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# POSITIVITY OF BROWNIAN TRANSITION DENSITIES

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*Abstract*

*Let $B$ be a Borel subset of $R^d$ and let $p(t,x,y)$ be the transition densities of Brownian motion killed on leaving $B$. Fix $x$ and $y$ in $B$. If $p(t,x,y)$ is positive for one $t$, it is positive for every value of $t$. Some related results are given.*

## 1. Statement of results.

Let $d \geq 2$, let $B$ be a Borel subset of $\mathbb{R}^d$, and let $p(t,x,y)$ denote the transition density of $d$-dimensional Brownian motion killed on exiting $B$. The purpose of this paper is to examine the question of when $p(t,x,y)$ is positive and when it is 0. If $B$ is an open set, this problem is easy, but we have in mind domains which are similar to the following. Let $B(x,r)$ be the open ball of radius $r$ centered at $x$ and let $B = B(0,1) - \cup_{i=1}^{\infty} B(x_i, r_i)$, where $\{x_i\}$ is dense in $B(0,1)$. In view of the fact that the set $\{x_i\}$ is dense, $B$ has empty interior. However, if the $r_i$ are small enough and go to 0 fast enough, there will be infinitely many pairs $(x,y)$ and $t > 0$ for which $p(t,x,y) > 0$ (cf. [BB, Sect. 4]).

In order for the problem of when $p(t,x,y)$ is positive to make sense, we need to define $p(t,x,y)$ precisely. We let

$$p(t,x,y) = \Phi(t,x,y) - \mathbb{E}^x\left[\Phi(t-\tau_B, X_{\tau_B}, y); \tau_B < t\right], \tag{1.1}$$

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where $\mathbb{E}^x$ is the expectation for Brownian motion $X$ started at $x$, $\tau_B = \inf\{t > 0 : X_t \notin B\}$ is the time of first exit from $B$, and $\Phi(t, x, y) = (2\pi t)^{-d/2} \exp(-|x-y|^2/2t)$. With this definition, $p(t, x, y)$ is symmetric in $x$ and $y$ and satisfies the Chapman-Kolmogorov equations:

$$p(t+s, x, y) = \int_B p(t, x, z) p(s, z, y)\, dz,$$

for all $x, y \in \mathbb{R}^d$ and $s, t \geq 0$. See [B], Sect. II.4 or [PS], Sect. 2.4 and 4.7 for proofs.

A key tool in our analysis is the eigenvalue expansion of $p(t, x, y)$ when the Lebesgue measure of $B$ is finite. In [BB] it was proved that $p(t, x, y)$ had an eigenvalue expansion $\sum_i e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)$, where the equality holds for almost every pair $(x, y)$. Our first result of this paper strengthens this equality to all pairs $(x, y)$. Define $\langle f, g \rangle = \int_B f(x) g(x)\, dx$ and $P_t f(x) = \int p(t, x, y) f(y)\, dy$. Set

$$B' = \{x \in B : \mathbb{P}^x(\tau_B > 0) = 1\}.$$

**Theorem 1.1.** *Suppose $B \subset \mathbb{R}^d$ has finite Lebesgue measure and $\mu$ denotes the restriction of Lebesgue measure to $B$. There exist reals $0 < \lambda_1 \leq \lambda_2 \leq \cdots < \infty$ and a collection of orthonormal functions $\varphi_i$ in $L^2(B)$ such that*

(i) *the sequence $\{\lambda_i\}$ has no subsequential limit point other than $\infty$;*

(ii) *for each $t \geq 0$ and every $x \in \mathbb{R}^d$, we have $P_t \varphi_i(x) = e^{-\lambda_i t} \varphi_i(x)$;*

(iii) *if $f \in L^2(B)$, then*

$$P_t f(x) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle f, \varphi_i \rangle \varphi_i(x), \qquad \forall x \in \mathbb{R}^d; \tag{1.2}$$

*the convergence is absolute and the convergence is uniform over $B$;*

(iv) *we have the expansion*

$$p(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y), \qquad \forall x, y \in \mathbb{R}^d; \tag{1.3}$$

*the convergence is absolute, and the convergence is uniform over $B \times B$;*

(v) *if for some $t > 0$ we have $p(t, x, y) > 0$ for $\mu^2$-almost every pair $(x, y)$, then $\lambda_1 < \lambda_2$ and $\varphi_1 > 0$, $\mu$-almost everywhere;*

(vi) *the $\varphi_i$ form a complete orthonormal system for $L^2(B')$.*

**Remark.** In [BB] it was incorrectly asserted that the $\{\varphi_i\}$ were a complete orthonormal system for $L^2(B)$.

