation theory of Lévy processes which is essentially the direct analogue of the random walk counterpart used in Novikov and Shiryaev (2004). In this sense our proofs are loyal to those of of the latter. Minor additional features of our proofs are that we also allow for discounting as well avoiding the need to modify the gain function in order to obtain the solution. Truncation techniques are also avoided as much as possible. Undoubtedly however, the link with Appell polynomials as laid out by Novikov and Shiryaev remains the driving force of the solution. In addition we show that the solutions show no contradiction with the conjecture given in Alili and Kyprianou (2004) that there is smooth pasting at the optimal boundary if and only if the boundary of the stopping reigion is irregular for the interior of the stopping region.

## 2 Results

In order to state the main results we need to introduce one of the tools identified by Novikov and Shiryaev to be instrumental in solving the optimal stopping problems at hand.

**Definition 1 (Appell Polynomials)** *Suppose that $Y$ is a non-negative random variable with $n$-th cumulant given by $\kappa_n \in (0, \infty]$ for $n = 1, 2, ...$ Then define the Appell polynomials iteratively as follows. Take $Q_0(x) = 1$ and assuming that $\kappa_n < \infty$ (equivalently $Y$ has an $n$-th moment) given $Q_{n-1}(x)$ we define $Q_n(x)$ via*

$$\frac{d}{dx} Q_n(x) = n Q_{n-1}(x). \tag{3}$$

*This defines $Q_n$ up to a constant. To pin this constant down we insist that $\mathbb{E}(Q_n(Y)) = 0$. The first three Appell polynomials are given for example by*

$$Q_0(x) = 1, \ Q_1(x) = x - \kappa_1, \ Q_2(x) = (x - \kappa_1)^2 - \kappa_2,$$

$$Q_3(x) = (x-\kappa_1)^3 - 3\kappa_2(x-\kappa_1) - \kappa_3,$$

*under the assumption that $\kappa_3 < \infty$. See also Schoutens (2000) for further details of Appell polynomials.*

In the following theorem, we shall work with the Appell polynomials generated by the random variable $Y = \overline{X}_{\mathbf{e}_q}$ where for each $t \in [0,\infty]$, $\overline{X}_t = \sup_{s\in[0,t]} X_s$ and $\mathbf{e}_q$ is an exponentially distributed random variable which is independent of $X$. We shall work with the convention that when $q = 0$, the variable $\mathbf{e}_q$ is understood to be equal to $\infty$ with probability 1.

**Theorem 2** *Fix $n \in \{1, 2, ...\}$. Suppose that we assume (H) as well as*

$$\int_{(1,\infty)} x^n \Pi(dx) < \infty.$$

*Then $Q_n(x)$ has finite coefficients and there exists $x_n^* \in [0,\infty)$ being the largest root of the equation $Q_n(x) = 0$. Let*

$$\tau_n^* = \inf\{t \geq 0 : X_t \geq x_n^*\}.$$

*Then $\tau_n^*$ is an optimal strategy to (2) with $G(x) = (x^+)^n$. Further,*

$$V_n(x) = \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq x_n^*)})$$

**Theorem 3** *For each $n = 1, 2, ...$ the solution to the optimal stopping problem in the previous theorem is continuous and has the property that*

$$\frac{d}{dx}V_n(x_n^*-) = \frac{d}{dx}V_n(x_n^*+) - \frac{d}{dx}Q_n(x_n^*)\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0).$$

*Hence there is smooth pasting at $x_n^*$ if and only if $0$ is regular for $(0,\infty)$ for $X$.*

**Remark 4** The theory of Lévy processes offers us the opportunity to specify when regularity of 0 for $(0,\infty)$ for $X$ occurs in terms of the triple $(a, \sigma, \Pi)$ appearing the Lévy-Khintchine exponent (1). When $X$ has bounded variation it will be more convenient to write (1) in the form

$$\Psi(\theta) = -i\mathrm{d}\theta + \int_{\mathbb{R}} (1 - e^{i\theta x})\Pi(dx) \tag{4}$$

where $\mathrm{d} \in \mathbb{R}$ is known as the *drift*. We have that 0 is regular for $(0,\infty)$ for $X$ if and only if one of the following three conditions are fulfilled.

**(i)** $\int_{(-1,1)} |x|\Pi(dx) = \infty$ (so that $X$ has unbounded variation).

**(ii)** $\int_{(-1,1)} |x|\Pi(dx) < \infty$ (so that $X$ has bounded variaiton) and in the representation (4) we have $\mathrm{d} > 0$.