Note that in the remaining results in this section we do not assume that $B$ has finite Lebesgue measure. The most general result we have on positivity is the following.

**Theorem 1.2.** *Suppose $x, y \in B$ are fixed. Then $t \mapsto p(t, x, y)$, $t \in (0, \infty)$, is either identically 0 or else everywhere strictly positive.*

In other words, if $p(t, x, y) > 0$ for some $t$, then $p(t, x, y) > 0$ for all $t$.

The example of two disjoint open balls shows that Theorem 1.5 below needs an assumption of "connectedness." Let $T_A$ be the hitting time of $A$ by $X_t$. Recall that a set $A$ has positive capacity if and only if $\mathbb{P}^x(T_A < \infty) > 0$ for every $x$. Sets with positive Lebesgue measure have positive capacity.

For our purposes, the following assumption is sufficient.

**Hypothesis 1.3.** *Suppose whenever $x \in B'$ and $A \subseteq B'$ has strictly positive capacity then $\mathbb{P}^x(T_A < \tau_B) > 0$.*

Hypothesis 1.3 is equivalent to some other statements.

**Proposition 1.4.** *The following are equivalent.*

*(i) Suppose whenever $x \in B'$ and $A \subseteq B'$ has strictly positive capacity, then $\mathbb{P}^x(T_A < \tau_B) > 0$.*

*(ii) Suppose whenever $x \in B'$ and $A \subseteq B'$ has strictly positive Lebesgue measure, then $\mathbb{P}^x(T_A < \tau_B) > 0$.*

*(iii) Suppose whenever $x \in B'$ and $A \subseteq B'$ has strictly positive Lebesgue measure, then $\mathbb{E}^x \int_0^{\tau_B} 1_A(X_s)\, ds > 0$.*

Recall that a point $x$ is regular for a set $A$ if $\mathbb{P}^x(T_A = 0) = 1$; see [B] or [PS]. We will show that $p(t,x,y) > 0$ if both $x$ and $y$ are irregular for $B^c$.

**Theorem 1.5.**

*(i) If $x$ or $y$ is regular for $B^c$, then $p(t,x,y) = 0$ for all $t$.*

*(ii) Suppose Hypothesis 1.3 holds. If $x$ and $y$ are both irregular for $B^c$, then $p(t,x,y) > 0$ for all $t > 0$.*

*Examining Hypothesis 1.3 further, we have*

**Theorem 1.6.** *There exists a countable family (possibly finite) of disjoint sets $B_1, B_2, \ldots$ such that $\cup_{j=1}^{\infty} B_j = B'$, each set $B_j$ has strictly positive Lebesgue measure, and for every $B_j$ we have*

*(i) if $x \in B_j$, $A \subseteq B_j$, and $A$ has strictly positive capacity, then $\mathbb{P}^x(T_A < \tau_{B_j}) > 0$;*

*(ii) if $x \in B_j$ and $A \subseteq B' - B_j$, then $\mathbb{P}^x(T_A < \tau_{B'}) = 0$.*

*Finally we have*

**Proposition 1.7.** *Suppose $\mu(B) < \infty$ and Hypothesis 1.3 holds. If $z \in B'$, then $\varphi_1(z) > 0$. Moreover $\lambda_1 < \lambda_2$.*

**Remarks**. (a) It is clear that our proofs will work for many other symmetric Markov processes. The key property is that the transition densities be bounded.
(b) We consider only Borel sets in this paper.
(c) We are grateful to Pat Fitzsimmons for pointing out to us that the conditions in Proposition 1.4 are also equivalent to

(iv) $B'$ is connected in the fine topology,

and that this may be proved readily from the results in [NW].

## 2. Proofs of results.

**Proof of Theorem 1.1.** By [BB, Th. 1.1], there exist reals $\lambda_i$ and functions $\overline{\varphi}_i$ such that $0 < \lambda_1 \le \lambda_2 \le \cdots < \infty$, $\{\lambda_i\}$ has no subsequential limit points other than $\infty$, the $\overline{\varphi}_i$ are orthonormal in $L^2(B)$,

$$P_t \overline{\varphi}_i(x) = e^{-\lambda_i t} \overline{\varphi}_i(x) \tag{2.1}$$

for almost all $x$, and $p(t,x,y) = \sum_i e^{-\lambda_i t}\overline{\varphi}_i(x)\overline{\varphi}_i(y)$ for $\mu^2$-almost every pair $(x,y)$.