**(iii)** $\int_{(-1,1)} |x|\Pi(dx) < \infty$ (so that $X$ has bounded variaiton) and in the representation (4) we have $\mathrm{d} = 0$ and further

$$\int_{(0,1)} \frac{x}{\int_{(0,x)} \Pi(-\infty, -y)dy} \Pi(dx) = \infty.$$

The latter conclusions being collectively due to Rogozin (1968), Shtatland (1965) and Bertoin (1997).

# 3 Preliminary Lemmas

We need some preliminary results given in the following series of lemmas. All have previously been dealt with in Novikov and Shiryayev (2004) for the case of random walks. For some of these lemmas we include slightly more direct proofs which work equally well for random walks (for example avoiding the use of truncation methods).

**Lemma 5 (Moments of the supremum)** *Fix $n > 0$. Suppose that the Lévy process $X$ has jump measure satisfying*

$$\int_{(1,\infty)} x^n \Pi(dx) < \infty. \tag{5}$$

*Then $\mathbb{E}((X_1^+)^n) < \infty$. Suppose further that (H) holds. Then $\mathbb{E}(\overline{X}_{\mathbf{e}_q}^n) < \infty$.*

Although the analogue of this lemma is well known for random walks, it seems that one cannot find so easily the equivalent statement for Lévy processes in existing literature; in particular the final statement of the lemma. None the less the proof can be extracted from a number of well known facts concerning Lévy process.

**Proof.** The fact that $\mathbb{E}((X_1^+)^n) < \infty$ follows from the integral condition can be seen by combining Theorem 25.3 with Proposition 25.4 of Sato (1999).
The remaining statement follows when $q > 0$ by Theorem 25.18 of the same book. To see this one may stochastically dominate the maximum of $X$ at any fixed time (and hence at $\mathbf{e}_q$) by the maxium at the same time of a modified version of $X$, say $X^K$, constructed by replacing the negative jumps of size greater than $K > 0$ by negative jumps of size precisely $K$. One may now apply the aforementioned theorem to this process. Note that one will use in the application that the assumption (5) implies that $X^K$ has absolute moments up to order $n$.
For the case $q = 0$ and $\limsup_{t\uparrow\infty} X_t < \infty$ the final statement can be deduced from the Wiener-Hopf factorization. By considering again the modified process $X^K$ one easily deduces that the descending ladder height process has all moments. Indeed the jumps of the descending ladder height process can be no larger than the negative jumps of $X^K$ and hence the latter claim follows again from Theorem 25.3 with Proposition 25.4 of Sato (1999) applied to the descending ladder height process of $X^K$. On the other hand, $X^K$ has finite absolute moments up to order $n$ and hence finite cumulants up to order $n$. Amongst other things, the Wiener-Hopf factorization says that the Lévy-Khintchine exponent, which is a cumulant generating function[1], factorizes into the cumulant generating functions of the ascending and descending ladder height processes. The ascending ladder height process of $X^K$ is therefore forced to have finite cumulants, and hence finite moments, up to order $n$; see for example the representation of cumulant generating functions for distributions which do not have all moments in Lukacs (1970). By choosing $K$ sufficiently large so that $\mathbb{E}(X_1^K) < 0$ (which is possible since the assumptions on $X$ imply that $\mathbb{E}(X_1) < 0$) we have $\overline{X}_\infty^K < \infty$. Since $\overline{X}_\infty^K$ is equal in distribution to the ascending ladder height subordinator of $X^K$ stopped at an independent and exponentially distributed time, the finiteness of the $n$-th moment of $\overline{X}_\infty^K$, and hence of $\overline{X}_\infty \le \overline{X}_\infty^K$, follows from the same statement being true of the ascending ladder height subordinator of $X^K$.
Note, that the above argument using the Wiener-Hopf factorization can easily be adapted to deal with the case $q > 0$ too. ■

[1]Note, the cumulant generating function is sometimes called the second characteristic function (cf. Lukacs (1970)).

**Lemma 6 (Mean value property)** *Fix $n \in \{1, 2, ...\}$ Suppose that $Y$ is a non-negative random variable satisfying $\mathbb{E}(Y^n) < \infty$. Then if $Q_n$ is the $n$-th Appell polynomial generated by $Y$ then we have that*

$$\mathbb{E}(Q_n(x+Y)) = x^n$$

*for all $x \in \mathbb{R}$.*

**Proof.** As remarked in Novikov and Shiryaev (2004), this result can be obtained by truncation of the variable $Y$. However, it can also be derived from the definition of $Q_n$ given in (3). Indeed note the result is trivially true for $n = 1$. Next suppose the result is true for $Q_{n-1}$. Then using dominated convergence we have from (3) that

$$\frac{d}{dx}\mathbb{E}(Q_n(x+Y)) = \mathbb{E}\left(\frac{d}{dx}Q_n(x+Y)\right) = n\mathbb{E}(Q_{n-1}(x+Y)) = nx^{n-1}.$$

Solving together with the requirement that $\mathbb{E}(Q_n(Y)) = 0$ we have the result. ■

**Lemma 7 (Fluctuation identity)** *Fix $n \in \{1, 2, ...\}$ and suppose that*

$$\int_{(1,\infty)} x^n \Pi(dx) < \infty$$

*and that hypothesis (H) holds. Then for all $a > 0$ and $x \in \mathbb{R}$*

$$\mathbb{E}_x(e^{-q\tau_a^+} X^n_{\tau_a^+} \mathbf{1}_{(\tau_a^+ < \infty)}) = \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq a)})$$

*where $\tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}$.*

**Proof.** Note that on the event $\{\tau_a^+ < \mathbf{e}_q\}$ we have that $\overline{X}_{\mathbf{e}_q} = X_{\tau_a^+} + S$ where $S$ is independent of $\mathcal{F}_{\tau_a^+}$ and has the same distribution as $\overline{X}_{\mathbf{e}_q}$. It follows that

$$\mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq a)}|\mathcal{F}_{\tau_a^+}) = \mathbf{1}_{(\tau_a^+ < \mathbf{e}_q)} h(X_{\tau_a^+})$$

where $h(x) = \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q}))$. From Lemma 6 with $Y = \overline{X}_{\mathbf{e}_q}$ one also has that $h(x) = x^n$. We see then by taking expectations again in the previous calculation that

$$\mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq a)}) = \mathbb{E}_x(e^{-q\tau_a^+} X^n_{\tau_a^+} \mathbf{1}_{(\tau_a^+ < \infty)})$$

as required. ■

**Lemma 8 (Largest positive root)** *Fix $n \in \{1, 2, ...\}$ and suppose that*

$$\int_{(1,\infty)} x^n \Pi(dx) < \infty.$$

*Suppose that hypothesis (H) holds and $Q_n$ is generated by $\overline{X}_{\mathbf{e}_q}$. Then $Q_n$ has a unique positive root $x_n^*$ such that $Q_n(x)$ is negative on $[0, x_n^*)$ and positive and increasing on $[x_n^*, \infty)$.*

**Proof.** The proof follows proof of the same statement given for random walks in Novikov and Shiryaev (2004) with minor modifications. (It is important to note that in following their proof, it is not necessary to make an approximation of the Lévy process by a random walk). ■

# 4 Proofs of Theorems

**Proof of Theorem 2.** In light of the Novikov-Shiryaev optimal stopping problems and their solutions, we verify that the analogue of their solution, namely the one proposed in Theorem 2, is also a solution for (2) for $G(x) = (x^+)^n$, $n = 1, 2, ....$
To this end, fix $n \in \{1, 2, ...\}$ and define

$$V_n(x) = \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq x_n^*)}) = \mathbb{E}(Q_n(x + \overline{X}_{\mathbf{e}_q})\mathbf{1}_{(x_n^* - \overline{X}_{\mathbf{e}_q} \leq x)}).$$

From the above representation one easily deduces that $V_n$ is right continuous. From Lemma 7 we have that

$$V_n(x) = \mathbb{E}_x(e^{-q\tau_n^*}(X^+_{\tau_n^*})^n \mathbf{1}_{(\tau_n^* < \infty)})$$

and hence the pairs $(V_n, \tau_n^*)$ are a candidate pair to solve the optimal stopping problem.
Secondly we prove that $V_n(x) \geq (x^+)^n$ for all $x \in \mathbb{R}$. Note that this statement is obvious for $x \in (-\infty, 0] \cup [x_n^*, \infty)$ just from the definition of $V_n$. Otherwise when $x \in (0, x_n^*)$ we have, using the mean value property in Lemma 6 that

$$\begin{aligned} V_n(x) &= \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq x_n^*)}) \\ &= x^n - \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} < x_n^*)}) \\ &\geq (x^+)^n \end{aligned}$$