Set $\widetilde{\varphi}_i(t,x) = e^{\lambda_i t}P_t\overline{\varphi}_i(x)$. By the eigenvalue expansion of $p(t,x,y)$, for all $t$ we have $\widetilde{\varphi}_i(t,x) = \overline{\varphi}_i(x)$, a.e. If $s<t$,

$$\begin{aligned}\widetilde{\varphi}_i(t,x) &= e^{\lambda_i t}P_t\overline{\varphi}_i(x) = e^{\lambda_i s}P_s(e^{\lambda_i(t-s)}P_{t-s}\overline{\varphi}_i)(x)\\ &= e^{\lambda_i s}P_s(\widetilde{\varphi}_i(t-s,\cdot))(x) = e^{\lambda_i s}P_s\overline{\varphi}_i(x) = \widetilde{\varphi}_i(s,x).\end{aligned}$$

Therefore $\widetilde{\varphi}_i(t,x)$ does not depend on $t$.

Let

$$\varphi_i(x) = e^{\lambda_i}P_1\overline{\varphi}_i(x) = \widetilde{\varphi}_i(1,x),$$

and let

$$q(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y). \tag{2.2}$$

Clearly $q$ is symmetric. Since

$$\int \varphi_i(y)\varphi_i(y)\,dy = \int \overline{\varphi}_i(y)\overline{\varphi}_i(y)\,dy = \delta_{ij}, \tag{2.3}$$

we have

$$\begin{aligned}\int_B q(t,x,z)q(s,z,y)dz &= \int_B \sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(z)\sum_{j=1}^{\infty} e^{-\lambda_j s}\varphi_j(z)\varphi_j(y)dz\\ &= \sum_{i=1}^{\infty} e^{-\lambda_i t}\sum_{j=1}^{\infty} e^{-\lambda_j s}\varphi_i(x)\varphi_j(y)\delta_{ij} = q(t+s,x,y).\end{aligned}$$

In other words, $q$ satisfies the Chapman-Kolmogorov equations for all $s,t,x$ and $y$.

Fix $x$. Define

$$Q_t f(x) = \int f(y)q(t,x,y)\,\mu(dy).$$

Suppose $f\in L^2(B)$. Let $h = f - \sum\langle f,\overline{\varphi}_i\rangle\varphi_i$. By [BB], Theorem 1.1,

$$P_t f = \sum\langle f,\overline{\varphi}_i\rangle P_t\overline{\varphi}_i, \qquad \text{a.e.}, \tag{2.4}$$

or $P_t h = 0$ a.e. Since $\varphi_i = \overline{\varphi}_i$ a.e., then

$$e^{-\lambda_i t}\langle\varphi_i,h\rangle = e^{-\lambda_i t}\langle\overline{\varphi}_i,h\rangle = \langle P_t\varphi_i,h\rangle = \langle\varphi_i,P_t h\rangle = 0,$$

which shows that $h$ is orthogonal to all of the $\varphi_i$. By the definition of $q$ and $Q_t$,

$$Q_t h(x) = \int h(y)q(t,x,y)\,\mu(dy) = \int\sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)h(y)\mu(dy) = 0.$$

This implies that,

$$Q_t f(x) = \sum e^{-\lambda_i t}\langle f,\overline{\varphi}_i\rangle\varphi_i(x). \tag{2.5}$$

From (2.4) and (2.1) with $t$ replaced by $t/2$,

$$P_{t/2}f(y) = \sum\langle f,\overline{\varphi}_i\rangle e^{-\lambda_i t/2}\overline{\varphi}_i(y), \qquad \text{a.e.}$$

By the semigroup property and the fact that $P_{t/2}$ has a kernel $p(t/2,x,y)$,

$$\begin{aligned} P_tf(x) &= P_{t/2}(P_{t/2}f)(x) \\ &= \sum \langle f,\overline{\varphi}_i\rangle e^{-\lambda_i t/2} P_{t/2}\overline{\varphi}_i(x) \\ &= \sum e^{-\lambda_i t}\langle f,\overline{\varphi}_i\rangle \varphi_i(x). \end{aligned}$$

Hence

$$P_tf(x) = Q_tf(x) \tag{2.6}$$

for all $f\in L^2(B)$ and all $x$. This implies

$$p(t,x,\cdot) = q(t,x,\cdot), \qquad \text{a.e.} \tag{2.7}$$

Therefore $q$ is a transition density for Brownian motion. Using (2.4), for all $x$ and $y$,

$$q(t,x,y) = \int q(t/2,x,z)q(t/2,y,z)\,dz = \int p(t/2,x,z)p(t/2,y,z)\,dz = p(t,x,y).$$

From the definition of $\varphi_i(x)$ we obtain

$$\varphi_i(x) = e^{\lambda_i t}P_t\overline{\varphi}_i(x) = e^{\lambda_i t}P_t\varphi_i(x),$$

which proves (ii).

Fix some $t_0>0$. Since

$$\begin{aligned} |\varphi_i(x)| &= \left| e^{\lambda_i t_0}P_{t_0}\varphi_i(x)\right| \\ &\le e^{\lambda_i t_0}\Big(\int_B p(t_0,x,y)^2\,dy\Big)^{1/2}\Big(\int_B \varphi_i(x)^2\,dy\Big)^{1/2} \\ &\le c_1(t_0)e^{\lambda_i t_0}, \end{aligned} \tag{2.8}$$

then if $t>2t_0$,

$$e^{-\lambda_i t}|\varphi_i(x)\varphi_i(y)| \le c_1(t_0)^2 e^{-\lambda_i(t-2t_0)}.$$

Set $s=t-2t_0$. By the eigenvalue expansion (2.2),

$$\sum e^{-\lambda_i s} = \int_B q(s,x,x)dx = \int_B p(s,x,x)\,dx \le \Big(\sup_{x,y\in\mathbb{R}^d}\Phi(s,x,y)\Big)\mu(B) < \infty.$$

Therefore the convergence in (1.3) is uniform and absolute on $B\times B$. This proves (iv).

Note that $\langle f,\varphi_i\rangle \le \|f\|_2$. This, (2.5), (2.6) and (2.8) imply (iii). Part (v) follows from [BB, Thm. 1.1].

It remains to prove (vi). Since

$$\varphi_i(x) = e^{\lambda_i t}P_t\varphi_i(x) = e^{\lambda_i t}\mathbb{E}^x[\varphi_i(X_t); t<\tau_B],$$

then $\varphi_i(x)=0$ if $x\notin B'$. Suppose $f$ is in $L^2(B')$, $f=0$ on $(B')^c$, and $f$ is orthogonal to each of the $\varphi_i$. We need to show $f=0$ a.e. on $B'$. By (iii) $P_tf(x)=0$ for $x\in B'$. Let $\varepsilon>0$ and take $g$ continuous so that $\|f-g\|_2<\varepsilon$. We have $P_t g(x)\to g(x)$ as $t\to 0$ for $x\in B'$. On the other hand, $\Phi(t,x,y)$ is the kernel for an approximation to the identity. From

$$|P_t(f-g)(x)|\le\int_B |f(y)-g(y)|\Phi(t,x,y)\,dy$$

and [B], Th. IV.1.6(a) and Th. IV.1.1(b), we conclude

$$\|\sup_{t>0}|P_tf-P_tg|\,\|_2\le c_2\varepsilon.$$

Since $\varepsilon$ is arbitrary, it follows that $P_tf(x)\to f(x)$ a.e. on $B$, hence $f=0$ a.e. on $B'$. $\square$

Before proving Theorem 1.2, we need the following lemmas.

**Lemma 2.1.** *Suppose $\sum_i |c_i|e^{-\lambda_i t}<\infty$ for all $t>0$ and $\sum_i c_ie^{-\lambda_i t}=0$ for all $t\in(a,b)\subseteq(0,\infty)$. Then $c_i=0$ for all $i$.*

**Proof.** First suppose $\sum|c_i|<\infty$. Let $\nu(dx)=\sum c_i\delta_{\lambda_i}(dx)$. Then

$$\int_0^\infty e^{-tx}\nu(dx)=\sum c_ie^{-\lambda_i t}=0$$

for all $t\in(a,b)$. By the uniqueness of the Laplace transform (see [F], p. 432), $\nu\equiv 0$.