where the final inequality follows from Lemma 8 and specifically the fact that $Q_n(x) \leq 0$ on $(0, x_n^*]$. Note in the second equality above, by taking limits as $x \uparrow x_n^*$ and using the fact that $Q(x_n^*) = 0$ we see that $V_n(x-) = (x^+)^n$ at $x = x_n^*$. That is to say there is continuity at $x_n^*$.
Thirdly on the event that $\{\mathbf{e}_q > t\}$ we have that $\overline{X}_{\mathbf{e}_q}$ is equal in distribution to $(X_t + S) \vee \overline{X}_t$ where $S$ is independent of $\mathcal{F}_t$ and equal in distribution to $\overline{X}_{\mathbf{e}_q}$. In particular $\overline{X}_{\mathbf{e}_q} \geq X_t + S$. Since, $\mathbb{P}_x$ almost surely, $Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq x_n^*)} \geq 0$ and $Q_n$ is positive increasing on $[x_n^*, \infty)$ it follows that

$$\begin{aligned} V_n(x) &\geq \mathbb{E}_x(\mathbf{1}_{(\mathbf{e}_q > t)} Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq x_n^*)}) \\ &\geq \mathbb{E}_x(\mathbf{1}_{(\mathbf{e}_q > t)} \mathbb{E}_x(Q_n(X_t + S)\mathbf{1}_{(X_t + S \geq x_n^*)} | \mathcal{F}_t)) \\ &= \mathbb{E}_x(e^{-qt} V_n(X_t)). \end{aligned}$$

From this inequality together with the Markov property, it is easily shown that $\{e^{-qt}V_n(X_t) : t \geq 0\}$ is a supermartingale. As $V_n$ and $X$ are right continuous then so is the latter supermartingale.
Finally we put these three facts together as follows to complete the proof. From the supermartingale property and Doob's Optimal Stopping Theorem we have for any $\tau \in \mathcal{T}_{0,\infty}$ that

$$V_n(x) \geq \mathbb{E}_x(e^{-q(t\wedge\tau)} V_n(X_{t\wedge\tau})).$$

Hence by Fatou's Lemma,

$$\begin{aligned} V_n(x) &\geq \mathbb{E}_x(\liminf_{t\uparrow\infty} e^{-q(t\wedge\tau)} V_n(X_{t\wedge\tau})) \\ &\geq \mathbb{E}_x(\liminf_{t\uparrow\infty} e^{-q(t\wedge\tau)} V_n(X_{t\wedge\tau})\mathbf{1}_{(\tau<\infty)}) \\ &= \mathbb{E}_x(e^{-q\tau} V_n(X_\tau)\mathbf{1}_{(\tau<\infty)}). \end{aligned}$$

Using the fact that $\tau$ is arbitrary in $\mathcal{T}_{0,\infty}$ together with the lower bound on $V_n$, it follows that

$$V_n(x) \geq \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}_x(e^{-q\tau}V_n(X_\tau)\mathbf{1}_{(\tau<\infty)}) \geq \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}_x(e^{-q\tau}(X_\tau^+)^n\mathbf{1}_{(\tau<\infty)}).$$

On the other hand, rather trivially, we have

$$\sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}_x(e^{-q\tau}(X_\tau^+)^n\mathbf{1}_{(\tau<\infty)}) \geq \mathbb{E}_x(e^{-q\tau_n^*}(X_{\tau_n^*}^+)^n\mathbf{1}_{(\tau_n^*<\infty)}) = V_n(x).$$

and the proof of the theorem follows. ■

**Proof of Theorem 3.** On account of the fact that $(x^+)^n$ is convex, it follows that for each fixed $\tau\in\mathcal{T}_{0,\infty}$ the expression $\mathbb{E}(e^{-q\tau}((x+X_\tau)^+)^n\mathbf{1}_{(\tau<\infty)})$ is convex. Taking the supremum over $\mathcal{T}_{0,\infty}$ preserves convexity (as taking supremum is a subadditive operation) and we see that $V_n$ is a convex, and hence continuous, function.
To establish when there is smooth fit at this point we calculate as follows. For $x<x_n^*$

$$\begin{aligned}\frac{V_n(x_n^*)-V(x)}{x_n^*-x} &= \frac{(x_n^*)^n-x^n}{x_n^*-x}+\frac{\mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<x_n^*)})}{x_n^*-x}\\ &= \frac{(x_n^*)^n-x^n}{x_n^*-x}+\frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q})-Q_n(x_n^*))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<x_n^*)})}{x_n^*-x}\end{aligned}$$