Now write

$$0=\sum c_ie^{-\lambda_i t}=\sum c_i'e^{-\lambda_i s},$$

where $c_i'=c_ie^{-\lambda_i\varepsilon}$, $s=t-\varepsilon$, and $\varepsilon<a\wedge(b-a)/8$. The above equality holds for all $s\in(a+\varepsilon,b-\varepsilon)$, so by the first paragraph, each $c_i'$ is 0. $\square$

**Lemma 2.2.** *Suppose $\mu(B)<\infty$. Let $T>0$, $y\in B$. Then $p(T,X_t,y)$ is continuous in $t$, almost surely.*

**Proof.** Let $U^\lambda f(x)=\int_0^\infty e^{-\lambda t}P_tf(x)\,dt$. Applying the eigenvalue expansion,

$$\varphi_i(x)=(\lambda+\lambda_i)U^\lambda\varphi_i(x)=U^\lambda((\lambda+\lambda_i)\varphi_i^+)(x)-U^\lambda((\lambda+\lambda_i)\varphi_i^-)(x),$$

or $\varphi_i(x)$ is the difference of two $\lambda$-excessive functions. Next use the fact [B, Sect. II.6] that a $\lambda$-excessive function composed with a Brownian motion yields a process with continuous paths, so $\varphi_i(X_t)$ is continuous, a.s. Our result follows from the fact that the convergence in the eigenvalue expansion is absolute and uniform. $\square$

Recall that the fine topology is the smallest topology with respect to which all $\lambda$-excessive functions are continuous. By the proof of Lemma 2.2, the $\varphi_i$ are finely continuous. Hence by the uniform convergence we have

**Corollary 2.3.** *$p(t,x,y)$ is jointly finely continuous in $x$ and $y$.*

**Lemma 2.4.** *Suppose $\mu(B) < \infty$. Suppose $p(T,x,y) \geq \delta > 0$. There exists $s_0$ depending on $T$, $x$, and $y$ such that if $s \leq s_0$ and $A = \{z : p(T,z,y) > \delta/2\}$, then $\mathbb{P}^x(X_s \in A) > 3/4$.*

**Proof.** Notice that $\tau > 0$, $\mathbb{P}^x$-a.s., because $p(T,x,y) > 0$. Let $M_t = p(T, X_t, y)$. Since $M_0 \geq \delta$ a.s.-$\mathbb{P}^x$ and $M_t$ is continuous by Lemma 2.2, there exists $s_0$ such that

$$\mathbb{P}^x(\inf_{s \leq s_0} M_s > \delta/2, \tau > s_0) > 3/4.$$

Our result is immediate from this. $\square$

**Proof of Theorem 1.2.** Suppose first that $\mu(B) < \infty$ and that $p(T,x,y) > 0$. Let $t \in (0,\infty)$ be fixed. By Theorem 1.1 (iv), $\sum e^{-\lambda_i t}|\varphi_i(x)\varphi_i(y)| < \infty$, for all $t > 0$. This, the assumption that $p(T,x,y) > 0$ and Lemma 2.1 applied with $c_i = \varphi_i(x)\varphi_i(y)$ imply that there exists $t_1 \in (0,t)$ with $p(t_1,x,y) > 0$. Let $s = t - t_1$. By the eigenvalue expansion (1.3) and the Cauchy-Schwarz inequality,

$$\begin{aligned} 0 &< p(T,x,y) = \sum e^{-\lambda_i T}\varphi_i(x)\varphi_i(y) \\ &\leq \Big(\sum e^{-\lambda_i T}\varphi_i(x)^2\Big)^{1/2}\Big(\sum e^{-\lambda_i T}\varphi_i(y)^2\Big)^{1/2}. \end{aligned}$$

This implies that $|\varphi_i(x)| > 0$ for some $i$, and by the eigenvalue expansion again, $p(s,x,x) > 0$.

From Lemma 2.4 with $\delta = p(t_1,x,y) \wedge p(s,x,x)$, there exist $A_1$, $A_2$, and $s > 0$ such that $p(t_1,z,y) > \delta/2$ if $z \in A_1$, $p(s,z,x) > \delta/2$ if $z \in A_2$, and $\mathbb{P}^x(X_s \in A_i) > 3/4$, $i = 1,2$. If $D = A_1 \cap A_2$, then $\mathbb{P}^x(X_s \in D) > 0$, which implies that $\mu(D) > 0$. Therefore

$$\begin{aligned} p(t,x,y) &= \int_B p(t_1,z,y)p(s,z,x)\,dz \\ &\geq \int_D p(t_1,z,y)p(s,z,x)\,dz \\ &\geq (\delta/2)^2\mu(D) > 0. \end{aligned}$$