where the final equality follows because $Q_n(x_n^*)=0$. Clearly

$$\lim_{x\uparrow x_n^*}\frac{(x_n^*)^n-x^n}{x_n^*-x}=\frac{dV_n}{dx}(x_n^*+).$$

However,

$$\begin{aligned}&\frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q})-Q_n(x_n^*))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<x_n^*)})}{x_n^*-x}\\ &\quad= \frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q})-Q_n(x))\mathbf{1}_{(x<\overline{X}_{\mathbf{e}_q}<x_n^*)})}{x_n^*-x}\\ &\qquad-\frac{\mathbb{E}_x((Q_n(x_n^*)-Q_n(x))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<x_n^*)})}{x_n^*-x}\end{aligned}\tag{6}$$

where in the first term on the right hand we may restrict the expectation to $\{x<\overline{X}_{\mathbf{e}_q}<x_n^*\}$ as the atom of $\overline{X}_{\mathbf{e}_q}$ at $x$ gives zero mass to the expectation. Denote $A_x$ and $B_x$ the two expressions on the right hand side of (6). We have that

$$\lim_{x\uparrow x_n^*}B_x=-\frac{dQ_n(x_n^*)}{dx}\mathbb{P}(\overline{X}_{\mathbf{e}_q}=0).$$

Integration by parts also gives

$$\begin{aligned}A_x &= \int_{(0,x_n^*-x)}\frac{Q_n(x+y)-Q_n(x)}{x_n^*-x}\mathbb{P}(\overline{X}_{\mathbf{e}_q}\in dy)\\ &= \frac{Q_n(x_n^*)-Q_n(x)}{x_n^*-x}\mathbb{P}(\overline{X}_{\mathbf{e}_q}\in(0,x_n^*-x))\\ &\quad-\frac{1}{x_n^*-x}\int_0^{x_n^*-x}\mathbb{P}(\overline{X}_{\mathbf{e}_q}\in(0,y])\frac{dQ_n}{dx}(x+y)dy.\end{aligned}$$

Hence it follows that

$$\lim_{x \uparrow x_n^*} A_x = 0.$$

In conclusion we have that

$$\lim_{x \uparrow x_n^*} \frac{V_n(x_n^*) - V(x)}{x_n^* - x} = \frac{dV_n}{dx}(x_n^*+) - \frac{dQ_n(x_n^*)}{dx}\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0)$$

which concludes the proof. ∎

# 5 Remarks

**(i)** As in Alili and Kyprianou (2004) one can argue that the occurence of continuous pasting for irregularity and smooth pasting for regularity appear as a matter of principle. The way to see this is to consider the candidate solutions $(V^{(a)}, \tau_a^+)$ where $\tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}$ and

$$V^{(a)}(x) = \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq a)}).$$

Let $\mathcal{C}^*$ be the class of $a > 0$ for which $V^{(a)}$ is bounded below by the gain function and let $\mathcal{C}$ be the class of $a > 0$ in $\mathcal{C}^*$ for which $V^{(a)}$ is superharmonic (i.e. it composes with $X$ to make a supermartingale when discounted at rate $q$). By varying the value of $a$ in $(0, \infty)$ one will find that, when there is irregularity, in general there is a discontinuity of $V^{(a)}$ at $a$ and otherwise when there is regularity, there is always continuity at $a$. When there is irregularity, the choice of $a = x_n^*$ is the unique point for which the discontinuity at $a$ disappears and the function $V^{(a)}$ turns out to be pointwise minimal in $\mathcal{C}$ (consistently with Dynkin's characterization of least superharmonic majorant to the gain) and pointwise maximal in $\mathcal{C}^*$. When there is regularity, the minmal curve indexed in $\mathcal{C}$ and simultaneously the maximal curve in $\mathcal{C}^*$ will occur by adjusting $a$ so that the gradients either side of $a$ match which again turns out to be the unique value $a = x_n^*$.

**(ii)** From arguments presented in Novikov and Shiryaev (2004) together with the supporting arguments given in this paper, it is now clear how to handle the gain function $G(x) = 1 - e^{x^+}$ for Lévy processes instead of random walks as well as how to handle the pasting principles at the optimal boundary. We leave this as an exercise for the reader.

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