If $\mu(B) = \infty$, let $B_n = B \cap B(0,n)$. Then $\tau_{B_n} \uparrow \tau_B$. It is easy to see from (1.1) that the transition densities $p_n(t,x,y)$ for Brownian motion killed on exiting $B_n$ increase to $p(t,x,y)$. So if $p(t,x,y) > 0$ for some $t$, then $p_n(t,x,y) > 0$ for some $n$. If $s$ is a positive real, the argument of the above paragraphs shows that $p_n(s,x,y) > 0$. Since $p(s,x,y) \geq p_n(s,x,y)$, the theorem follows. $\square$

We prove the remaining assertions of Section 1 in this order: Theorem 1.5, Proposition 1.7, Theorem 1.6, Proposition 1.4.

**Proof of Theorem 1.5.** If $x$ or $y$ is regular for $B^c$, then $p(t,x,y) = 0$ by [B, Sect. II.4].

Next suppose $x, y \in B'$ and Hypothesis 1.3 holds. Since $\mathbb{P}^x(\tau_{B'} > 0) > 0$, Fubini's theorem implies that $p(t,x,z) > 0$ for some $z \in B'$ and $t > 0$. Fix some $z$ and $t$ with these properties. For $a \in (0,1)$ let $V_a$ be the set of $v \in B'$ such that $p(t,x,v) > a\,p(t,x,z)$. By Corollary 2.3, the set $V_a$ is a fine neighborhood of $z$ for every $a \in (0,1)$. It follows that $\mathbb{P}^z(\tau_{V_{1/2}} > 0) = 1$. Therefore $V_{1/2}$ is a non-polar set, and so by Hypothesis 1.3, $\mathbb{P}^y(T_{V_{1/2}} < \tau_{B'}) > 0$. Let $S$ be the first exit time from $V_{1/4}$ after time $T_{V_{1/2}}$. Using Corollary 2.3 again we obtain $\mathbb{P}^y(T_{V_{1/2}} < S < \tau_{B'}) > 0$. This shows that the process starting from $y$ spends a positive amount of time in $V_{1/4}$ before exiting $B'$ with positive probability. Now apply Fubini's theorem to see that $p(s,y,v)>0$ for some $s>0$ and all $v$ in a set $A\subset V_{1/4}$ of positive measure. Theorem 1.2 implies that $p(u,x,v)>0$ and $p(u,y,v)>0$ for all $u>0$ and all $v\in A$. We obtain for any $u_0>0$,

$$p(u_0,x,y)\geq\int_A p(u_0/2,x,v)p(u_0/2,v,y)dv>0. \qquad \square$$

**Proof of Proposition 1.7.** Since $z\in B'$, then there exists $\delta>0$ such that $\mathbb{P}^z(X_t\in B)>0$ if $t\leq\delta$, or $\int_B p(t,z,w)\,dw>0$. Suppose $A\subseteq B'$ has positive Lebesgue measure. Recall $U^\lambda$ from the proof of Lemma 2.2. Since we are assuming Hypothesis 1.3, Theorem 1.5 shows that if $x\in B'$ and $w\in A$, then $p(t,x,w)>0$ for all $t$. So

$$\begin{aligned} U^\lambda 1_A(x) &= \mathbb{E}^x\int_0^\infty e^{-\lambda t}1_A(X_t)\,dt \qquad (2.9)\\ &= \int_0^\infty e^{-\lambda t}\int_A p(t,x,w)\,dw\,dt>0. \end{aligned}$$

By the Krein-Rutman theorem, the first eigenfunction of $U^\lambda$, which is $\varphi_1$, is positive a.e. on $B'$. Also the first eigenvalue, namely $(\lambda+\lambda_1)^{-1}$, has multiplicity one, which implies $\lambda_1<\lambda_2$. Since $\varphi_1>0$ a.e. on $B'$, then

$$\varphi_1(z)=e^{\lambda_1\delta}P_\delta\varphi_1(z)=e^{\lambda_1\delta}\int_B p(\delta,z,w)\varphi_1(w)\,dw>0. \qquad \square$$

**Proof of Theorem 1.6.** We define an equivalence relation on the set $B'$ by saying that $x$ and $y$ are in the same equivalence class if and only if for every set $A\subset B'$ we have either

(a) $\mathbb{P}^x(T_A<\tau_{B'})>0$ and $\mathbb{P}^y(T_A<\tau_{B'})>0$; or

(b) $\mathbb{P}^x(T_A<\tau_{B'})=0$ and $\mathbb{P}^y(T_A<\tau_{B'})=0$.

Fix some $x$ and let $B_1$ be the equivalence class of $x$. First we show that $B_1$ has positive Lebesgue measure. Since $x\in B'$, we have $\mathbb{P}^x(\tau_B>0)=1$. Find $s>0$ small such that $\mathbb{P}^x(\tau_B>s)>0$. Then, by Fubini's theorem, there exist $t\in(0,s)$ and a set $C\subset B'$ which has positive Lebesgue measure such that $p(t,x,y)>0$ for all $y\in C$. We will show that $C\subset B_1$. Let $A\subset B'$ be such that $\mathbb{P}^x(T_A<\tau_{B'})>0$. Then we must have $\int_V\mathbb{P}^v(T_A<\tau_{B'})\,dv>0$ for every fine neighborhood $V$ of $x$. Suppose that $y\in C$. Since $p(t,\cdot,\cdot)$ is jointly finely continuous, we have $p(t,v,y)>p(t,x,y)/2$ for all $v$ in some fine neighborhood $V$ of $x$. Then

$$\mathbb{P}^y(T_A<\tau_{B'})\geq\int_V\mathbb{P}^v(T_A<\tau_{B'})p(t,v,y)\,dv>0.$$

An analogous argument using a fine neighborhood of $y$ shows that: if $\mathbb{P}^y(T_A<\tau_{B'})>0$, then $\mathbb{P}^x(T_A<\tau_{B'})>0$. Hence $x$ and $y$ are in the same equivalence class. We conclude that $C\subset B_1$ and so $B_1$ has positive Lebesgue measure. Since this is true of every equivalence class and since Lebesgue measure is $\sigma$-finite, the family of all equivalence classes is at most countable. We will denote the equivalence classes by $B_j$.

It remains to prove (i) and (ii).

Suppose that $x\in B_j$, $A\subset B'-B_j$ and $\mathbb{P}^x(T_A<\tau_{B'})>0$. We must have $\mathbb{P}^x(T_{A\cap B_k}<\tau_{B'})>0$ for at least one $k\neq j$, so we may assume without loss of generality that $A\subset B_k$

for some $k \neq j$. Since $A \subset B'$, we have $\mathbb{P}^x(\tau_{B'} > T_A + s) > 0$ for some $s > 0$. By Fubini's theorem, there exist $u > 0$ and $z \in B_k$ such that $p(u, x, z) > 0$ (recall the set $C$ from the first part of the proof). The argument used in the first part of the argument shows that $x$ and $z$ are in the same equivalence class, i.e., $k = j$. This proves (ii).

Suppose that $A \subset B_j$ has a positive capacity. The set of points in $A$ which are not regular for $A$ is polar ([B, Th. II.5.5]), so we may find a point $y \in A$ with $\mathbb{P}^y(T_A > 0) = 0$. Since $y \in B'$, we have $\mathbb{P}^y(\tau_{B'} = 0) = 0$ and so $\mathbb{P}^y(T_A < \tau_{B'}) > 0$. Consider any $x \in B_j$. Since $x$ and $y$ are in the same equivalence class, $\mathbb{P}^x(T_A < \tau_{B'}) > 0$. By (ii), $\mathbb{P}^x(T_{B_k} < \tau_{B'}) = 0$ for $x \in B_j$ and $k \neq j$. This implies (i) because

$$\mathbb{P}^x(T_A < \tau_{B_j}) = \mathbb{P}^x(T_A < \tau_{B'}) > 0. \qquad \square$$

**Proof of Proposition 1.4.** Since sets of positive Lebesgue measure have positive capacity, (i) implies (ii). Suppose (ii) holds. By Theorem 1.6(ii) the family of sets $\{B_j\}$ consists of a single set. Then Theorem 1.6(i) shows that (i) holds.

It is obvious that (iii) implies (ii). Let us suppose (i) holds and we will prove (iii). If $x \in B'$ and $A \subseteq B'$ has positive Lebesgue measure, then by (i) and Theorem 1.5, $p(t, x, w) > 0$ for all $t > 0$ and all $w \in A$. Statement (iii) now follows by applying (2.9). $\square$

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